Classical Field Theory
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Lecture - 14

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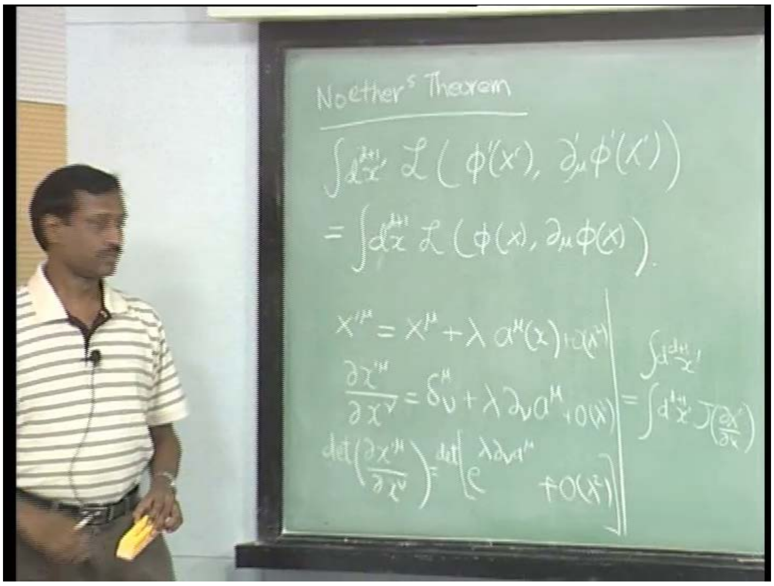

So, we know the statement you have heard it at many many times in this course. So, I will not state it. So, the idea is that if you have a continuous symmetry we should get a bunch of conserved currents depending on how many symmetries are there ok. So, the statement of a symmetry is that. So, so prime coordinates are suppose to be the changed the symmetry act and changes the coordinates from x to x prime. So, this should be equal to. So, this is the statement that the action should be invariant . So, this will be a starting point and so, but since we will be very very general we need to even worry about things like does the measure change. So, let us recall that we will write that the coordinates change as follows lambda is a book keeping parameter and a mu is some general function of x if it were a constant then it just play an usual translations and then of cause d for x prime is equal to d for x.

So, first step is we will do is to work out what this measure is. So, the the standard thing is to say that the d d plus one. So, lets write out. So, this will be the Jacobian and we need to work out what we changed the determinant we need to work out the determinant of the Jacobian because integral d d plus 1 x prime this measure will be equal to integral d d plus 1 x prime the Jacobian of which is the by j of this I mean just the magnitude of this determinant. So, we need to work out that.

Now, this looks very complicated, but what I am going to do is this is only to order lambda square what I will do is I will use a simple device which is we since is only to order lambda square I can take this and write it as exponential of this is like the identity matrix plus some other matrix I can write that as an exponential. So, this is equal to e power lambda d mu a mu plus of course, errors of the order of lambda square and I need to take the det of this, but now I will use the simple formula which says that det of any matrix is equal to the trace ok.

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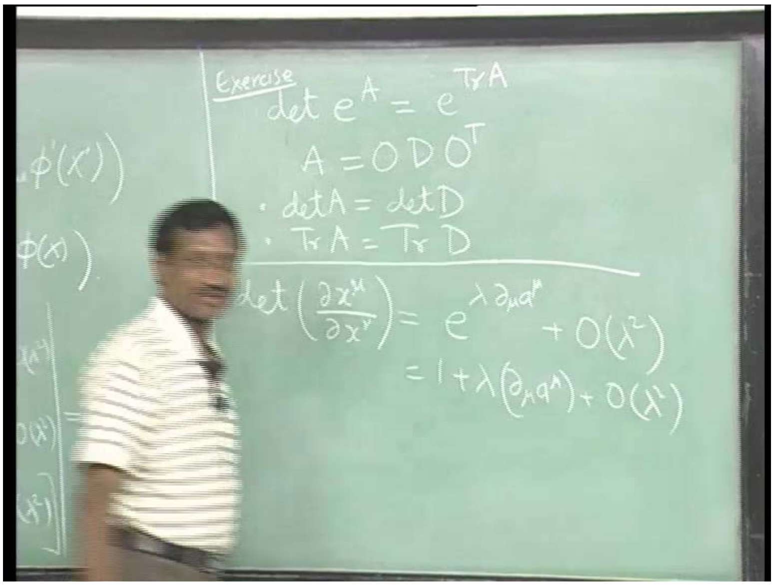

The easy way to see this is to realize that if a is a symmetric matrix for simplicity if a is the symmetry matrix you can always write it as some orthogonal matrix times the diagonal matrix O o transpose. Now, when you go ahead and look at these thing. So, so what we see is that the determinant of a is equal to determinant of d and of course, trace of a is also equal to trace of d by using stander properties of determinant and traces. So, for a diagonal matrix it is easy to check that this is true and because of this relationship you can see that this is true for any symmetric matrix for that matter you can convince yourself that any matrix can be brought towards called the Jodan convenical form that this formula continues to hold. So, I just leave it as an exercise to shows this. So, fun exercise easy to show for this.

