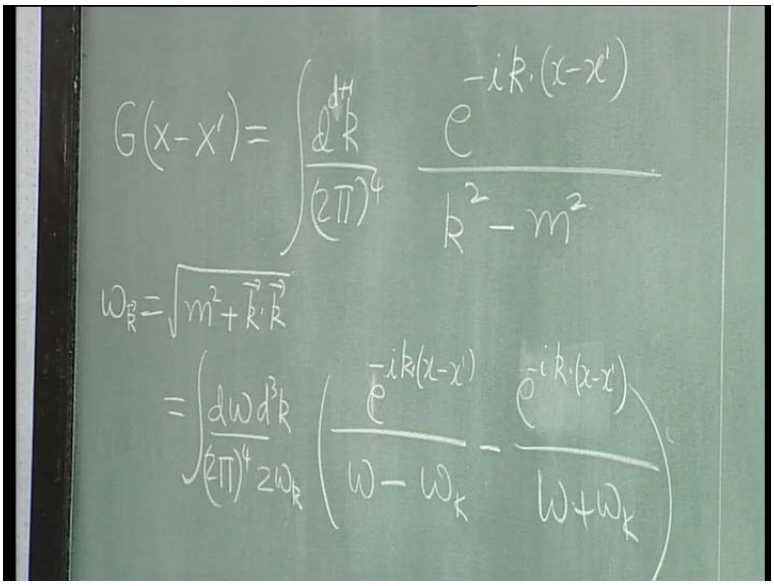
Classical Field Theory
Prof. Suresh Govindarajan
Department of Physics
Indian Institute of Technology, Madras

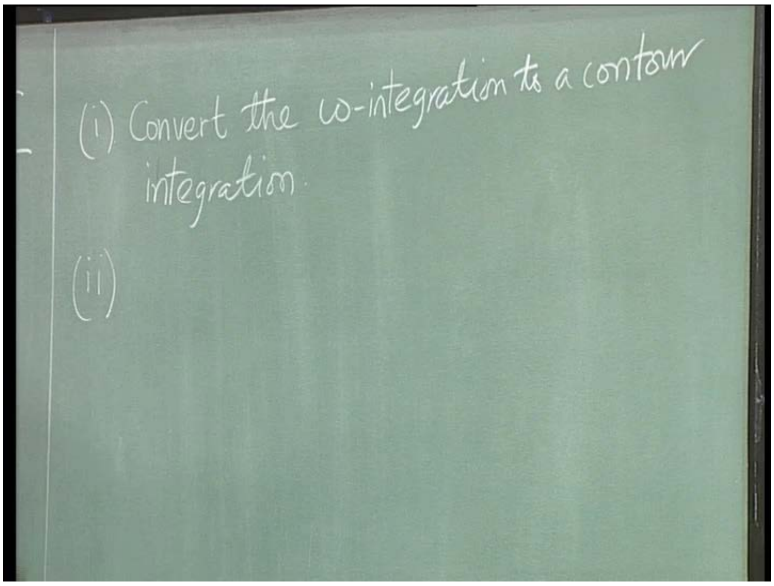
Lecture – 12

(Refer Slide Time: 00:23)



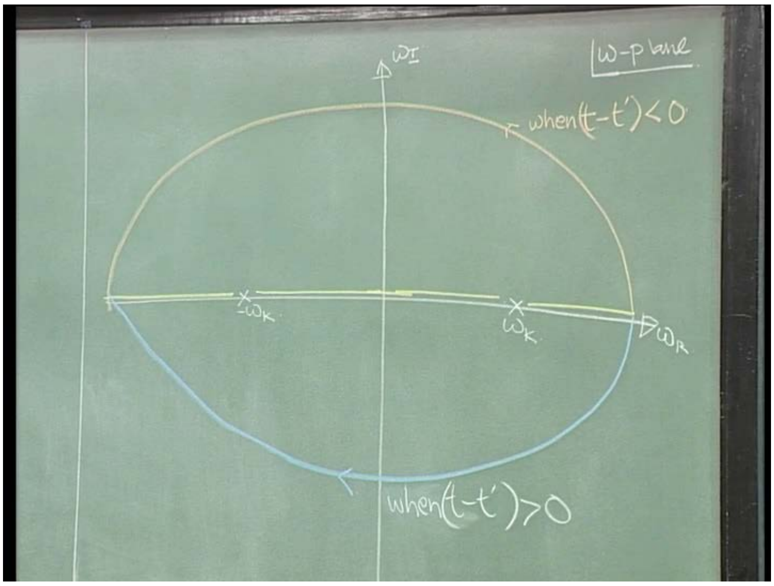
Functions which differs of course by the homogeneous solution, and they correspond to different kinds of boundary condition that you want to impose. So, what we were doing at the end of the day, was to just compute an integral of the following kind. So, we wanted to carry out this integration, and so we said we will first do the omega integration, so we define a few things, if you remember omega of 3 vector k actually, which I will not indicate all the time. This is equal to square root of the positive square root of n square plus k dot k. So, what we did was to rewrite this, so there is. We were in arbitrary dimension at some point I will specialize to 3 plus 1 dimensions, were right now you can keep the form lay very general. So, what we have done, is to use just partial faction expansion, and re written these terms where we reached. So there are there are four steps, I guess to completing this integral. So. let we have finished one, but let me state all four of them.

(Refer Slide Time: 02:20)



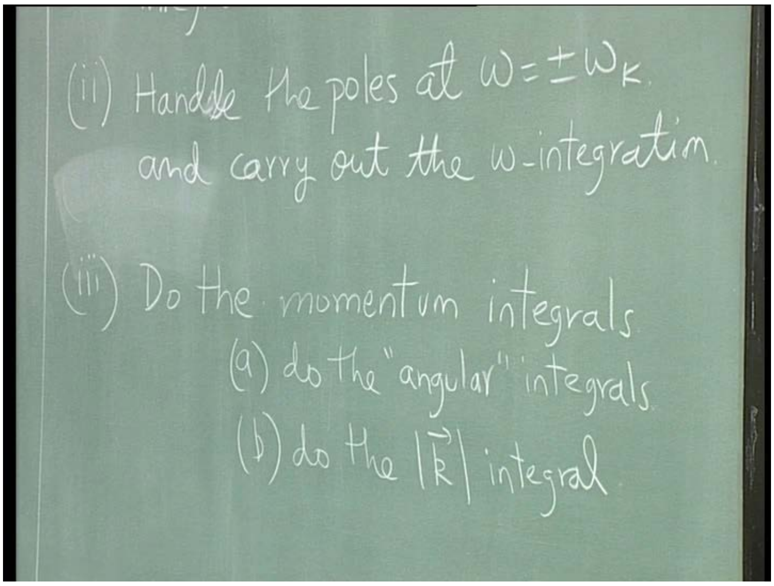
The first step is to now carry out the. So, we would like to carry out the omega integration, but we will do that into parts; one is to convert, the omega integration to a co integration. This we had finished last lecture, but I will remind you of what our conclusions were. The next bit is to actually handle. So what we see here is a, there are two poles, at plus omega k, and minus omega k. So we need to take care, and the contour here is, omega contour goes from minus infinity to plus infinity along the real axis, so these are real numbers, so it is going to hit both of them.

(Refer Slide Time: 03:18)

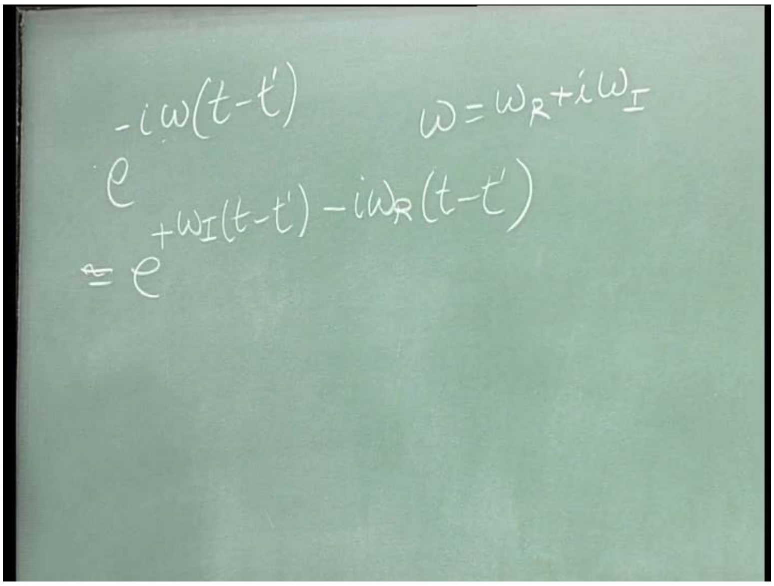


And I indicate that by a gap out here, which we need to handle at some point in time. So the integration is from minus infinity to plus infinity, in the omega, this thing. And what we have done, is to write it, think of omega as a complex number. So what we mean by omega k here, is a real part of that thing.

