Classical Field Theory
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Lecture - 13

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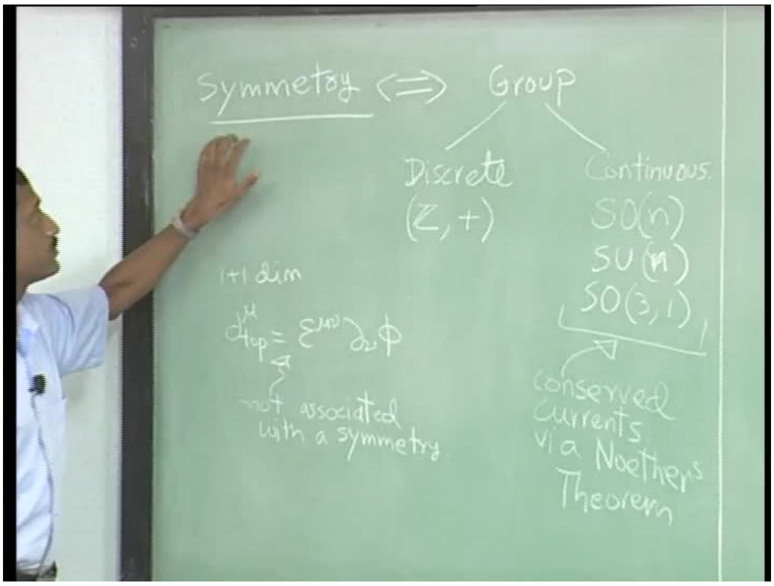

The following fact, I mean which is not the theorem; that is given a continuous symmetry, we will get that conserve current. So we will write this formula where the only input we have to do is we have to explicitly write out, what is symmetry is, and out, where you will get conserved current. And we will discover that, the one which we did couple of lectures ago, for getting the stress tenser or the energy momentum tensor would be an example of that category, and the general master formula. And we will keep coming back to the master formula, through this course, at different points in time. So let us sort of recap on symmetry first, so by now hopefully, you are comfortable with the fact that, symmetries are best captured by some, by the idea of group.

So when we mean a symmetry you should, immediately think of some group. So, we have already saw that groups fail into two categories, but I slightly change my definition of that, we will call group either discrete or continuous, and these are best illustrated though example, rather than looking at the definition etceteras. Discrete is a, I mean finite group are special case of discrete groups. So for instance, we were the group z with plus is a is not it has infinite number of elements, but it is not a continuous group, like this. It is not like we have a parameter to change. So, I want to expand the definition of finite groups, which include even things like this, and continuous groups of course, are S O n, we have already seen several examples, S U n, there are many more, tolerance group so on so forth. So, these are all continuous groups.

So symmetries is also we will classify into these two categories; and the reasoning is just, like I mention. You know this theorem actually refers to, you will get conserved current, only for continuous groups, because here we have a notion of varying some parameter, so roughly speaking you think of the identity element, which just nothing, and your you can talk of neighborhood, of the identity elements. We have some parameter or parameters, which you can take to be small, and so there is some notion of expanding about some parameters, and that will be very important in what we will, in our derivation of these things. So these are that two kinds of groups, that we will locate, and you know this theorem, tell you that you get only symmetries for this.

And we already seen that, just because you have seen a conserved current, it does not necessarily follow, that there is a continuous symmetry associated with it. We already saw one current, which we call topological, which was in one place one dimension, let us call it topological. This was a current, which was trivially conserved in one plus one dimension of course, because this is the levitate in one in one plus dimension. So this has, at least not associated with symmetry.

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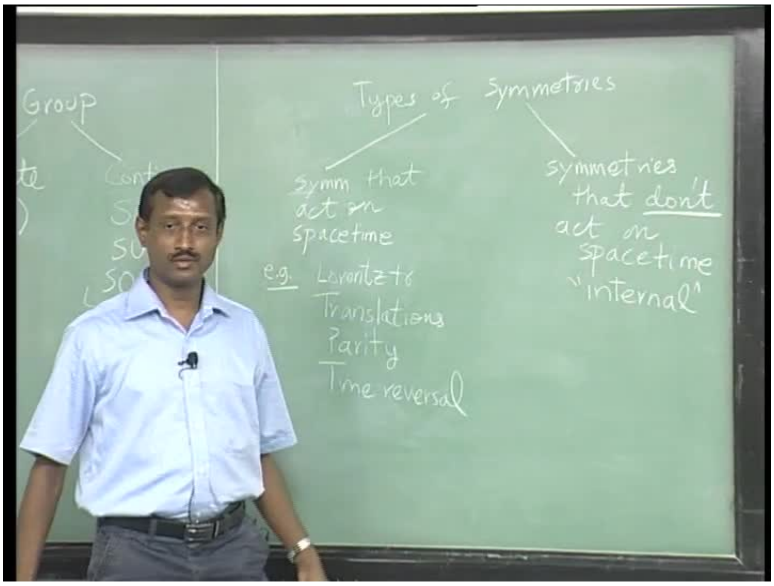

We will also look distinguish between two kinds of symmetries. So types of symmetries; these symmetries that act on space time, and symmetries that do not. So such symmetries are actually called internal symmetries. So these we have already seen, examples of these would be, you know Lorentz transformation for instance, also translation in space time. So these two are example; parity, time reversal. All these are examples of symmetries, that do act on coordinates. The coordinates means pacing time coordinate. Well this will act on field space, so I will give you an example.