So, now, you can see the merit in what I have done out here I want determinant of e power this matrix and that is just becomes. So, so what we get using this formula as the simple application of this is that the determinant of is equal to e power trace of this thing. So, equal to e power lambda trace is nothing, but now in this case its one upper and one lower. So, there is no issue I just stay mu mu that is what I get plus order lambda square and this is equal to one plus. So, I exponentiate and then expand the exponential because life is a lot easier because we want only things to first order ok. There are other ways of proving this, but I like this method. So, we have taken care of this. So, now, we just what I will do is I can write 0 equal to this thing and I just work always to order lambda.

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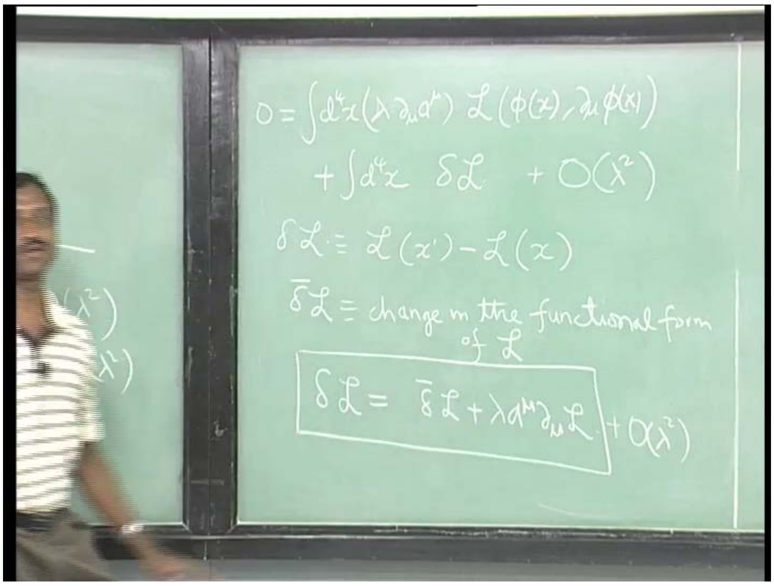

So, the bucky ping device is very nice thing. So, we can see that what we get is 0 equal to and since I am going to yeah let me write it out and then explain. So, so let us for a moment assume since we are going to work only to order lambda plus order lambda square since we are going to work only to order lambda square what I am doing out here is a term like this is already order lambda. So, if I am going keep actually I should not have one out here. So, one term of order lambda comes when this is this contributes a d mu a mu times then this will be to first order will be just the same as this. So, that is what I have written out here, but what is delta l delta l is the part which actually says forget the take the measures to be the same to order lambda there should be they it should be the different of these two things. So, delta l is by definition just I will just write it in short to reminds you what it is instead of writing this is for everything being prime by l of x prime I mean this object and l of x I mean this objects this is just to save space, but we have already seen something that that delta since l is a scalar we saw we introduced delta bar of l which was the change in the functional form ok.

So, that is not this object, but we had proved something we had shown that in several lectures ago we had shown something like this. So, this is what we get and the advantage of working with delta bar is that that commutes with derivatives special derivatives while this does not. So, this take cares of these things plus order lambda square of course, so. So, now, we have we have a nice formula which has in terms of delta bars and explicit derivatives acting on things and. So, we can collect these things and show that 0 equal to ok

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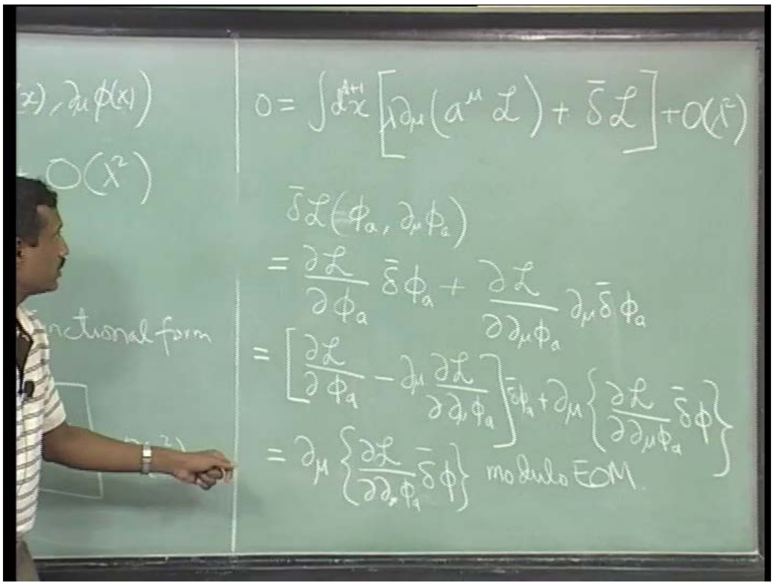

So, I am. So, I am using just formula and plug in it here and you can see that this particular term this and this joint to give you this term plus. So, this will have a lambda going with it plus order lambda square. So, I have just substituted for delta l out here and that is about it. So, we can now look at what delta bar of l is. So, delta bar of l and l is suppose to be a function of y and let us take many scalars or it could be even a vector field. So, this will get terms from plus I have done noting I have just written out how delta bar acts on these things and now what I can do is to integrate. So, rewrite this as follows and then just take this on to this side write something and then we will see what we get.