(Refer Slide Time: 03:58)



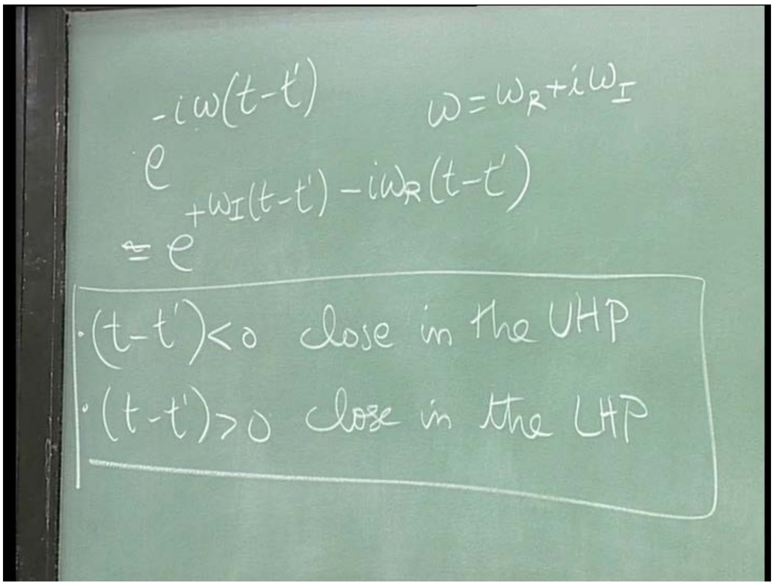
(Refer Slide Time: 05:29)



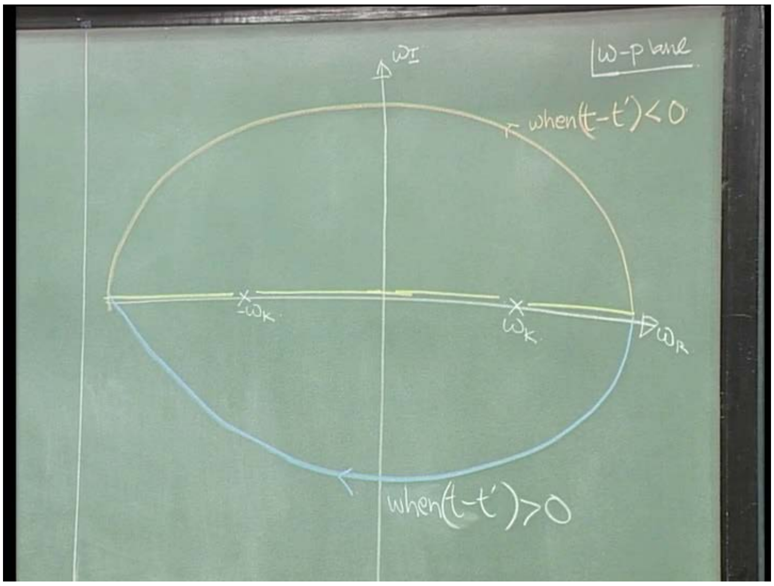
So we have to poles; one here at minus omega k, another at plus omega k. We need to handle that, also we need to close the contour, and I will just remind you later, but let us go through the process which we will go through, and we will complete this in this lecture. So, next bit is to, handle the poles at omega equal to plus or minus omega k. The third bit is to carry out, and maybe I should leave it as one, if omega integration. Once we have done that thing, the next bit is to, do the momentum integrals, which we will do in two parts. You go to do there do the angles, what I call the angular integrals, and the last bit which is, do the mod k, or the radiant in case pace. And in fact in this problem, all these steps can be carried out, and we actually finish we have discussed how to do this in last lecture, by looking at e power i omega minus i omega of t minus t prime, and then we said we should write omega, as some real part plus i times some imaginary part. Then this exponential will look like e power plus omega imaginary into t minus t prime and plus accelerating part, which you know exists.

So the idea was that, we should close the contour, so this is a semi circle of infinite radius, and so the point is that we want these contributions to go away, and we do that by looking for situations, where this is exponential suppressed. So if you if you take t minus t prime less than zero; that means this is negative. Then you can see for large t minus t prime, or for that large omega i, this is vanishing, in the upper half plain. So the rule here is, if t minus t prime is less than zero, you close in the upper half plain, so the contribution of this goes away. And when you have the opposite thing; that is when t minus t prime is greater than zero, then you need omega i to be negative. (( )) Yes, you can go ahead, or you could do things like you could choose, if you are unhappy with this kind of contour, you choose the square contour, and you worry, I mean I would. I will leave it as an exercise for you to convince yourself that, those things do not matter.

(Refer Slide Time: 07:26)

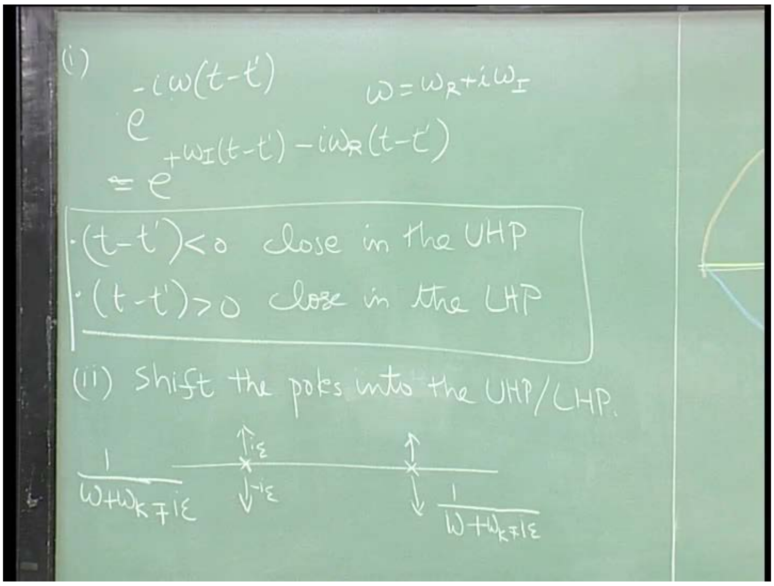


(Refer Slide Time: 07:28)



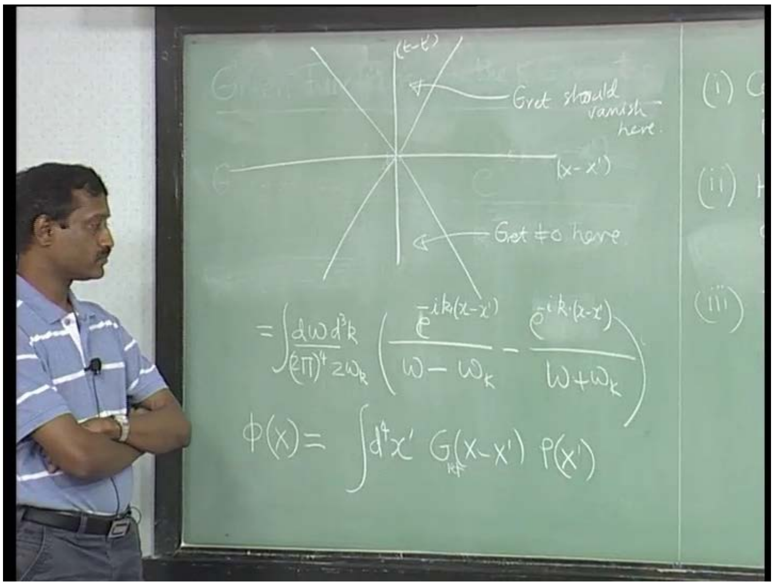
So, once, you can see right, if you choose this kind of contour, if you convince yourself, that it does not contribute, then it, then just deforming the contour, does not change the story so, but I am just sort of this intuitively the way, to understand how it works, but you can convince yourself that, there is any problem what I am doing. So this gives you a unique this thing, so it says t minus t prime less than zero close in the upper half plain, and if t minus t prime is greater than zero close in the lower half plain. So this is a nice rule, which you can just quickly verify. There is no need to memorize the thing. So this is what you get, and one point remember, is when you are closing in the lower half plain, the contour goes clockwise, so it is a. So it will give a minus sign, when you do the residue of the pole. So we have this is part one, now we need to still handle, the poles at omega equal to plus or minus omega k. This is standard trick, which is to, say that we either push this up here, or down here, into this plain, or you do these things, so there are four possibilities, so this is part one.