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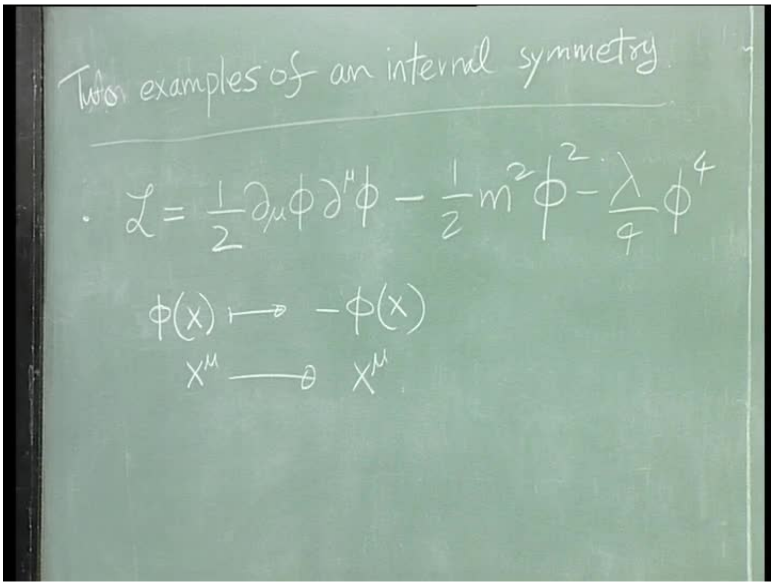

So let us look at a example, of an internal symmetry, rather I will give you two examples; one will be discrete, and one will be continuous. So I do not want you to think that, I mean certain symmetry, internal symmetric cannot be discrete, or whatever. So let us fist do the discrete thing, so let us consider a system with a Lagrangian, with one scalar, but this is a simple, this is the one which give a tragedian equation. And now there is a symmetry which just a following things, suppose it take phi goes to minus phi. It was nothing to the coordinate. So it is very easy to see, that this is the quadratic in phi, so I could even go ahead, and add an interaction of this kind, as long as I do not write a cubic, or quentech kind of term, I am ok. Now this is the z 2, so this is the example of an internal symmetry, it does not act on this thing.

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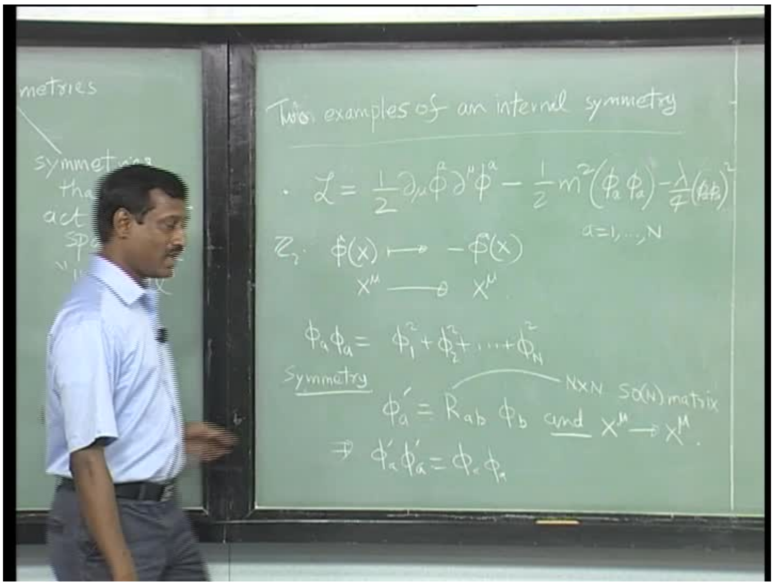

So let us do something, make it more interesting, let us just modify this Lagrangian, where I add. So let us write this way. Let me modify this a bit. No square out here, it is still quadratic, if an a is some index, which runs from one to n. We just summing over them, so there are n spices of identical, if you look at identical scalar, and they interact such that, the this term here is just a quadratic, it is like the length of vector of the main dimensional vector, Euclidian vector. So by phi a, phi a I just mean, and this object is square of, for square of this things. So this remains a symmetry out here, I can still keep it. So this is the z 2, so this only acting in field space .So we think of this a index has nothing to do with space time, so we call this an internal index.

So now it is very easy to see, that there exist a continuous group, which acts as rotation in this internal space, and dimensional space. So there is a new symmetry, which says that, it is better to write it like this, and I am not really being particular about upper or lower, because it is Euclidian, it does not matter deliberately. So phi a prime equal to r a b, but nevertheless I mean repeated index is summed over. So where r is nothing to do with space time, it is just some n by n orthogonal, this is some n by n s o n matrix. Very easy to see that phi prime a.

Now this implies that phi prime a, phi prime a equal to phi a. See whether way you can see that, the mu of phi a also transform like a vector of s o n. So you have a continuous symmetry, which does not act, and of course, there is nothing to the space-time coordinate. Only acts in fictitious internal space. I should not say this fictitious, because as real as the field so. Is this clear, because lots of people have confusion with this thing. I mean if you say the word rotation, it has to be rotation in physical space time; not necessarily by rotation here what I mean is, s o n, the group s o n, I mean. And you know that this n could be one million, nothing wrong.

So this is an example of an internal symmetry, but the key point here is, again this is a continuous symmetry, while this is the discrete internal symmetry. And No ether's theorem does not distinguish between, whether symmetry is internal, or whether it is acting on the space time, coordinates. I do not think this is called external by the way. So hopefully you get the setting of the problem, here we see that there are. So, we have to focus on, continuous symmetries to understand. So, let me sort of illustrate, how the smallness would be used. We will find I mean, for instance for orthogonal matrices.