So, the most important thing out here is since we have delta bar here it commutes with this. So, I i do not have to I mean that is the only reason to work with delta bar is because you have this freedom of what I just did you you commutes with derivatives and. So, now, I can just act on this . So, I am writing it as a total derivative plus this object I am missing something out here I think its delta bar. So, so for we did not make a we only use the fact that it was a symmetry, but now I am going to use the fact that this term is 0 modular of equations of motion. So, this is equal to modulo equations of motion because equation of motion is this equal to zero. So, if I am looking at trajectories which are solutions to the equation of motion then this. So, it is on a solution. So, now, you can see that delta bar of l can be replaced with this object out here and just like this was a was a d mu on something you can see that the nice good news here is this is also working like this. So, now, you can see that you get 0 which is an of something which is d mu times something. So, we will get a master formula that is not yet the current because there will be several currents we need details and. So, so what I will call a master formula is almost divested formula, but not quite…

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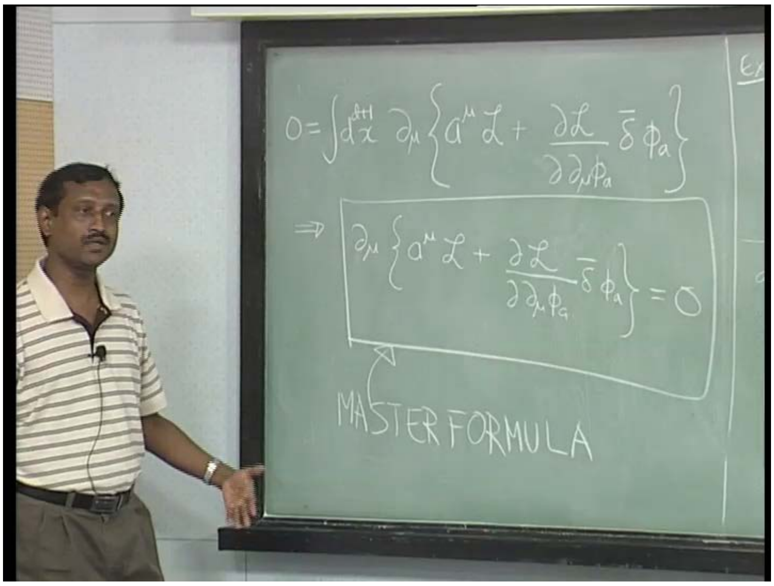

So, I will get 0 and it is not difficult to see that we can argue that the if this is 0 plus order lambda square now I am dropping that, but to order lambda this term has to be 0 and. So, what we will argues is that this the integrant is 0 we are argue something stronger. So, another words this will imply that . So, this is what we will call a master formula ok. So, what will do now is to take various examples and extracts currents from it. So, the first one to do would be translation which we have already seen, but nevertheless we can still do it in this set up and we will see…

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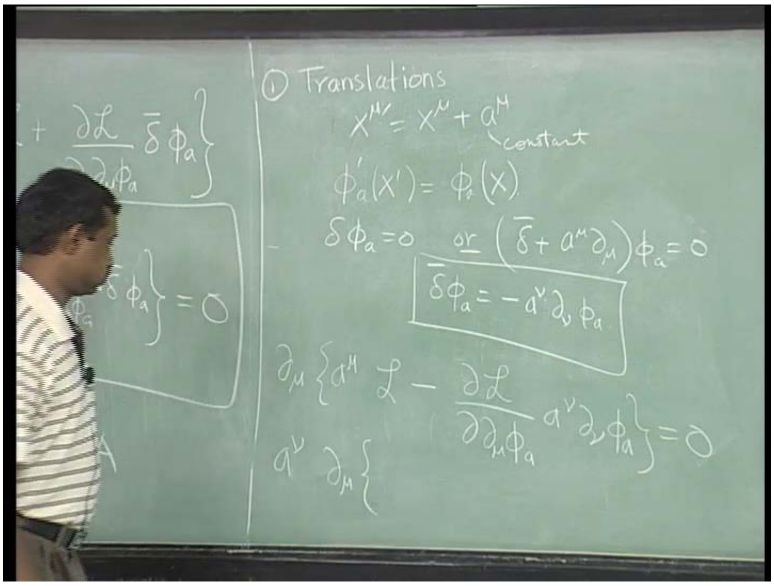

So, we will do three examples today the first one is something which we have already seen and the other two would be new once. So, the first one is just... So, under that x mu prime equal to x mu plus now a mu is a constant it is not a arbitrary function I use the same notation, but I i do not think it will be confusing for you and how does the scalar field for a scalar field I will do it here and for a vector field it will be in the assignment you would be receiving later today. So, this implies that delta of phi a is zero, but delta equal to or delta bar plus because we need in this formula we need to use delta bar. So, what we get is delta bar phi a is equal to minus a mu d mu phi a I will not I will call this some other index let us call this.

So we have everything we needed to put delta bar phi which is just worked out and we already these things. So, we get this. So, the master formula. So, you plug this into the master formula and what do we get we get t mu of a mu l minus excuse me forgotten times delta bar phi a with I put the minus sign here. So, this is what the master formula implies for this now since a is a constant I can just pull out this a mu outside a is a constant. So, it can go though the derivative. So, this is just a mu times d mu times l minus yeah I will put out a nu rather than this. So, you can see for every value for every a there are four parameters I get one and goes out current and this is what we call t mu. So, up to a sign this is exactly what we wrote.