(Refer Slide Time: 08:29)



Part two, this shift into upper half plain or lower half plain. So these are the two choices, you have two poles, so there are four possibilities, which I will sort of indicate by this picture, if you wish. So the way to go about doing this is, suppose you wanted to so let us focus on this guy, this pole is one by omega plus omega k. Suppose we wanted to move this to the upper half plain, will use will use something called the i epsilon prescription, which corresponds to just adding an i epsilon this way, or this would be minus i epsilon. So you just put it out here, and say, if you want to push it out here. So minus or plus i epsilon. So let us look at the top sign, that would convert it to minus omega plus i epsilon, which means pushing it upward. So this minus, this up first guy, comes from the top sign, and similar thing happens out here. We will just get one by minus or plus i epsilon.

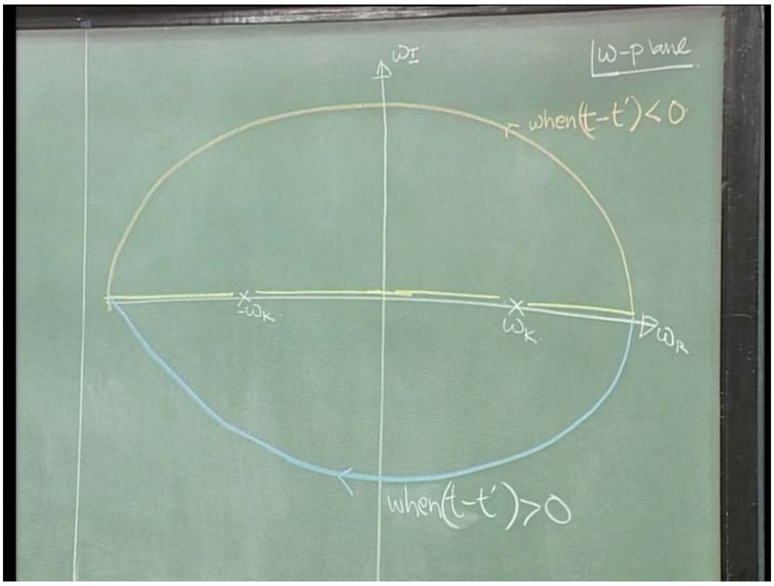
(Refer Slide Time: 10:44)



So i epsilon is, taken to be infinitesimal and small, but in positive more importantly I need not switch signs on you, so you can see it is a positive number. So you can see that there are possible sign choices, and this is where our boundary conditions, the kind of things we want, become important. So, first thing is to look at, what we will call the retarded green functions, so let me draw this picture of the I will need this, I look like I need everything, so I have to give up something. So let us draw now the picture. So again I will just draw it in one special dimension.

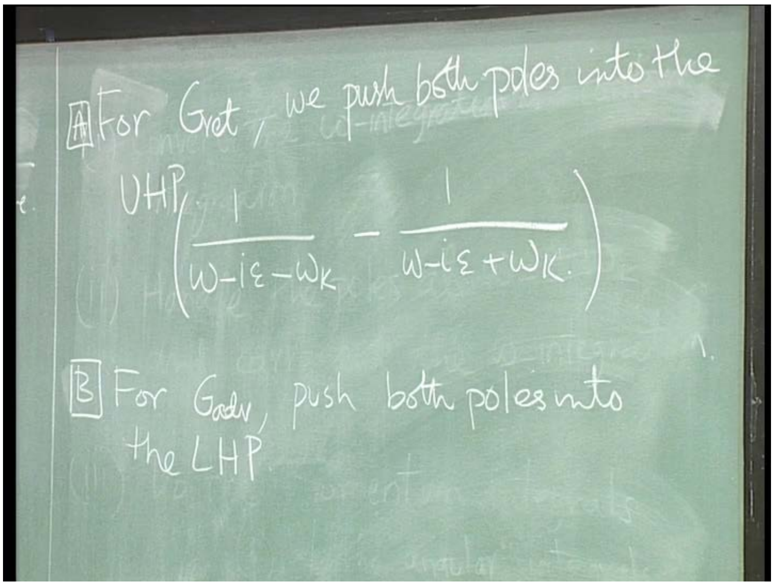
So this is t minus t prime, so the retarded green function should be one, which should be, so g retarded should vanish here. So it should something out here, can only be affected things in the past, so in the future things it should not be affected, so that is what we will call g retarded should vanish. So let us see how we would go about doing that, so g. So you can see out here, in the future light cone t minus t prime is greater than zero. So the thing is that, so the way to solve this out, is to look go back to, what we wrote, so we have phi at location x, which was e equal to some integral over d four x prime g of x minus x prime and rho of x prime. Signs are not important factors, but this is so, this is the source, and this is. So we are getting the field at point x, and you should be determined by sources which are in the pasting, and that is what I call the retarded guy.

(Refer Slide Time: 13:31)



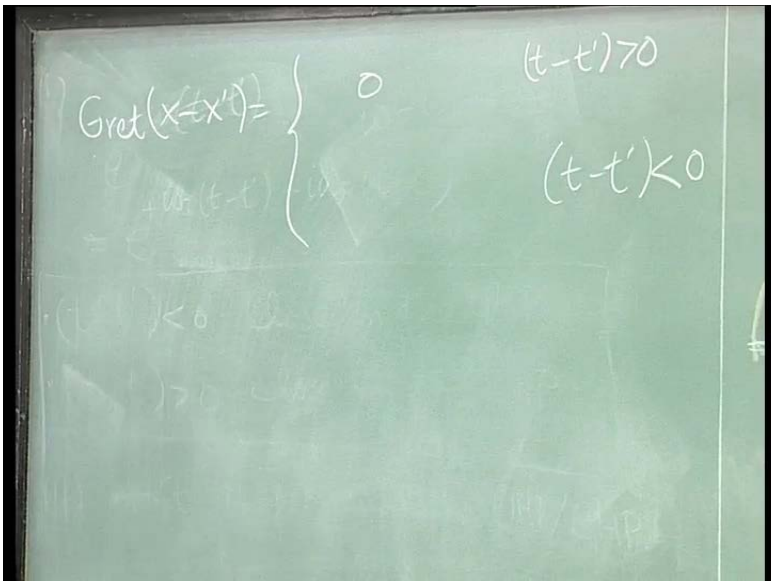
So, the retardation in taken into effect, this is what I call retarded, something in the past, so there. So what I am saying it is non zero here, so this is what we want. So let's look for something which vanishes, when t minus t prime is greater than zero, (( )) yeah we will look at, we will see that, I mean no x minus x prime is not so important right now, it is only future or past, I mean that is been only determined by this, we will handle everything else, it has to vanish here as well, but that is a different story. Right now we are just looking at, where this should vanish in the past or the future light cone. So coming back to this, so we want to vanish, when t minus t prime is greater than zero. So what happens when t minus t prime is greater than zero. We close the contour in the lower half plain. So if we do the following thing, if you push both of them, into upper half plain, then you will get, there are no poles, and the residue is zero. So that is what it tells you.

(Refer Slide Time: 14:03)

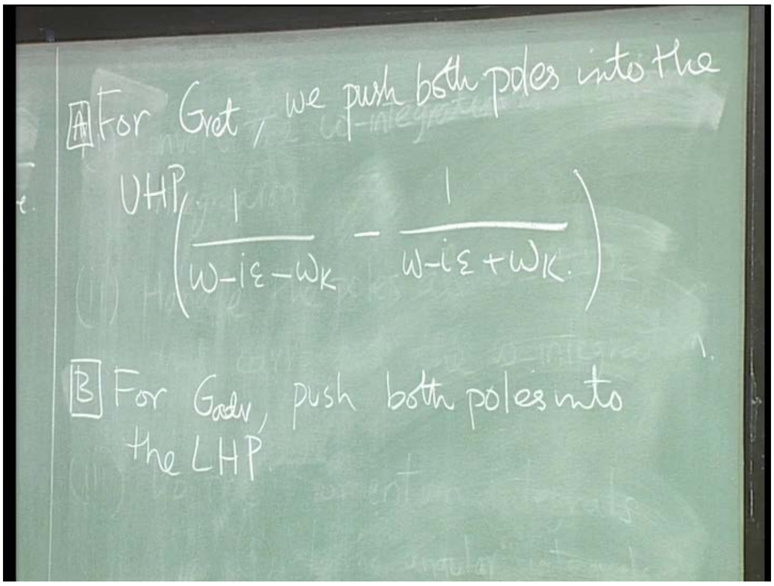


So we see that, for the retarded, so we push both poles into the upper half plain, and what how does that work, you have to choose the negative sign out here. So that is equal into saying that, you would right this thing. So one by omega minus omega k minus. So I will just keep track of these things, exponentials are still there, minus. So this is the prescription. So what about the opposite if you want it, for g advanced, we just do the opposite, which is to push both poles, but intuitively. If you see up to a sign, if you give me the retarded guy, and the advanced guy, they will only differ, if I just exchange the roles of x and x prime, so I really do not need to, I could do it by the roles of x and x prime, and then I can go to this.