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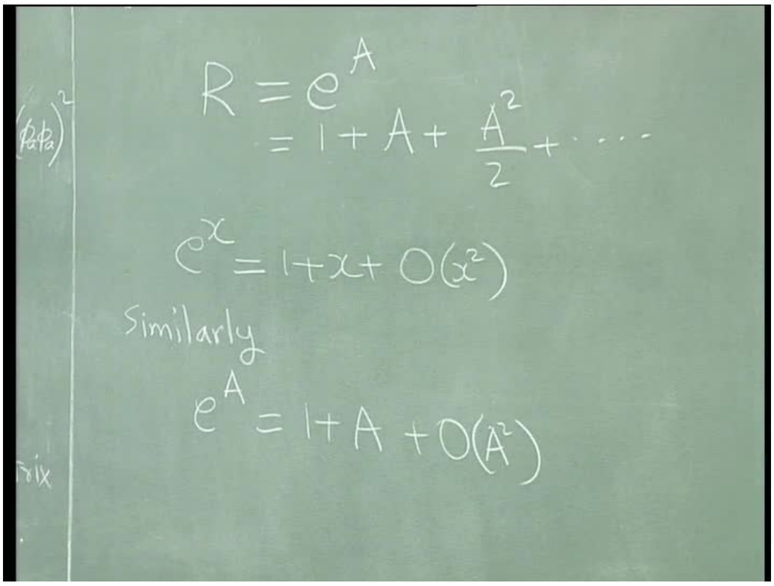

So, let us consider, so consider s o n matrices. Let us call them o, just for. I called it r out here, so let me just call it r, so r we have already discussed this, when we discussing, can be written as the exponential of some anti symmetric environmental, and it is a one to one correspondence, it is a bisection, and now I can use the definition of this thing, so on so forth. So the idea is, what would be notion of something being small. Small would tell you that it has to arbitrarily close to identity. So if you are doing an exponential; e power x, you would write it as one plus x plus order x square, and this is good when x is small, compare to what, to one. So the same things hold out here, you will. So you should think of this, as being small, and you so you neglect this term.

So I will just use similar notation, and but really what you should understand is, this here will contain a bunch of parameters, and each parameter is taken be small. So it is not one condition, so let us I will be more explicit, so I will write here. Similarly, I will write. Similarly, we can write e power a equal to one plus a plus order a square.

Another way to do this, is to introduce some auxiliary parameter, which is just a scalar, which we write like this. At the end of the day, you can set lambda to be one. So lambda is like a book keeping parameter, and then you would see that, I could write it, in a slightly nice of fashion, but I do not promise to stick to this notation all the time, I might use the one which I just erased. I will keep going back and forth, but it should be obvious.

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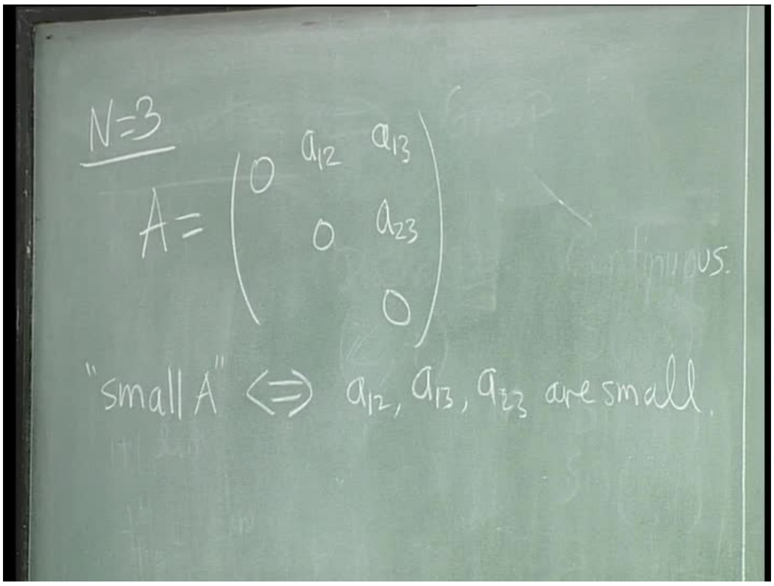

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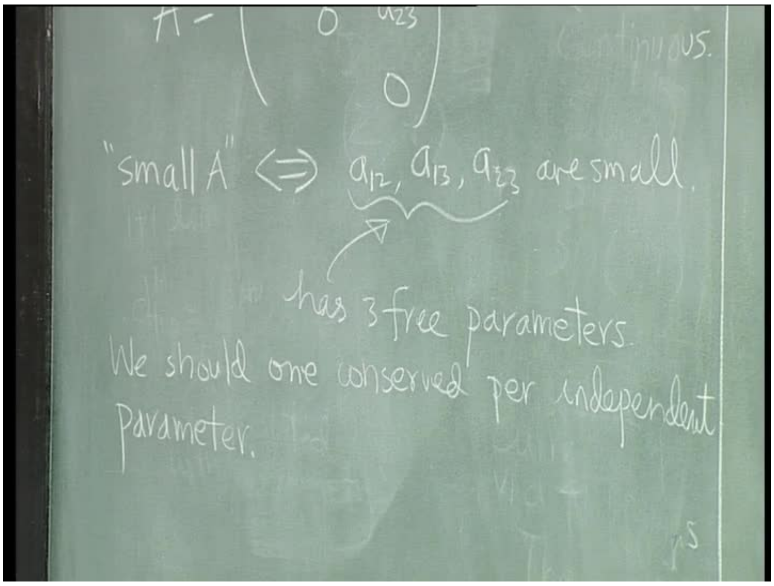

So let us again come back to this a, and lets choose a to be three dimensional. So let us consider to the case of. Maybe I will go to that side of the board. So let us consider the case, when n is 3, so a is an anti symmetric matrices, 3 by 3 matrix. So let write it out, and this part is just obvious, so I am not going to write it, it is an anti symmetric matrix, so I specify a 1 2, I write minus a 1 2 so on so forth, so these are obvious. So this has, so when, what we means by small a, is the same as saying that both a 1 2 a 1 3 and a 2 3 are small. So you can see now, whenever when we talk of things like a square, it will always involves product of these guys, and each one is small, or order lambda if you think that way, so if you look at the next term, it will be order lambda square. But, now you can see that, out here, there are three distinct ways of going away from zero, from identity, or in term so this. So, thing is, so this symmetry, such a symmetry has three form, three free parameters, which are clearly independent of each other, and in fact you can think of nice, another nice way of thinking about, this is the space of a, form a linear vectors space, and since we are only to odd. So the space of a is linear vector space, which is three dimensional, and in this case its over real.