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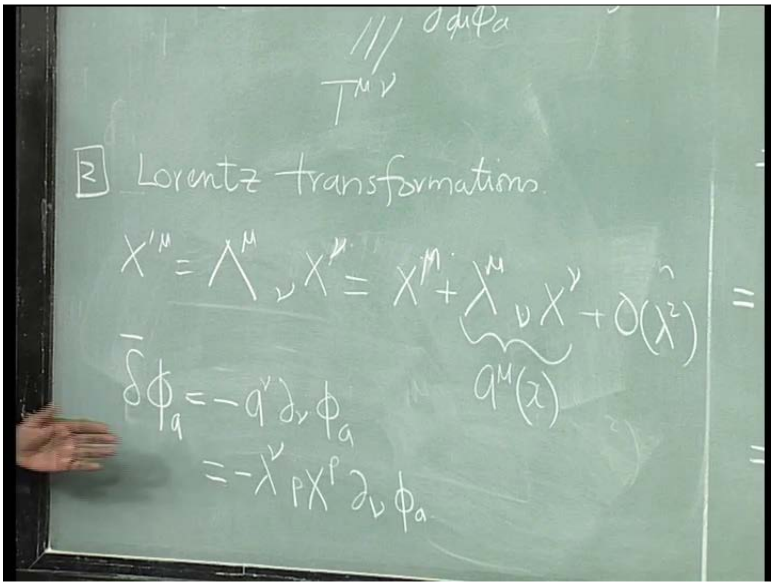

So, this is minus t mu nu one upper lower. So, I can if you wish correct things by lowering this and raising this and writing an eta may be I will write it once and this is the object inside the curly braces which I will call this is exactly what we had got earlier. So, now, we can see that we have four currents or d plus 1 currant arbitrary dimension. So, let us do. So, this is the first example second one is Lorentz transformations. So, x prime mu nu would be mu nu mu, but we are only interested to for infinite decimal things and we solve last lecture that I can rewrite this as and now this lambda itself is my order lambda square not repeat my myself, but again for. So, we can see that what we have called. So, this is what I would have called a mu of x now unlike this case its here it is it is a function of x. So, its. So, I cannot do things like pull it out side.

So, this is obvious what about delta bar of phi a this formula will just go though except a nu should become. So, so a nu d nu of phi a we should be equal to now I have to plug in what the, but this gives me. So, this is minus lambda should be little bit more carful this mu index becomes nu. So, I should call this some order index rho x rho times. So, we just have to be careful to see that the dummy in this is do not coincide and things like that. So, I will just taken care of hat. So, I get delta bar of phi a and I also know what this is. So, all I have to do now is to go to the master formula and plug and charge which is what I will do next I will just bit of notation here I am going to follow the convention that my first index is a current index and this is the nu which came from the parameter just remember that, but now we have what we will see is that our parameters have already two indices. So, we will have some which will have three index structure of where the current will have three indices two coming from the from the parameters and one is the current itself ok.

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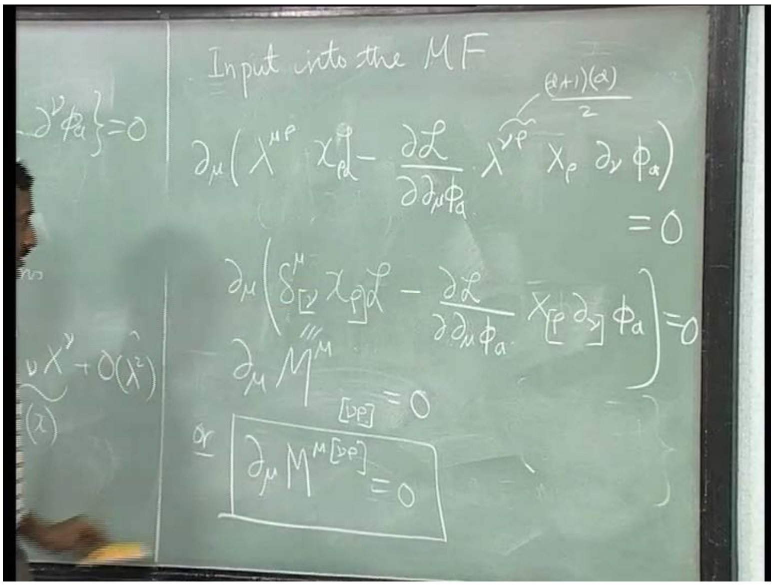

So, we should be a little bit more careful about how we do things and we will be. So, let us just first step put things into the master formula the master formula m f. So, we have this. So, we get d mu of a mu lambda. So, a mu of b already have it out there. So, it is lambda mu nu x nu and minus delta bar of this thing which I have here. So, we are. So, now, we have to what we have to do is analog of what we did here which is to full out the parameter. So, that is lambdas, but that is some certainty here the lambdas if I pull this down or pull this up is anti symmetric. So, I have to keep the track of that issue. So, let me first do the following which is to call this dummy index rho for both first thing this is just nothing I am just renaming dummy indices then I will also pull this up compensate for it by pulling this down same thing out here I have done nothing I mean I just relay I mean I want I want to remember that this is anti symmetric.