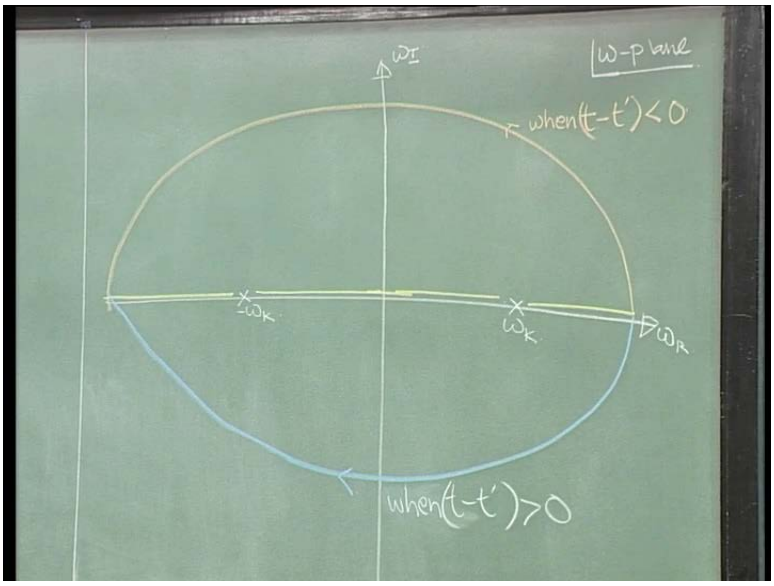
(Refer Slide Time: 15:53)



(Refer Slide Time: 16:15)

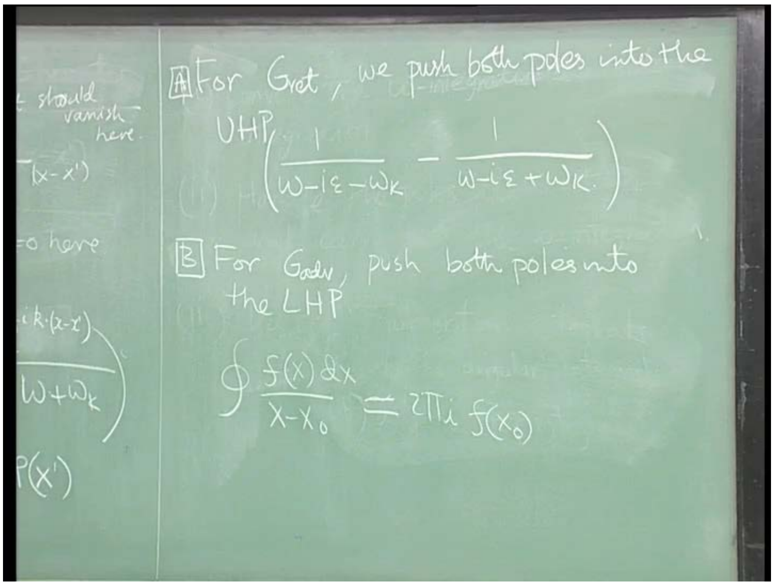


(Refer Slide Time: 16:19)

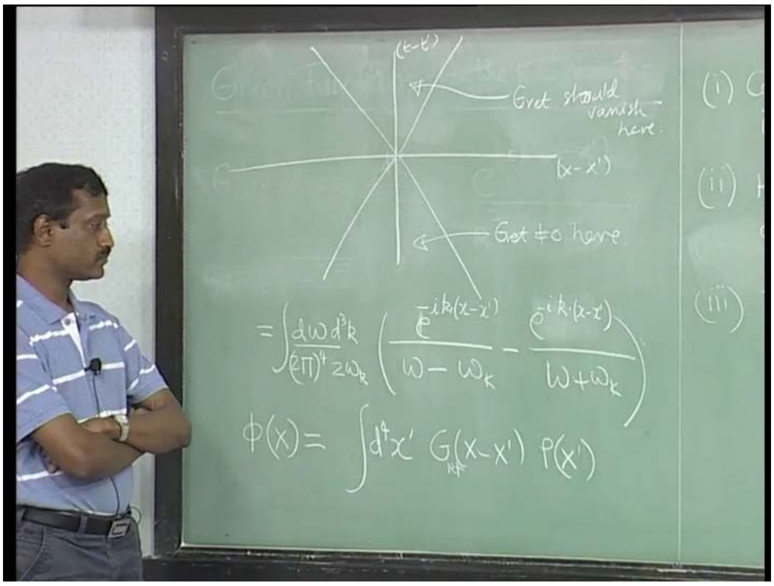


So let us choose g retarded and carry out the integral, omega integration. So, now we need to carry out the omega integration, so the. So clearly we have seen that g retarded of x minus x prime, by our choices is zero, when t minus t prime is greater than zero, so we still need to work out, what it is, when t minus t prime is less than zero, and that where we have to the integral. So that is when we close in the upper half plain, and we have to take the residues of both of these guys.

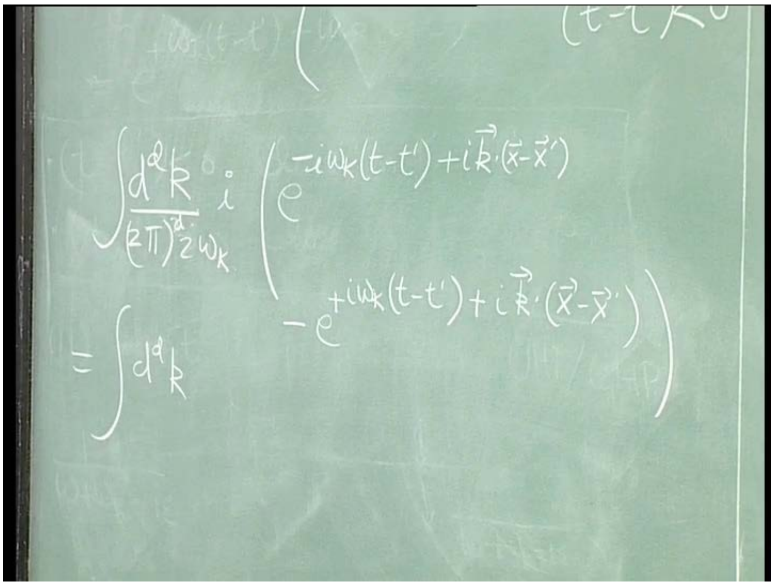
(Refer Slide Time: 16:24)



(Refer Slide Time: 16:57)

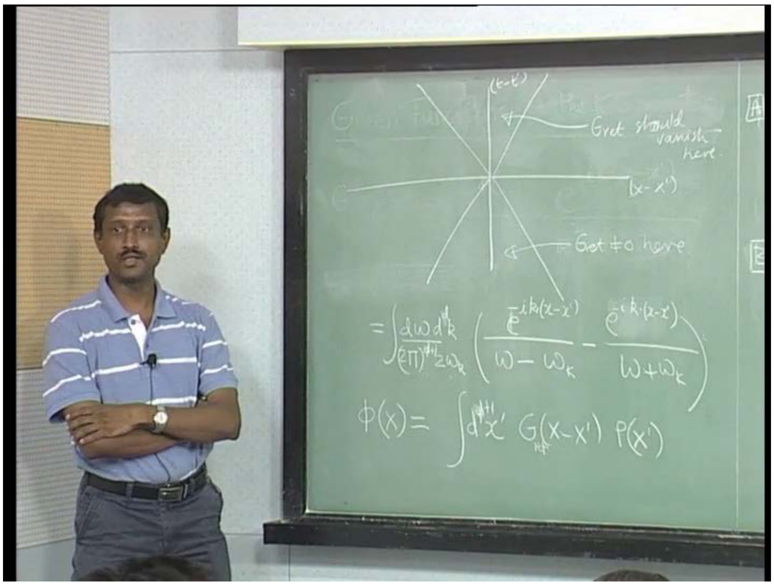


(Refer Slide Time: 17:21)