So it has three free parameter, and so what Noethers theorem tells you. Is that for every independent parameter. You get one conserved current. So this, so those are theorem, so we are its funny we are discussing a theorem, which we have not proved, but the idea is that, I want to see where we are getting at. So, we should expect, one conserved current per free parameter. By free or independent to be more precise, independent parameter.

So one more thing about one more nice thing about looking at small things, is that we will see later, is that the structure, the linear vector space structure that I have said, emerges just about all the examples, and this structure is lot simpler, adding vector in a linear vector space is very easy. It is a lot easier than multiplying through, what do you call, two orthogonal matrices.

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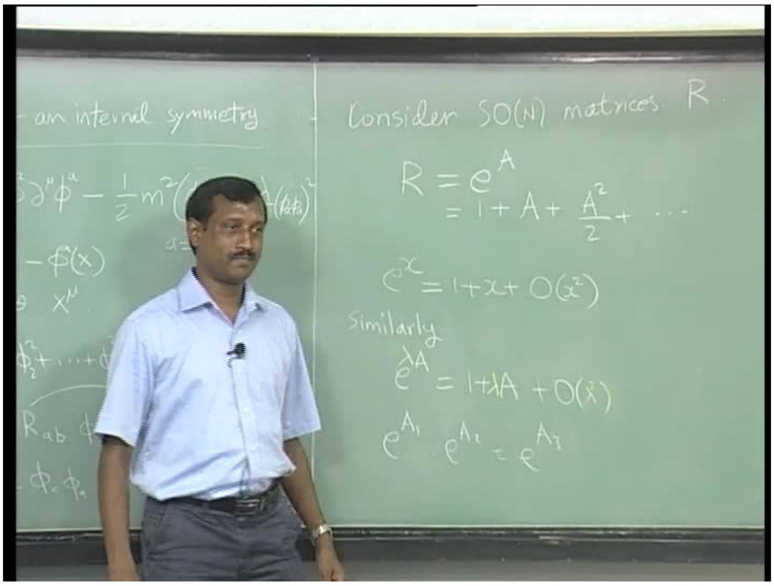

I mean just proving, if you have to say; that e power a 1 times e power a 2, and I ask to work out what a 3 is. You know that there is a long formula for this thing, but what we will see for many of things, it is enough to understand, what happens at, this order and everything rest of its sort of carries over. So this captures a lot of stuff, and we will see the whole theories of lee algebras, emerge from this simple way of looking at things.

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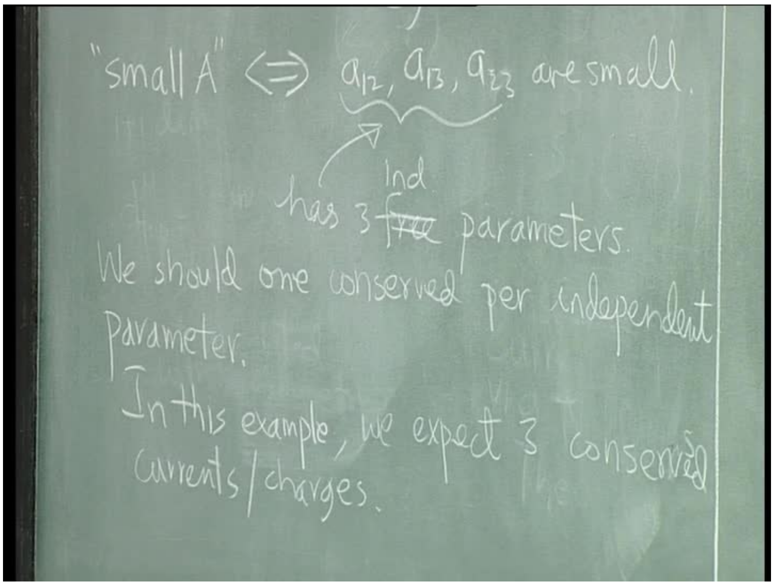

So here we should get, so I should again say, this is independent parameter. So we expect, in this example. I mean we expect three conserved current, three conserved currents or charges. This is exactly similar, again another example is translations. There was one translation is space in time, and one for every special degree of freedom, so we aspects that many current, and we did get that many currents. So lets us sort of see how this same structure, holds for the Lorentz group, because you may think there are, you know there is signature change out there, can we understand that. And what we will see, again it will have the same structure, and there is a sense in which, just as this matrices were orthogonal, where anti symmetric, we will find that, there is an analogous thing which will be anti symmetric, but there is an issue of keeping track of indices, so I will work this out, and its good example of why keeping one lower, one upper, etceteras makes sense.

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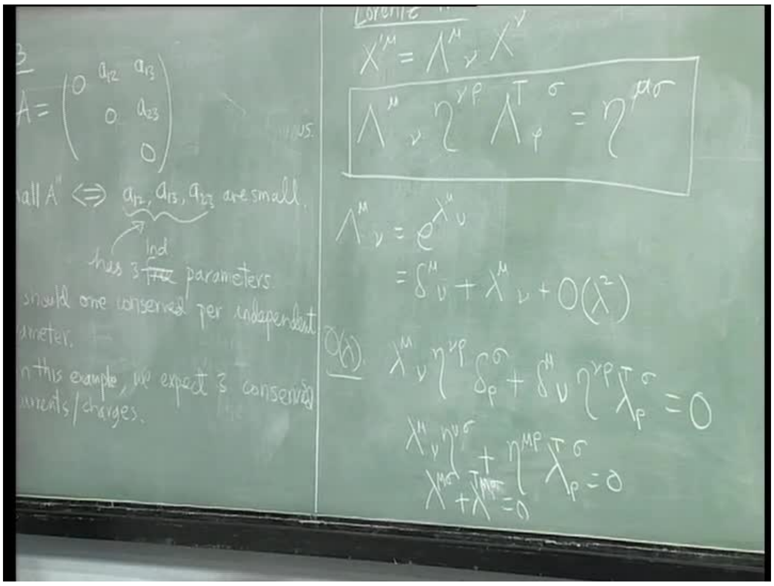

So let us go back, let us look at the definition of Lorentz transformation, which is just that, so wrote something like this, so there. So the idea is said we are going to work out, the infinitesimally small Lorentz transformation, so you will have x prime mu, and this is a lambda mu nu x nu, where lambda mu nu eta rho lambda transpose, which has the opposite structure rho sigma. So this is the definition of a Lorentz transformation. There is of course, determinate condition etceteras, but what w e wants to do is, we want to write, we would like to write. So this little bit of Greek this thing, this is capital lambda in Greek, and this is lower case lambda in Greek.