So, I pull out lambda nu rho. So, now, nu and rho are the two parameters associated with the currents and how many are there because this is anti symmetric there are d into. So, it is d plus 1 into d by two. So, if you are in three plus 1 dimensions you will have six currents which is exactly what you know the number of lorentz transformation are six.

So I pull out. So, I can rewrite this as lambda nu rho d mu of thing out here delta mu nu x rho I forgotten l somewhere yeah here. So, this is what I get, but should be a little bit more I cannot just erase this because I have to impose the the fact that things have to anti symmetric under mu rho exchange. So, I have to keep track of that I do it by explicitly writing this out in this form by by this what I mean is if you give me some tensor because neuro index structure when I write something like this what I mean is t nu rho minus t rho nu and that is the similar notation which is to symmetries yes yes yeah yeah since the lambdas are anti symmetric it follows that only that anti symmetric the symmetric part of it of this does not contribute because symmetric contracted with anti symmetries is not. So, once I do this equation of I means the the conservation is just this you have to write this again in your book, but for me that about it, but the caution is that if I did not put this square bracket that is wrong I would have more conditions because it will save it will give you it is like the matrix worth of conditions d plus 1 whole square which is too many conditions you got more conditions than parameters in your now I do not if I take mu nu and rho be the same this is 0 symmetric part is trivially true.

So, now you can see I can defined I can I can write this whole thing as follows. So, yeah I can write this as I like it I like it I can write this as m nu nu rho I have to conserve upper and lower indices there are one upper index and this thing, but I can also raise it which is what I will do or raise both of them, but if you look at this structure this looks very I mean it looks like x with going with t. So, let me just write it out and then I will write a claim which is easy to check does this look familiar look at.

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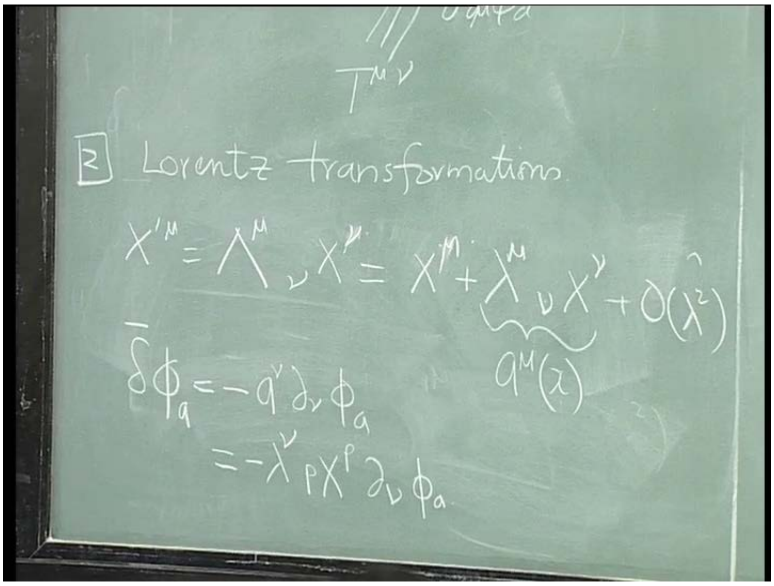

Let us look at the charges. So, this looks exactly. So, let us let us in particular take mu to be I or 0 and then look at. So, yeah. So, so let us let us let us just look at what we have written for angular momentum in normal this thing angular momentum l would be equal to r cross p, but if you used if you wrote out in.

Terms of this you will see l I or l one is equal to x two p three minus p three sorry p minus x three p two this looks exactly like this you see that there is an x out here and t is like a p. So, that if I put a 0 out here. So, in. So, even better. So, let us look at m 0 nu rho would be equal to t 0 nu. So, if you think of this as the conserved four vector density then you will see that this looks. So, in particular n 0 I j looks exactly like it were right. So, the 0 I j part looks like a normal orbital angular momentum. So, this object is, but of course, it has other parts to it you know nu could be 0 also. So, it is a relativistic generalization of this and this is called the generalized angular momentum because it has some of its component match our normal angular what we would have called angular momentum in in mechanical systems kind of and So, that is what we get which one.

So, this is the current associated. So, yeah you are right. So, I should call the charges as the generalized angular momentum density the integral of that over space. So, yeah. So, let us do that lets be more this is what I should call this thing and the charges would be. So, this is. So, integral d d x would be the corresponding what you should call generalized angular momentum. So, it is kind of nice that we see here that this is of the form of r cross p, but what we will see is that that is because we choose our fields to be scalars. So, in your assignment you will be replacing this by the the a case of electromagnetism where you have a four vector field a mu and then you will find that you get an extra piece. So, in in such cases…