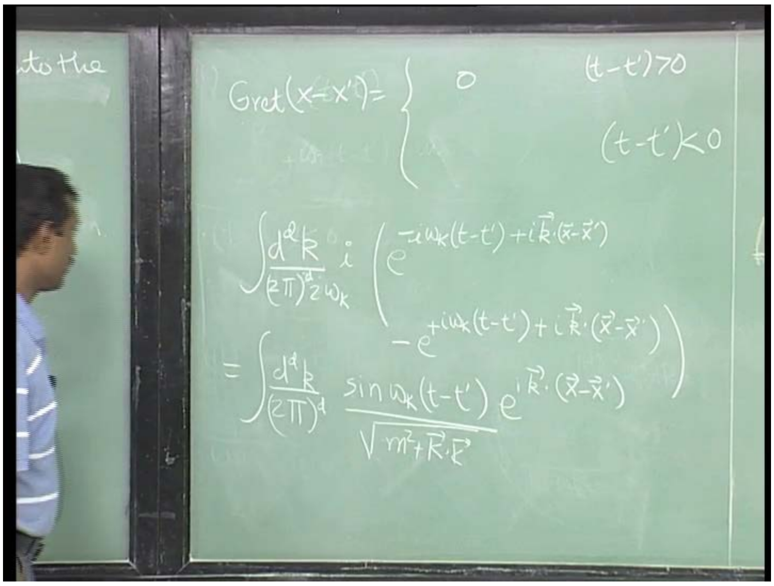


So let us go ahead and do this. So that a so what a so what does a. So, let us residue theorem, if you have something like this; f of x some quantum integral, of some analytic thing. This should be equal if the contour is clockwise; whatever its two pi i times a residue which is, f of x naught. So let us see what a what is the analog of f of x here. It is e power minus i k dot x minus x prime times this 2 pi power 4 2 omega k. So the residue will just tell you, whatever is the value of omega, it will replace that. So you will see that for this guy, I would put omega equal to omega k, but first for guy we would put minus omega k, that is all I have to do. So if I do that, I can I here, let us do that and write the answer out here. So we will carry out the into. So I am carrying out the integration, and but of course, you will have the special integration, and remember that there is a 2 pi coming from here, so it will eat up one of the two pi cubed or whatever, but there will be an i, so two omega k, and the residue will give you an i and then you will get e power minus. First term will give us plus omega k.

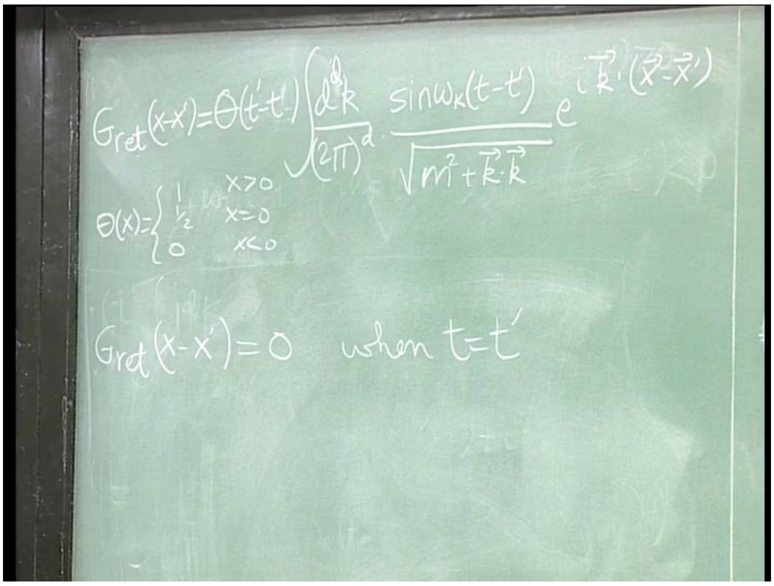
(Refer Slide Time: 18:51)



(Refer Slide Time: 19:01)



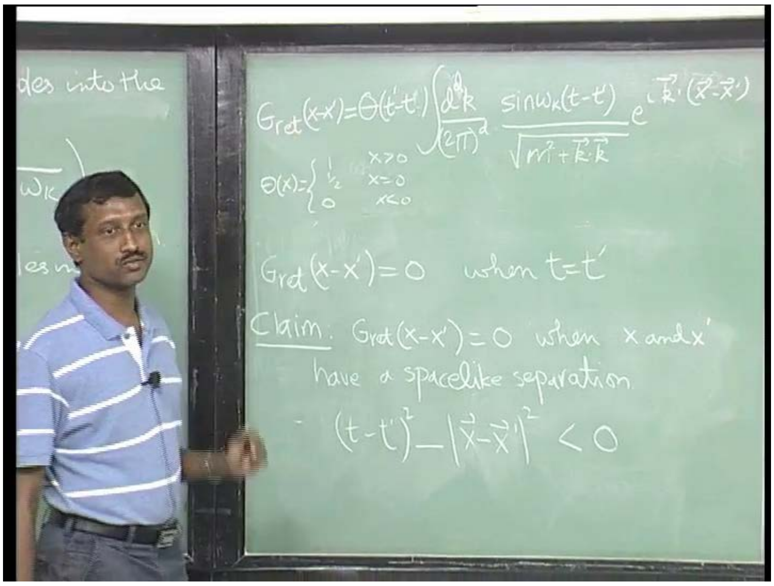
(Refer Slide Time: 19:54)



And the next term, this sign change comes, because the residue is, here it is minus omega k. So this is what I get, there is a minus sign, and that is it. But life is good, there is an i out here, there is a 2 out here. So you can see, that the time dependent part, becomes a sign of omega k t. So this we can see becomes equal to. I think I made a mistake here, this I have been writing for three dimensions so, times sin of omega k t minus prime into e power, omega k which we can write out for once. So this is what we get. Clearing this space is not enough for, what we have written.

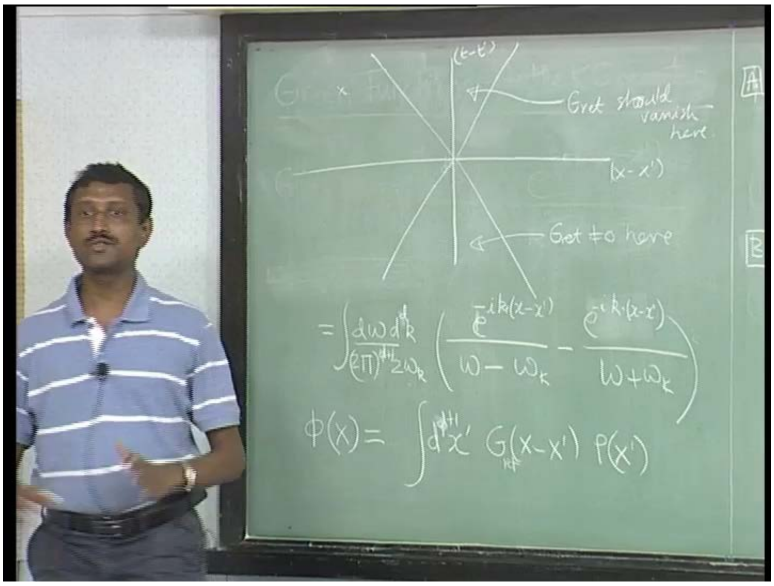
So we can put these two things together, by as follows. I need to check, so this is the heavy side theta function, so this should vanish when t minus t prime is, less than zero, so I should change this. So theta, so the heavy side theta function zero for x less than zero one for x greater than zero and a half for x equal to zero, step function or the this thing. So you can see now when t prime minus t is greater than zero it is one, and zero for the other things, and let us look at this. And ask what happens, when t equal to t prime. So we need not get into the semantics or what happen at theta, but you can this sign function zero, so we can see that regarded. So what does this imply, this implies, so what is the line t equal to t prime. This is this real line. So any two points on this are space like separated. So what this tells you, is that this green function vanishes on this line, on this line, actually it is in the full space.

(Refer Slide Time: 22:04)



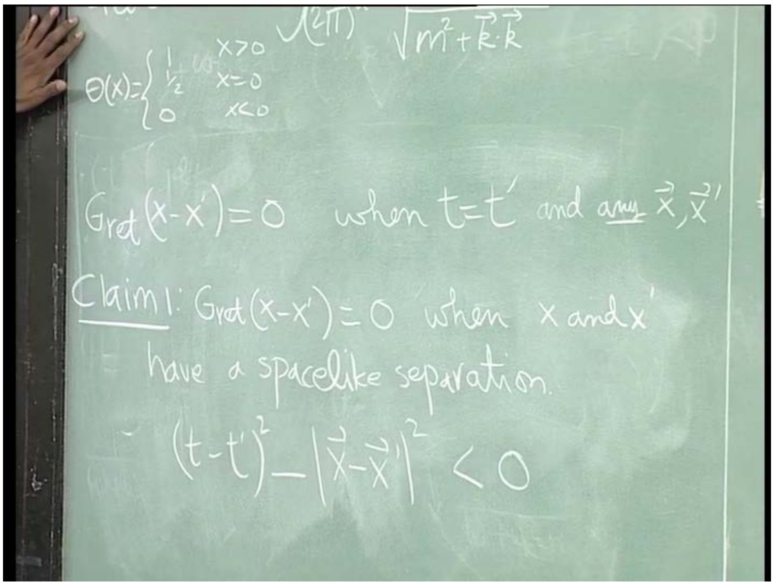
So in other words, the claim here; g regarded of x minus x prime is equal to zero when two point, when x minus x prime x and x prime have a space like separation. We say that a separation is a space like, if this combination; that is t minus t prime square minus is less than zero, which is satisfied of course, when t equal to t prime which is zero, and this is...