So I am using the same letter, but the lambda indicates that it has to do with the smaller part, roughly speaking. So the thing is, like we want to do here. What we want to do, is to expand this about, and then I keep track only of the first terms, and ask what are the conditions. If you did the same thing out here, we will see that the condition, that this be orthogonal implies that, a should be anti symmetric. So what we will get, is the analog of that condition on lambda, and let us see what that is. So I just write here, one plus, so by one I mean really delta.

Another thing is to remember, is that if you are one upper and one lower, it has to be a chronic delta. If it is too upper, it should be an eta, if it is too lower; it has to be an eta again. Can let me remind you that, this lambda out here, have not this parameter kind of things it is, there are so many of this guys. So we plug these things in here, and we expanded, and we only check. The first entry is trivial, when you put a delta its trivially true, so it is really the next one, but what I have to do, is to take work only to order, keep these terms, and I am throwing away order lambda square terms. So for instance the only way that can happen, is this might be the lambda, and then this has to be a delta, and if I aid the opposite, if this is giving a delta, this has to give a lambda. So I left two such terms, there is no lambda dependence here, so this is at zeroth order, which is already taken care of. So at order lambda, I only have two term; one is replacing this, and this has to be now a delta rho sigma plus. Now the next one has to be delta. So this has to be transpose, so this will be exchanged, these two indices, so this would be rho, just remembering.

So we can get rid of these chronic delta is easily, so this jut tells you that, this rho become sigma, and out here this nu becomes mu. So let me write one formula, where I have taken care of that. So I get lambda mu nu eta nu rho sigma plus. I have done nothing so far, but one more thing you can, I mean I claim that there is no need for me to say this is lambda transpose because if you look at the index structure it exactly, the first index is lower out here, so this is upper, so you can these things, but it is to be. Since I am being very pedantic of whatever, I am putting the transpose out here, but I claim there I would not need to do that, but let us leave it that way.

So this what we get, and now we can think of defining. So from the special theorem relativity, we know that we think of eta has raising and lowering thing. So here this, what we can think of these, what this is doing out here, is to raise this things up here, and replace making it a sigma. So that is the definition if you wish, and I should keep track of where, this raise index goes, it is a second index. So this term I will write as lambda mu sigma, and what is happening out here. Same things is happening, but now it is raising the first index, so this should go up here, and now I will do what I said, I will get rid of t. I will not show that it is obvious I claim, it has done, I mean it is already taken care of, then you get.

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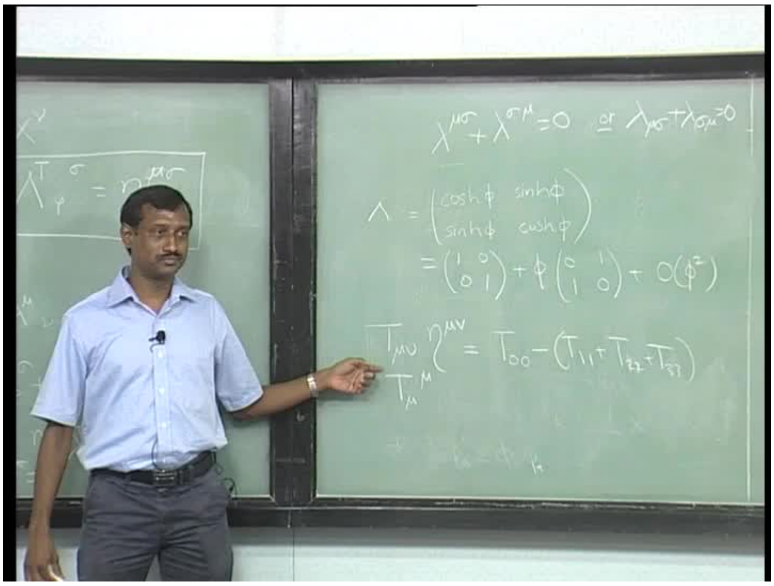

So what this does is, it raise this rho and make it mu, so I get this things. So you see that yeah yeah when I wrote, the minute I wrote lambda transpose I did exchange. So (( )) which one, on the lambda transpose, on this lambda transpose the first index is lower index, and the second index is upper; so I opposed to be here. It gives me contracting with eta mu rho, which one, so it has to (( )) correctly. The next step… The one below, below this, yes so this mu is raising this guy, but it will not change any position or anything, it just makes it go up, and makes it mu (( )) mu sigma. So when you getting transpose should we. No wait so wait, let me write this, so let us do this. So let us write whatever we get mu sigma, are we ok now.

So, if you want to get rid of this, I have to, you are right. So, this tells you that mu sigma plus lambda sigma mu, now got rid of transpose. So that implies that this is anti symmetric, when you raise the indices, but thinking about, I mean there is no way of of saying, that something which has this kind of dissimilar index structure; one covariant, one covariant, and one contra variant, they cannot come. I mean anti symmetry comparing you have to change the identical objet, you get as sin that is what it says, but as it sin here, there is no way for you to exchange it. So what this tells is you need an eta to raise it, and that is what the anti symmetric.