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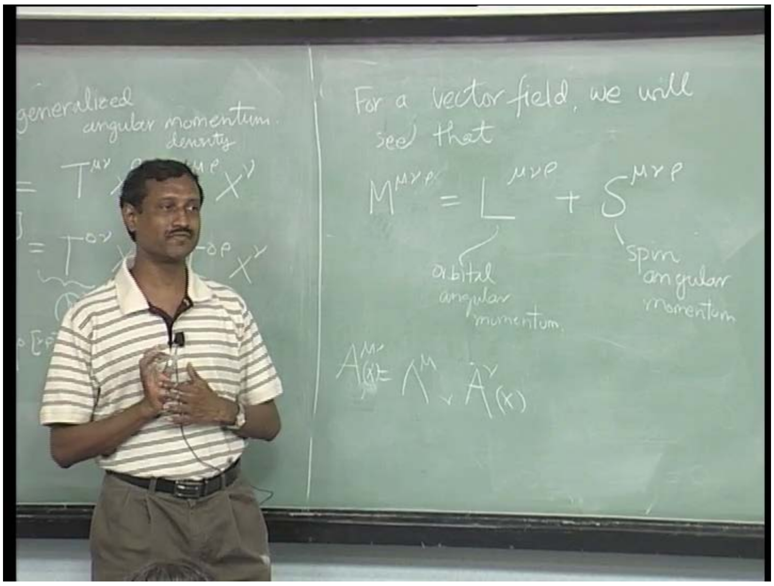

So, for the for the vector field or four vector field in your assignment you will see and we will discuss it for a four vector field for instance that we will have two parts what we will see is that it will consist of two parts one l will be exactly like what I wrote here r cross p kind of generalization, but then it this is this would not be conserved the conserved current has an extra price. So, we call this this part is called the spin angular momentum in this we will be called the orbital the analog of the orbital angular momentum on it will have two parts and that is it and there is no…

So, now comes the neat thing. So, symmetry tells you that there is a conserved current and this is that conserved current you do not have a freedom in this we will see that there is some simple freedom which will not change the (( )) charges. So, you can modify things, but ignoring the such an issue this is the object which is conserved and we find that it has two parts we have not proved it, but we will see you will see it in this thing and because of which you will see that what you might think is orbital angular momentum is not generally conserved what you need to include some internal this thing and this is the related to fact that the the four vector field a mu has a non trivial transformation law or lets write in this way over and about the transformation of the coordinates it also has these extra piece which was not there out here. So, delta bar of of a four.

Vector similarly if you have a (( )) field which is that will also have a similar term it would not be exactly this it will be something else key point is that it will have components which will mix under Lorentz transformation. So, even there you will find that there is a space and what we will see is when we go to when you do your course in quantum field theory and you quantizes these things you will see that every field gives you excitations you will find for for instance if you quantize a vector field you will get the quanta would be spin one.

So, this object will also in quantum mechanics satisfy the same angular momentum algebra that you heave, but the fact that it transform like a vector will tell you that it is something like a spin one particle I am being little bit emphasize because most of the time you would have done non relativistic quantum mechanics and I am trying to connect up with that, but we will make a make thing much much precise part of it in this cause and part of it in a course in quantum field theory if and when you do it ok. So, what you will see is that this will happen. So, the that leaves us with one third example which would be of an internal symmetry. So, this would be the s O n internal symmetry ok

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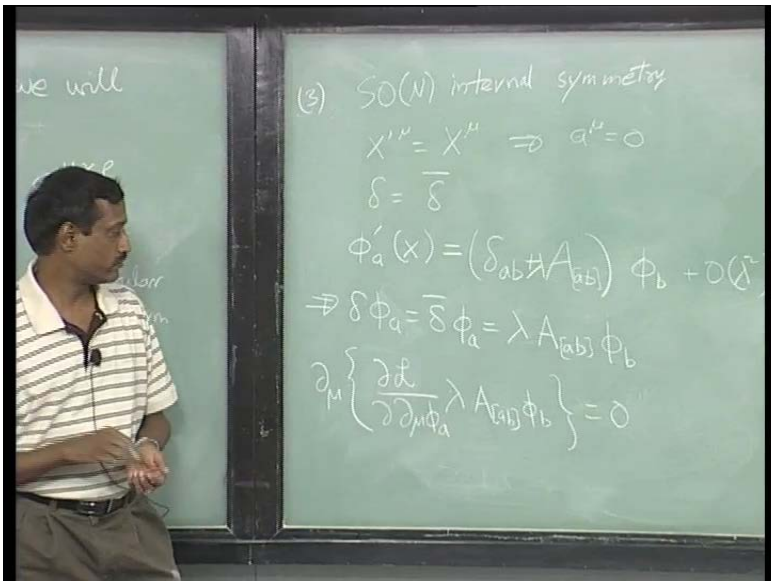

So, we already discussed this last time. So, first thing is that x prime mu equal to x mu. So, this implies mu is zero. So, in the master formula which I just erased that first term will not be there. So, the the only thing which would happen is and further delta equal to delta bar because a mu is 0 there is no distinguish need not worry about whether its delta or delta bar. So, we need to. So, phi prime a of x will be equal to some rotation matrix which I will write as delta a b plus some anti symmetric a a b this is what you we will get. So, this implies that delta of phi a equal to delta bar of phi a equal to and remember this is again anti symmetric. So, now, in the master formula we just need to what we have remaining is d nu of delta times delta bar of phi a, but have it here, but a a b and this is a constant matrix. So, I can just pull this out, but again like before I need to anti symmetries with resects to a b if I am going to element remove it fully out of here. So, I will just do that. So, what I get, I have anti symmetries by hand ok.