(Refer Slide Time: 22:58)



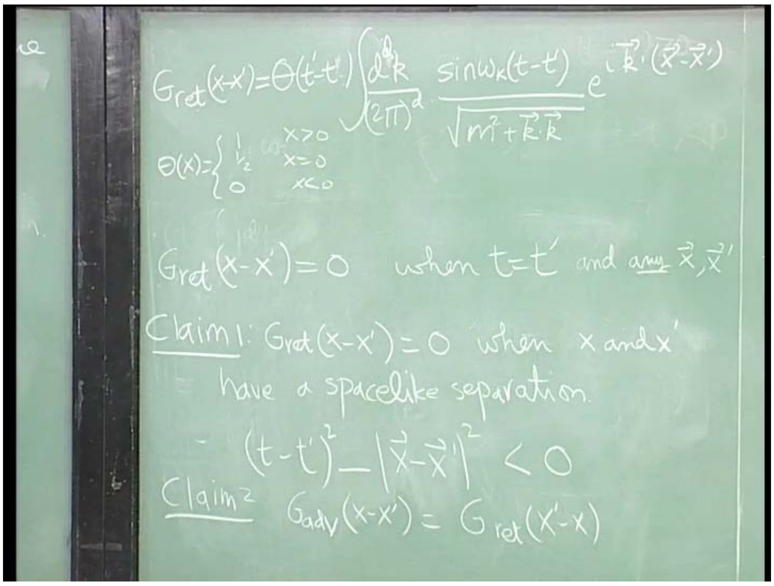
So here we are, here I am making use of. All I have shown is that it is vanishing on this line, or to be correct space like separated plain hyper plain, if you go to a higher, the d dimensions, but I claim that any Lawrence transformation that you make, can never take points from here to there, but you can take points here to anywhere here. Convince yourself you pick up point here, and work out the Lawrence transformation, which will take some point on this line out here, there exist one. So in other words, it vanishes not just on this line, it vanishes in this whole region, outside the light cone it vanishes. This is good news. This is something you wanted. So it vanishes not just here, g retarded does not vanish just here, it vanishes even in these regions, its only non vanishing out here, but it is quite nice.

(Refer Slide Time: 24:00)



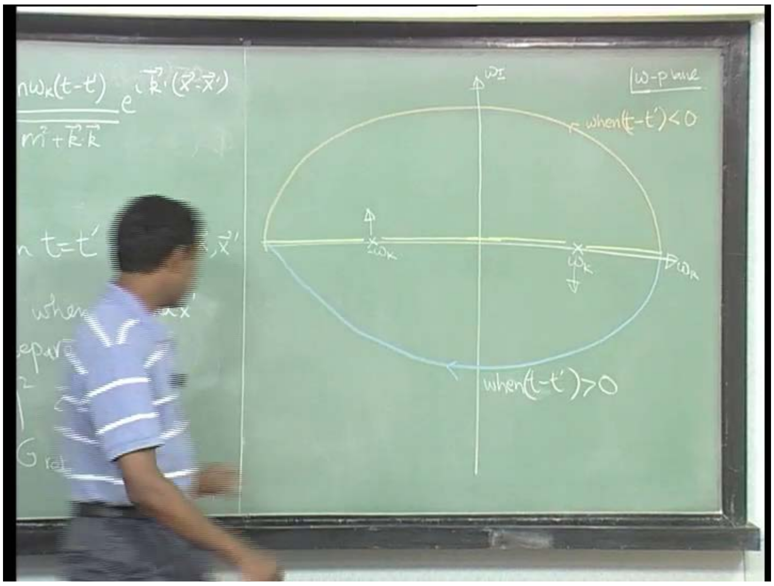
So, the claim is that, g advance is just minus, you just exchanged the roles. So this is claim one, which you can prove, because it is easy to see, I only proved this, for this sort of a thing, for any, and any x prime, any pair x x, but the fact that g retarded this is a Lawrence scalar, this is what I am using, and the existence of a Lawrence transformation. So this is a simple claim, it is a fun exercise, do it only in one plus one dimensions. It is enough for you to see that, and then you can convince yourself that holds always.

(Refer Slide Time: 24:46)

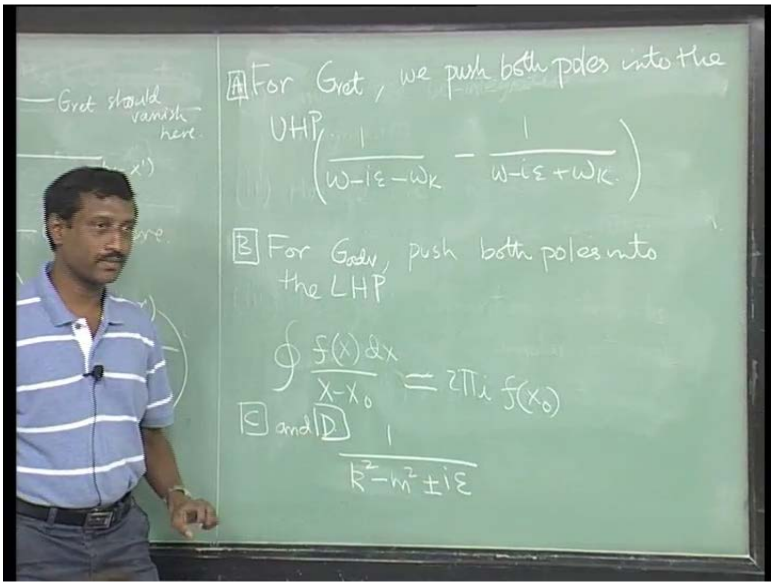


And the next claim is that, this is even easier g advanced of x minus x prime equal to g retarded of x prime minus x, sorry changed. So this sort of completes what we wanted to do with respect to this, but we need to do a little bit more work, which is to carry out the k integration, there is that bit, but before we get into that, we can look at another possibility which comes about, which is.

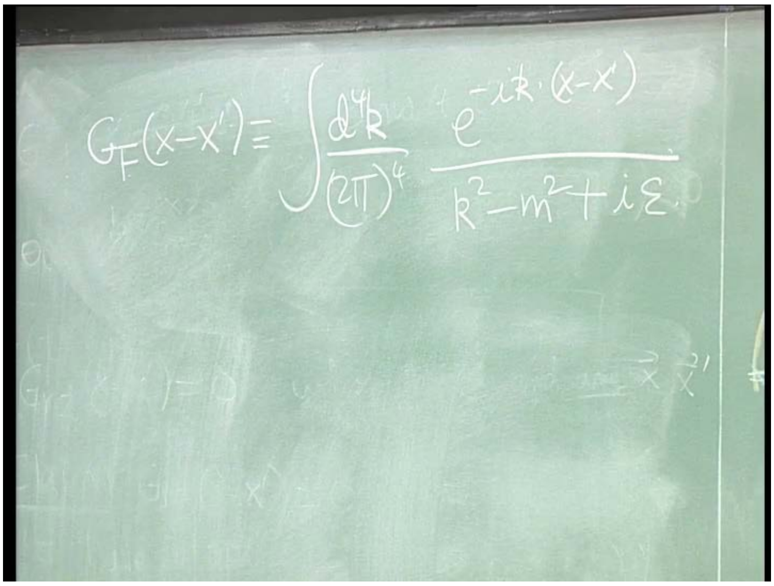
(Refer Slide Time: 25:31)



(Refer Slide Time: 25:49)

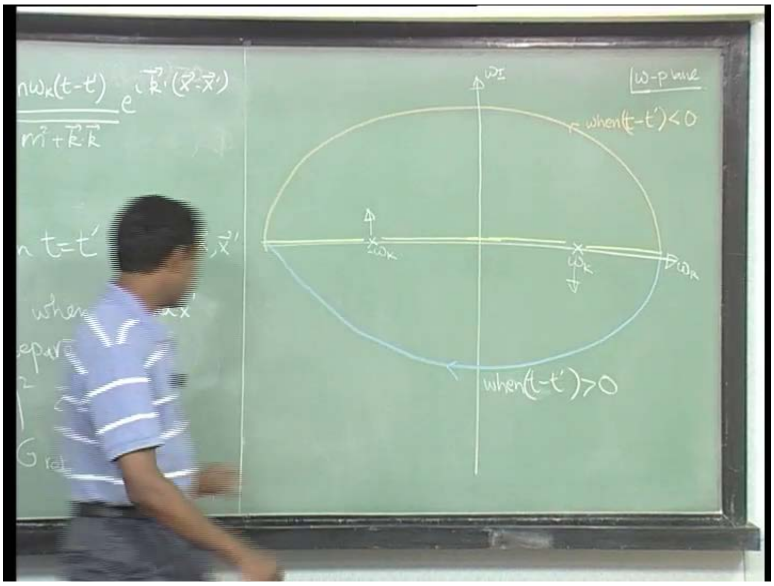


(Refer Slide Time: 26:48)