In fact if you just went ahead, and look at take. Let us take simple example of boost matrix, in one plus one dimension. So the phi here, the rapidity, is the analog of these a out here; that is small. If you take this to be small, so this would be some lambda, this matrix, with one upper and one lower, to expand this, for small one one, then the order phi term. Now, you naively thing finds this looks like symmetric matrix, but transposition does not make sense out here, because these this formally if you look at matrix, it does not, you cannot, I mean this, as its written you cannot do a transpose, and its looks like it is symmetric things, but actually that has no meaning. If you look, if you keep track of the index structure, what has meaning is only a up to you raise it, and nice exercise is to raise this, and you will see one of these pick over, and in one you will be raising a zero index, in other case you will be raising one special index. So one of them any way, all you care of about its relative minus sign, so its independent of signature if you wish. So this will, so this will become an anti symmetric matrix. So what is anti symmetric is, thing which likes indices. I could have a equally have written lower both of theme, when that will hold, or this is also true, both of them will be true, is this clear?

Another warning is, with for instance if you have, something like t mu nu, and you ask to take the trace of it. Can you tell me what is the trace of t mu nu? What would you do if I ask you to trace t mu nu. Just say whatever I mean, that is nothing about right or wrong we will see, how about you what. Exactly so the natural thing you would have in mind is take the sum of the diagonal elements, but that is not correct, the reason being that would say that you have want to trace, sum over both these two elements mu nu, you think, but t mu nu does not make sense.

So the way to do this, is to write it like this. You contract with the eta, which is the natural metric in the space, and this will tell you, so this will. If you look at this and expand it out, you will see that this will tell you that what you are summing over is t zero zero minus or another way of saying this is what I do out here, is to raise the second index up, and then I sum, this is what we should sum over. These are all related to the same thing, but you can see that if you are in Euclidian space, where eta, the matrix is just the chronicle delta. Then it is exactly what you said, it is sum of all the terms. Now if you are in some weird curved space, then the trace would be done by whatever is the matrix in that space. It would not be even so simple. So just remember that this is a, certain thing, but it is an important issue, and this is the. See, the key point is, what you want. You want this object to be Lorentz scalar, and this is what, when you trace it is the Lorentz scalar; is that clear.

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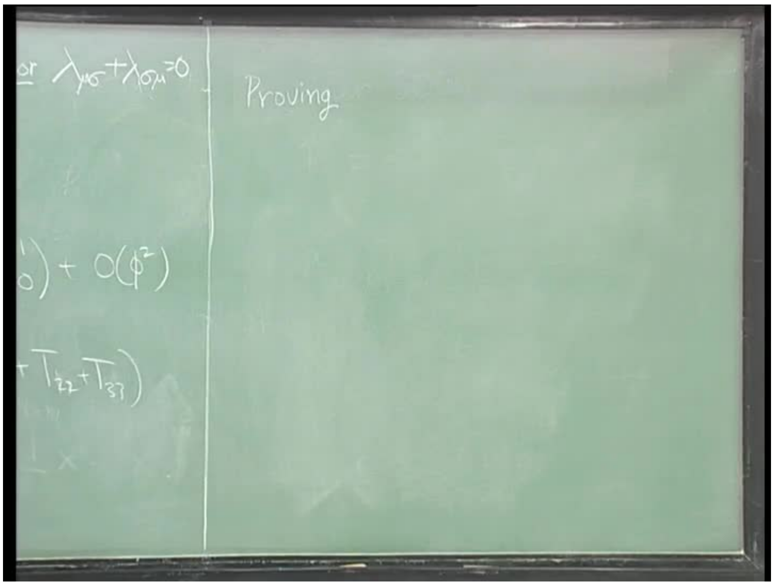

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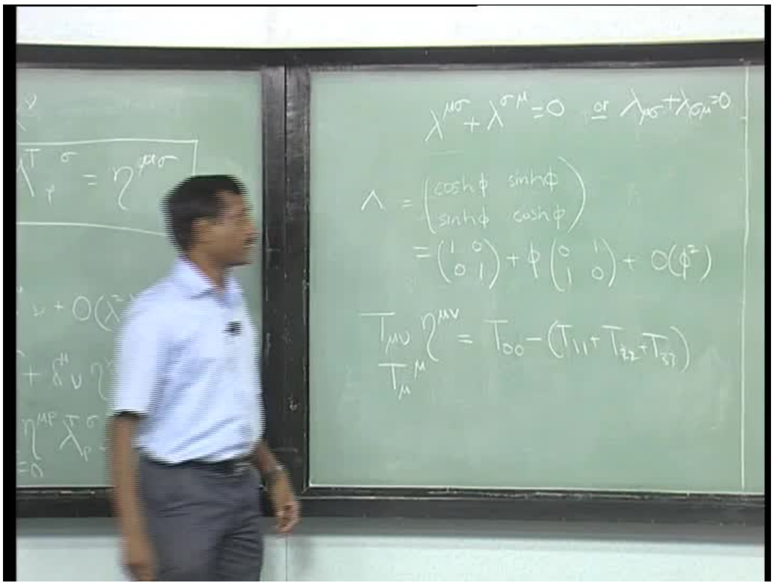

So let us now start the process of getting towards, proving Noethers theorem. (( )) No not necessarily, because ii could for instance take another example, I could have something else like this; t mu rho nu rho, which is a third rack tensor, then you have to tell me when you say trace, over which two indices, you have to pick a pair. And when you trace here, the answer here would not be scalar like here, but it would be a vector or convector, in this case a convector, because it is a. (( )) any pair, but if you, I mean you can trace over pairs of indices, like that, I mean that perfectly perfect.