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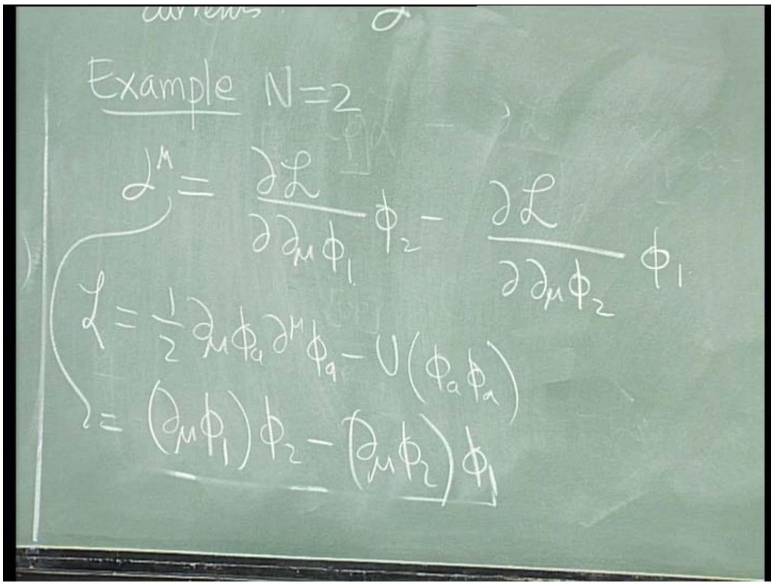

So, that I can now just do the simple devise of erasing this. So, how many currents do I get I get n into n minus one currents we will just call this here upper lower do not mean anything. So, so we get as many conserved currents as they are independent parameters. So, let us just take an example further example within this we take n equal to two. So, then there is exactly one current yeah thank you. So, you will have exactly one current. So, we will write that as a mu one two, but there since a only one current I will not write any index if you just see.

That this would be since this only involves derivatives of of the field in this thing you can see only the kinetic energy part will contribute. So, and the Lagrangian was just the normal d mu phi a d mu phi a minus some mu which was invariant which only dependent on that s O n invariant combination. So, now, you can see that d l by d d mu phi one will just be d mu upper of phi one. So, this will imply this is equal to have you seen this anywhere if anybody see this kind of a currant before ok.

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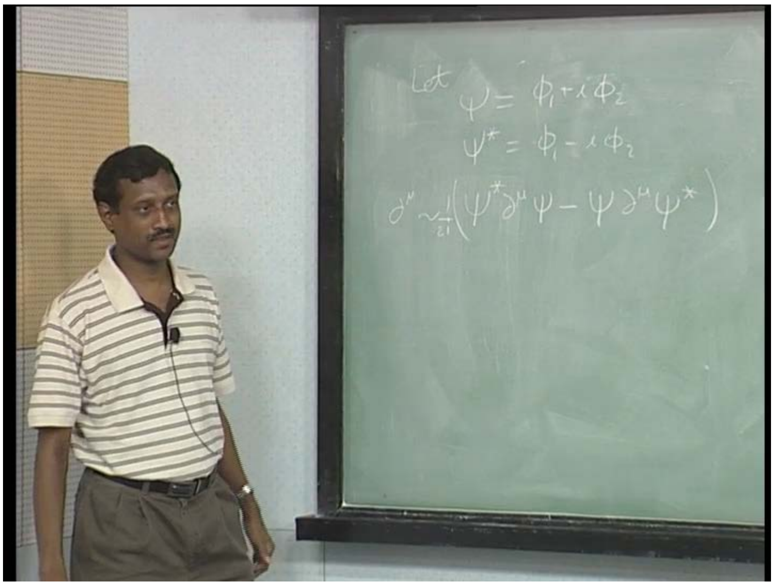

Let me just play a little bit of a trick convert to complex coordinates let us just define that up to some factor in this our messed up yes it should be upper now we are yeah. So, up to some factor this this is nothing, but may be one over two I or something like that what would have you seen this kind of a current before. So, this little bit like the (( )) current which you have seen in non relativistic quantum mechanics its looks like that right.

So, what we see out here is that is that this current. So, you get a whole bunch of conserved currants and things are actually little bit easier when when you have internal symmetry and I will show you a trick by which we can derive the same current and this is a nice trick which works only for internal symmetries for symmetries which act on space time coordinates it will missed the other piece. So, it is a fun trick which I use all the time. So, what we did in when we are getting this currants was to assume that this the coordinate this lambda where is it yeah. So, these were constants global constant ok

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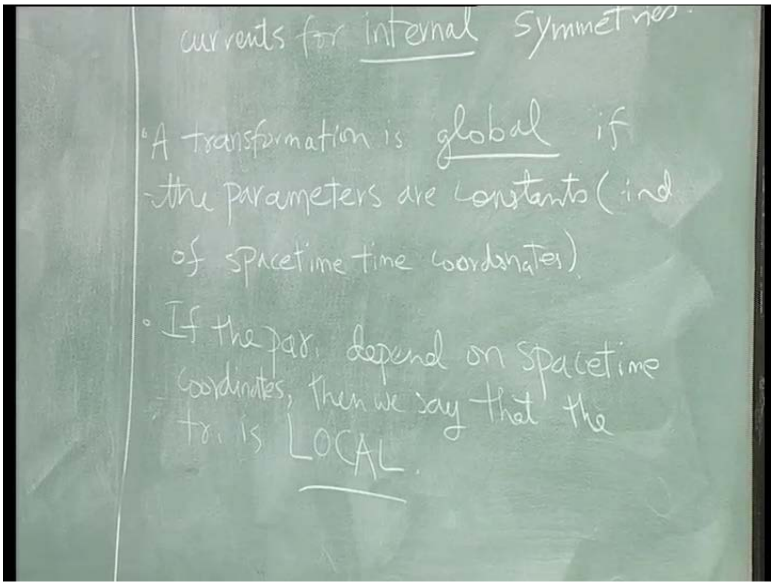

So, I need a little need to define a little bit I will call a symmetry a transformation is global if the parameters are constants that is independent of space time coordinates or. So, for instance that what one what one would say is that there is a s O n global symmetry out here as oppose to if the parameters depend on space time coordinates then we say that the transformation is local.