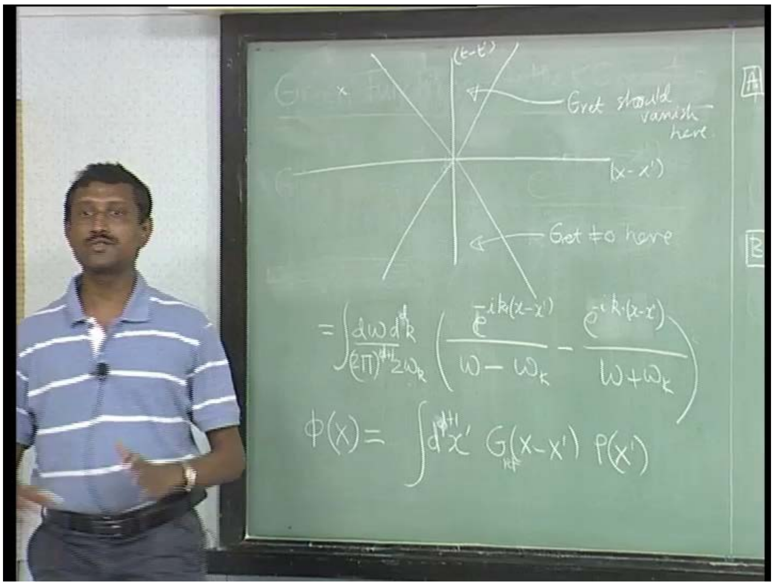


There are two other possibilities; one up and one down, you could have done this, or the opposite. This one going down, and this one going up. So the way to understand those two possibilities, is to. So the other two possibilities, so let us call it c and d, and which I will write in a slightly different manner, plus or minus i epsilon. Again epsilon is some positive quantity, I leave it as an exercise for you to convince yourself that; one of the signs does pushes, does this what I have indicated here, and the other sign does this, but we will focus on the plus sign out here. This plus sign, is something which appears in quanta fill theory, and it leads to something called the Feynman propagator. So let us call c the plus sign. So, c is taking plus sign, remember epsilon is positive. So that gives you something which we call g f, f for Feynman.

(Refer Slide Time: 27:24)

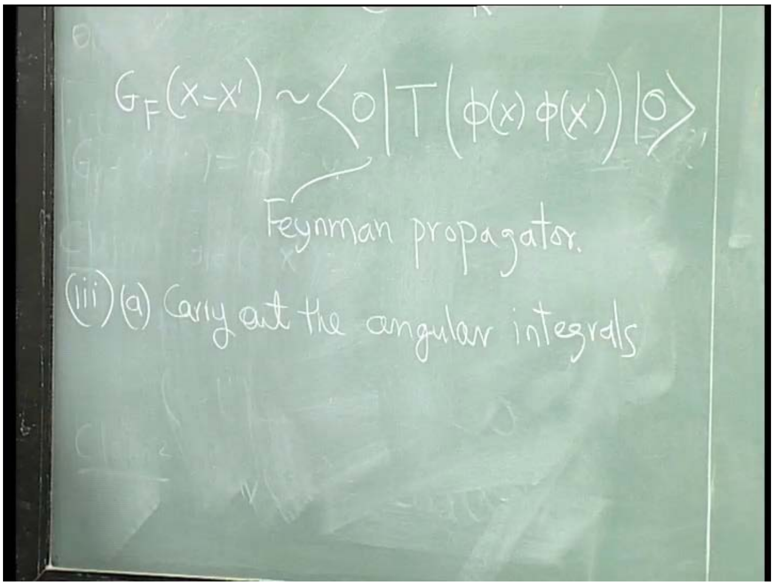


(Refer Slide Time: 27:46)



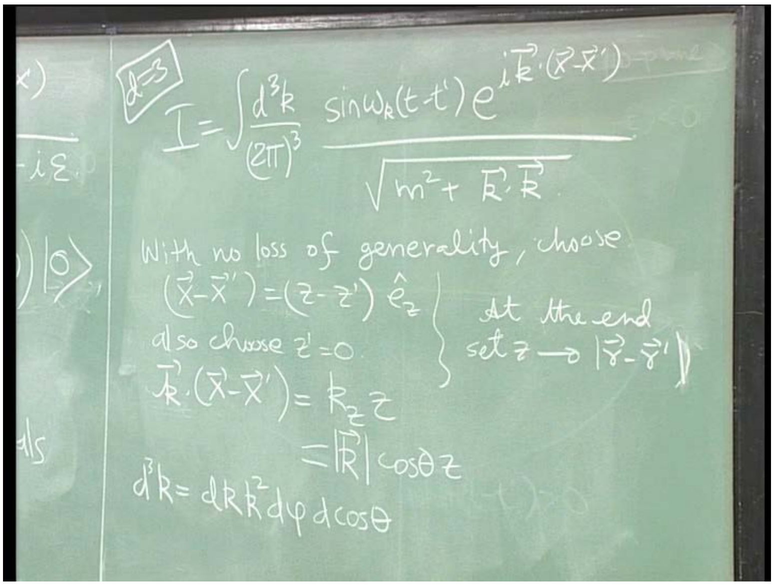
So, you will see that both, first thing is to notice is that, because there is always one pole in the upper half plain, and one pole in the lower half plain. You find that, when you do the omega integration, it does not vanish, in either one of them. It does not vanish here, nor does not vanish here. So it is not vanishing out here, but even more interesting is a fact, that actually does not vanish even in the, even in space like suppression, and that I will leave it as an exercise ,where i indicate you how it goes.

(Refer Slide Time: 27:55)



So this object actually g f of x minus x prime up to some multiplicable factor, in quantum field theory, is the time ordered of two fields; x pi of x phi of x prime. So this is, this is called the Feynman propagator, and so this also appears in, they will definitely appear in the next course. So time ordering here, is just that you want to put. So time ordering is that, you may have seen this in quantum mechanics, you need to. So you order the, these things just that the, one which is of lay of earlier time should be to the right, and the one later comes to these things. So that depends on whether t minus t prime is greater than zero, or less than zero. So now what we will do in the next 15 20 minutes, is to complete the carrying out the integration. So we have only done the omega integration, you need to do the. So the third part, is to carry out the, a bit is to actually carry out what I would call the angular integrations. You will become clear why I am calling them angular integrals.

(Refer Slide Time: 29:33)

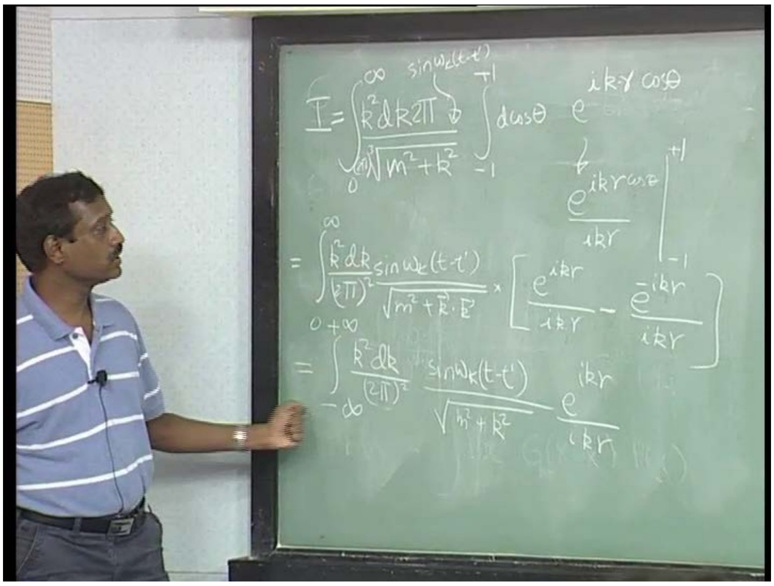


Coming back to… after we carry out the omega integrations, so we have an integral which look something like this d. So let us out d equal to 3, because I can be very explicit with regards to the angles. You can generalize to any dimension, so you had d cube k here divided by 2 pi cubed, some sign omega k of t minus t prime e power i k vector dot x minus x prime divided by square root of. So this is the integral we need to carry out, but there is something very nice going on out here. This is only the function of magnitude of k, so this is omega k, this also goes the same thing and. So the only dependence on any, on k the other parts of, this would come through this sort of a thing, and it is a standard way of there is a standard method of doing this. You, since your averaging all k, you can, and this is for a fixed x minus x prime, with no loss of generality you can say that x minus x prime is along z direction.