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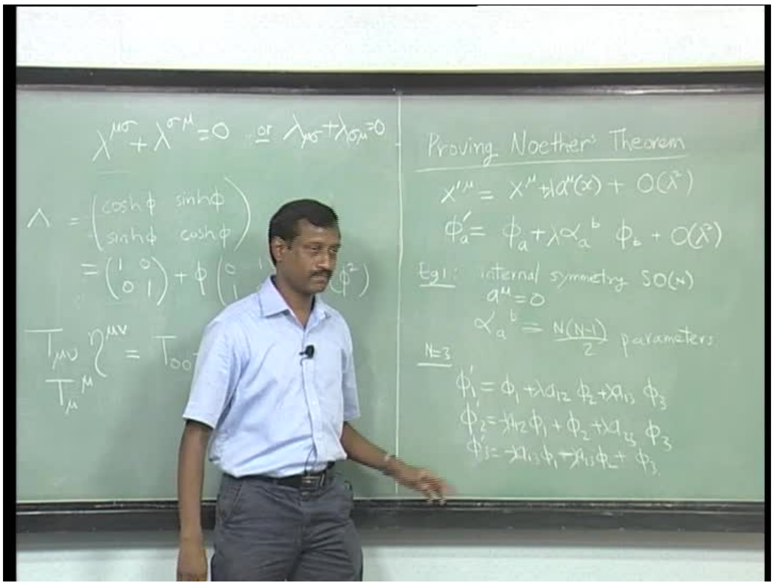

So I mean, I just shows the simplest, and this is something which will come, in this course, so I just I am just warning you, because this is a common mistake people make, so we want to prove Noethers theorem. Now we got the machinery going, so we will not assume we will, we will write out the most general thing. So let us say that symmetry acts on both space time, as well as the field space, so in delta x mu equal to, or lets write it this way, x prime mu x mu plus a mu of x. This is not translation like, we did earlier, it could be x dependent, for instance under under Lorentz transformation, you will see that it is linear in x, and.

So let us let us keep a book keeping device now, and in let let it act on the field space as well, we do not know how it acts. So just for concreteness, I just put some index a out here, and just r is no longer rotation matrix, it is just some matrix, that acts not only on space time, also act on things like this. Now, what I will do is again I want to assume, this is this is. I would not write it this fashion, so it would be some phi a plus some bunch of parameters, so let us just call it, that is this free parameter, I am just thinning I just I am just thinking I am looking for some parameter which will not, plus order lambda square, so I am telling you, how it acts on field space. So at the end of this there will be some parameter, which I...

So, these will be, well defined function. So, let us just consider, let us look at the two examples, which we have already seen. So let us do first example; one is, internal symmetry, which is s o n, so in such cases, for this example a mu would be zero, it does not act on it, but it does act on this; alpha a b would be those parameters which I wrote, so will, actually let me here, leave it this way. Alpha a b will be the 3 or n into n minus 1 by 2. So if n equal to 3 for instance, I can do more explicit, I would write phi prime 1 equals to phi 1 plus a 1 2 phi 2 plus a 1 3 phi 3 phi prime 2 would be equal to minus a 1 2 phi 1 plus phi 2 plus e 2 3 phi 3. And similarly, of course plus, so they should all have a lambda going with it, so that that is a book keeping device, any more lambda; so is this clear. So this alpha a b will be track of these, so in this case this should be this.

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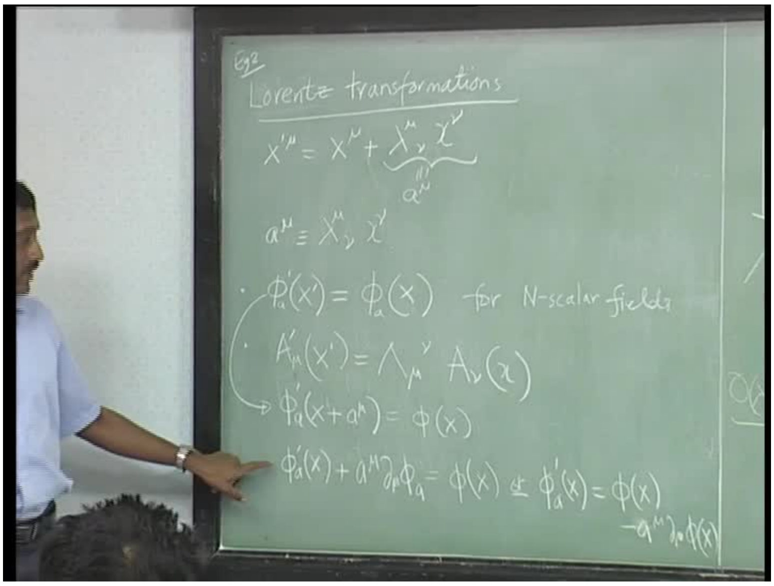

Now, let us do the same thing for. So let us do for Lorentz transformation, so this is example, what did I call it example one, example two. So what should I write here, so this is what I should what I call there. Actually it is important to realize that, you know even if the parameter lambda is small, this combination mu need not be really small, think about a rotation. I make a rotation by very small angle, but if you ask how much distance it moves, it is like, if you if you go to a distance r away, it changes by r times the angle, theta could be small I could make r arbitrally large.

So there is a formal sense in which this aim, so that is one reason to think of his lambda parameter, which I am writing out here as a book keeping devices as formal parameter. So I am not really, I mean I would be wrong, in assuming that a mu is small, even though lambda is. It depends on the axis, which go. What about, let say we have a single scalar field, but a single scalar field we know that we get something like this phi prime, of x prime should be equal to phi of x. If it even if they had the indices, it does not act on the indices at all, the Lorentz transformation, if it is scalar field, this is for, but if you have a vector field like mu, they should be equal.

Now, I will have two action; one coming from the fact is this index is the vector index, and the Lorentz transformation do act on that, so I would write something like this lambda mu nu, this yet not in the form, which I have written out there, I have to massage things a bit, so we will do that for both of these cases; is this clear what I have written. So here a is just for some n scalar field, each one of them is a scalar field, this a is some internal index. So rotations do not have anything to do with them, rotation or boost or whatever, so this is what happens.