In other words what we have a some symmetry which is acting on some internal space internal coordinates or internal quantum numbers and what is going on is that you in local things up each space time point it has a different rotation. So, here I could just go ahead and write a as a function of x. So, now, of course, if you go back to your action it is not invariant under local transformations because we cannot pull it out side this for instance the action is invariant under global transformations, but there is a nice thing which happens what you do is if you have a global symmetry just for fun you say that a depends on space time coordinates and then you will you will find a non invariance. So, let us see how that goes here I will not use the master formula, but I will just take the action I will. So, let us look at this I do not need to worry about major changes because it it does not act on space time coordinates I just need to consider the Lagrangian.

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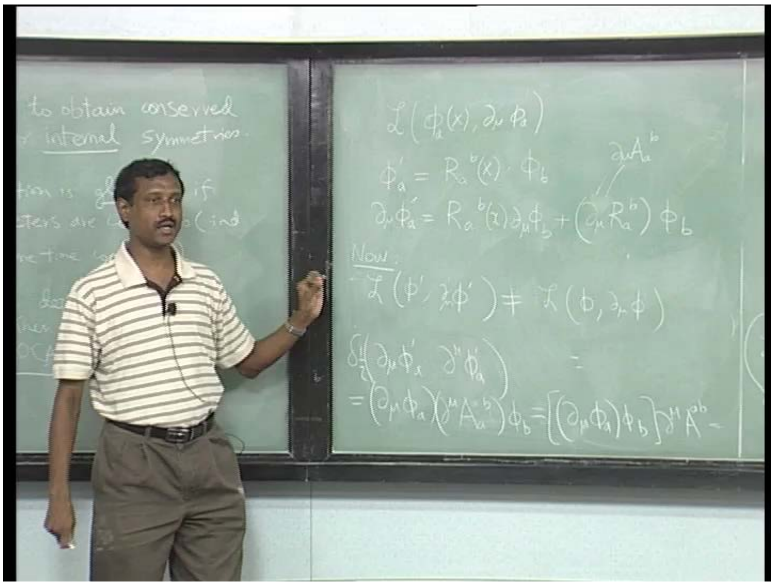

So, let me do this for yeah it does not matter. So, let us look at how. So, phi a prime was equal to r a b of x now I am making it local. So, it is a rotation which is in internal space, but now question is how does d mu change d mu of I a prime will have two pieces one would be, but the d mu can act on r also now because it is a take it to be a function of x ok.

If if this term were not there like in a global transformation then d mu of phi a is transforming in the exactly like this. So, d mu phi a whole square will be invariant, but this will not be because of precisely this this so. So, now, and what is the term what is the part of it which is not first thing is you can see that the that the potential energy parts still continues to hold even though its invariant not under just global under stronger thing which is local transformation, but it is a kinetic energy part. So, we just need to ask how it is not invariant and again we will do bucky ping we will only work to order lambda square. So, let us see what we get. So, so I get pieces. So, I get and just looking at the structure of r a b this is nothing, but you can see that this is equal to d mu of two order lambda square of course,. So, this is d mu in and I can write this and this is symmetric.

So, I can just write this so, the half out here so, I get this thing so, now, you can see that the coefficient of d mu a b is precisely the current. So, I can rewrite this whole thing as something is messed up oh no noting is messed up. So, we do not worry about upper and lower indices right. So, times we get exactly what we got here so. In fact, you can see in general you can go back and prove that you get exactly the terms which were I mean this I did for a particular case for any internal symmetry you can prove that you work to order lambda and whatever is the coefficient of the non invariance by making a global symmetry into a local one that gives you the current and this is something in practice in practice we use all the time.

So, if somebody comes by and give you some new Lagarngian whatever it is I just quickly I mean and a new symmetry which you have never seen before you just do it this way, but the most important thing to remember is there should be an internal symmetry if you try to do it for a symmetry which is not internal that is active on space and time coordinates you get. So, you can prove that this is equal to the actual thing given by the Noethers master formula. So, this strategy is the following if you the master formula is the key thing whatever is you if somebody gives you a symmetry you can you have to work out what delta bar of the fields are and just write out what and figure out what a mu is for that transformation and just put those things in there and just take care that you do not over determine your these things. So, make sure if some parameter anti symmetric you want symmetries whatever it is the thing is some parameter symmetric you symmetries the thing. So, on. So, forth. So, modulo that thing you find that you do get a current. So, this does not require any detail in the rest of this course whenever come across situations like this we just I will come back and refer to the master formula. So, here is the master formula and this is what I would do.