So now let us work out what. So what we would not to do is rewrite it in terms of the small parameter, which is in this case lambda mu nu. So we just go ahead plug things, so what we get from for the first, for this we get. We can write x prime as a x plus mu. This should be equal to phi of x; is that clear. Now if you expand things so what we get is. So I have just expanded this, to first order in a, but really first order in that formal parameter. Now we can even rewrite this, so we can rewrite this as a. Now you may worry, you know this is with the, what is the derivative with respective, or whatever x x prime, but really it is not a big deal, because it is an order a square.

So all I do is when I take this to the other side, I put a minus sign, and then you just said that everything here, with respect to variables, not x prime, when anyway just no x anymore; that is four. Now we have finally, written in that form, and what is a mu a, mu is defined already here. So let us work out, what happens for this thing. Now you see here, there are extra pieces coming, just one coming from here, so we can.

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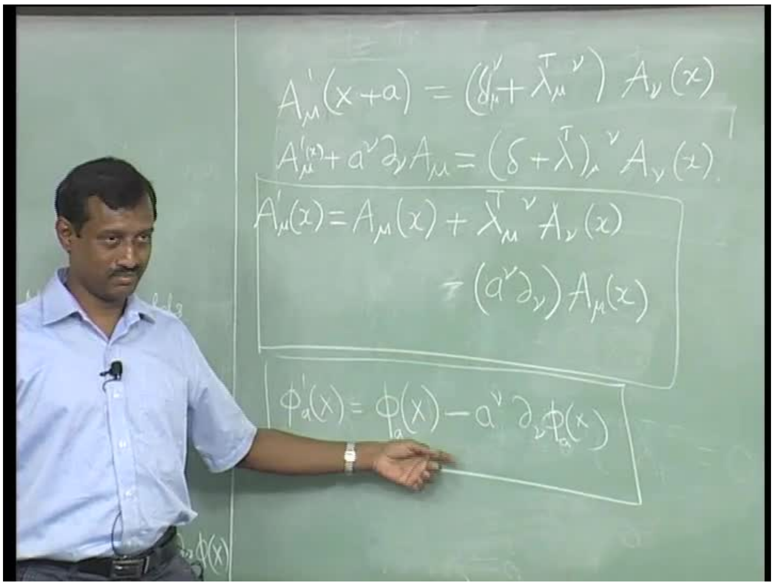

So there will be, let us write that a mu prime of x plus a equal to 1 plus lambda mu nu into, it should not be 1, it should be… Now again I expand this, so I cannot use the index mu as my dummy index, because it is a free index. Should I be careful here, should I have written transpose. Yes. Just remember, because I mean I am using a covariant guys, so I should write the transpose. So now again I can rewrite this, by taking it to the other side. First term of course, and you get one term like this. And let me rewrite this here, so that you can see and compare them. And just to be, to make the match more perfect, replace the dummy index.

Now you see that, this term is common to both, but the fact is this, this is a vector field, actually gives you extra piece. So what I call alpha out here, is something more complicated than, but you can see that it roughly fits into that thing. So alpha there, in this example is delta a b, but there is more to this story. There I just said that I keep track of various. This only is, what it will doing internal index space, so the alpha is just a delta a b, but there is this piece, because I mean, I am written that, how would I hide that, in what I wrote, but we do. I mean now there I put everything nu, you can see that this is what you get, and there are extra pieces, because this field as an index, which transform on trivially, under the (( )), is this clear?

So if you give any symmetry, the first step, is to say how does it act, and since its continuous group, you have notion of what you mean by infinite decimal, go ahead and do that.

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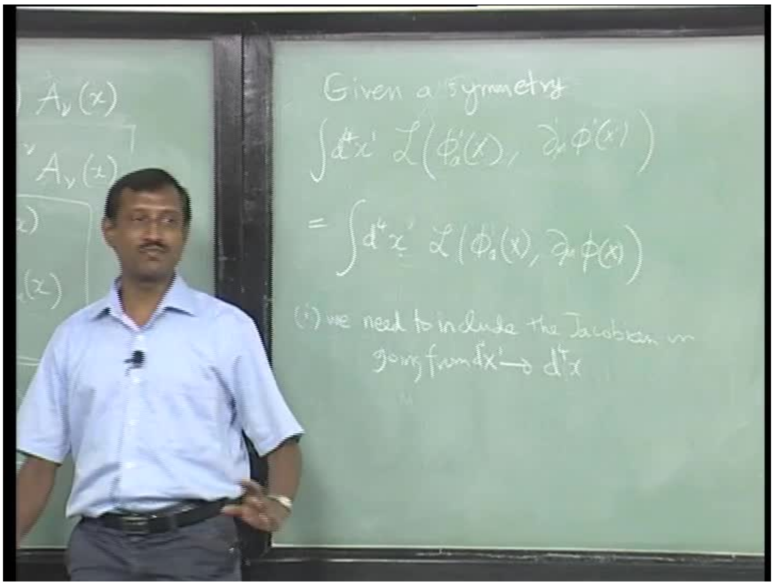

So the idea here is that. So, one thing we know is, given a symmetry, the action will be invariant under that, so what do you mean is. So the action in terms of… Let us tick to 4 only, d 4 x prime l of. I will write it only for, I mean, I will just choose a notation phi a, but I do not think it should be hard for you to replace phi a with a mu, and the derivatives everything will be with the primes. No more primes here. So it is just a statement that, the action is invariant and one new ingredient which come here, is because if you. When things act on space time, then that could be a non trivial change in the Jacobian, between x prime and this things. So we need to work out.

So we need to include the Jacobian, in going from x prime d, say d 4 x prime to d 4, but life is easy, because we do not want the exact expression for Jacobian, we are going to work always to first order. We will see there has vary very simple formula, which you can write, for Jacobian, without any problems. So we need, we will have to include this. But another thing is, if you may wonder why did, we did not look at the action when we discuss the energy momentum tensor. The reason is there, that this was a constant, and there was no Jacobian, Jacobian, rather than Jacobian is one. So that is why we did not. So, but in general, I mean in fact somebody I think I have been addressed the issue, that about the measure, and except for one, but he was right, we need to start with the full action, which I did not do, just to simplify things.