So, with no loss of generality, there is a theta here, but I am at this point on the focusing, on the integration part of this thing. With no loss of generality, choose minus x prime to be z minus on z prime, along the z direction. And to even more simplify things, I will just set z prime to be zero, because it is only on the difference, also choose. Then these objects actually can be written, if you wish. Actually I do not even need to do this, let me just.

Let us since I have made the statement let me use it. So what we see here is, now k. This would just become the exponential of k dot x minus x prime is k z z, after I do these two things, but k z can be written as magnitude of k times co sin theta into z. At the end the day what I will do is, z will become mod of z minus, mod of r minus r prime. At the end, so this to recover the whole thing at the end, set z to. So, this is the angular part in case pace, you can see that I need to do the co sin theta integral, its independent of phi. So d cubed k can be written as d of mod k, which I will write as d k. Please do not confuse this with the four vector k out here. This is four vector square, but this is the magnitude of the three vector k. Just for, otherwise I have got a key putting mode k k square d phi d cos theta. I prefer to write it as d cos theta, because it is a sin theta d theta, and but cos theta goes from minus one to plus one. So instead of making one more step, where I change variables, you can see it depends only on the combination cos theta, which runs from minus one to plus one. So now it is very easy to do the phi cos theta integral, the phi integration is the easiest, it is independent of phi; that will give you by two pi, and I need to do this guy.

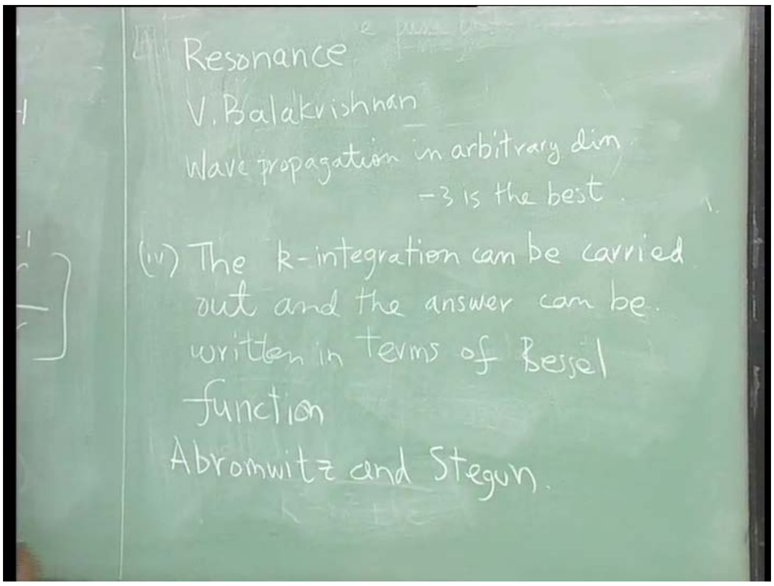
(Refer Slide Time: 34:25)



So let me write out and do this, do the integration. So let us just call this quantity I, and that leaves me with, so I get i equal to. I forgot a sign, so I just need to do this but this is very easy, and so in fact I will, let me do one thing, I just call that combination r instead of calling it z, so this just gives me e power i k r divided by cos theta evaluated between minus one and plus one. And now the k integration goes from zero to infinity, so what does this give you. It gives you e power i k r minus e power minus i k r, because cos sin theta plus one, so this is equal to. There was a 2 pi cubed, now that becomes 2 pi square, so, we get two terms and, but the k integration goes from zero to infinity, but let us ask what happens to, if I go ahead and take k to minus k on this term, then the exponential k becomes power i k r, it looks exactly like this, this k goes to minus k, so this becomes plus sign, and what happens to the limits. The limit becomes zero to infinity zero to minus infinity.

So let me write this out as two terms, they should not cancel, they are the first term is just integral zero to infinity, so this. Yes I am I am so let me just say it in words. So what happens is that this term, so this becomes, if I take k to minus k out here, this becomes e power plus i k r, which is similar to this. This becomes minus k r this flip sign out here, but let us come back to this guy, it is odd, so this will also change sign, so that will convert integration of zero to minus infinity to, flip it and make it minus infinity to zero. So you can see that I can rewrite this term by putting it back into this, with by changing the limits to go from minus infinity to plus infinity. Now you can see dimensionality matters. Now we are only left to do one, one dimension integral. So we have finished the angular integrals, and you can see in the other dimensions, there might be more angular integrals, but they will give you some constant functions; like we got integration gave you a two pi or whatever, you need to be little bit more careful about, how the angles work out and, so I leave it as an exercise for you to do it.

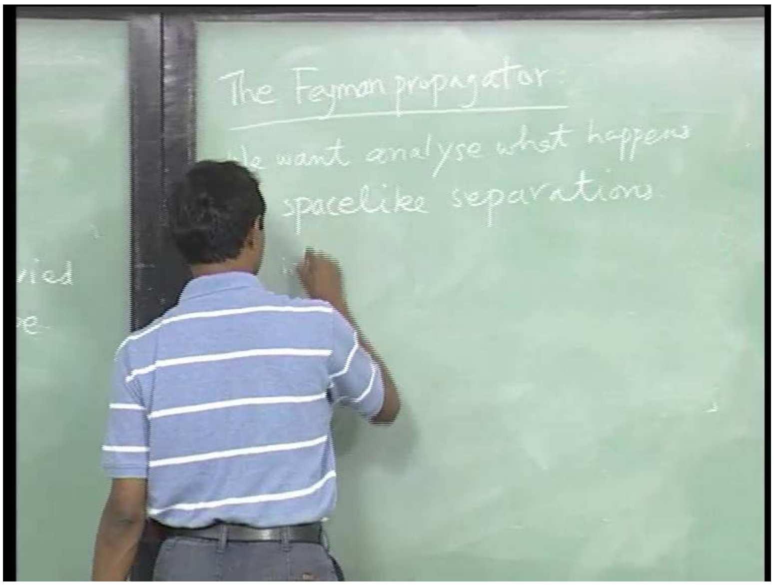
(Refer Slide Time: 39:17)



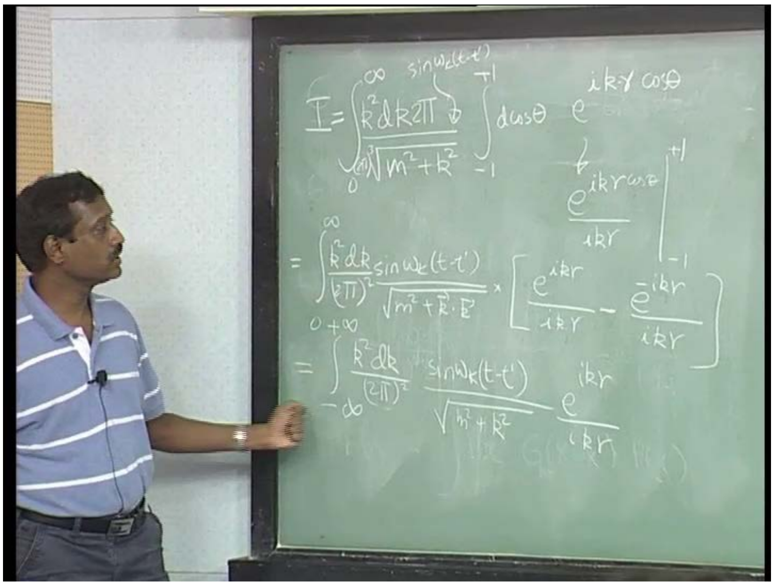
But there is a nice article by Dr. Bala Krishnan in the two in this nice journal called resonance, there are two articles by him. I think it is titled something like wave propagation, in arbitrary dimension or something. I will put up the exact reference on the course web page, and it is a two part thing, and there is something about three is the best. So he actually carries out discusses things in more arbitrary dimensions I do not want to do that, because I just want to work out one example, but it is a two part series, which is a fun series articles. But now coming back to this, this integral can done, and the answer. The last, which is the fourth part; the k integration, can be carried out, and the answer can be written, in terms of Bessels functions. So the way to go about understanding this is, I have already given you the hint.

I have told you this can be done written in terms of Bessels functions, and so you need to know an integral representation of Bessel function. Now you are not expected to remember this, but there is a wonderful books on special functions; one of which is Abromwitz and Stegun. In fact the people, that I guess n i s t whatever. They have actually put this online now, and it is updated, so you go, I mean you just search for this. In fact I think there is even a Wikipedia entry for this, and you can get look for Bessels functions out there, look for integral representations, and you will have to play with it a bit. It is not like every integral that you see here it should be there in Abromwitz and Stegun, you may have to massage it a little bit, but I think I have given you enough of a hint, that it can be written in terms of Bessels functions. So that is the. So you can indeed complete this.

(Refer Slide Time: 42:05)

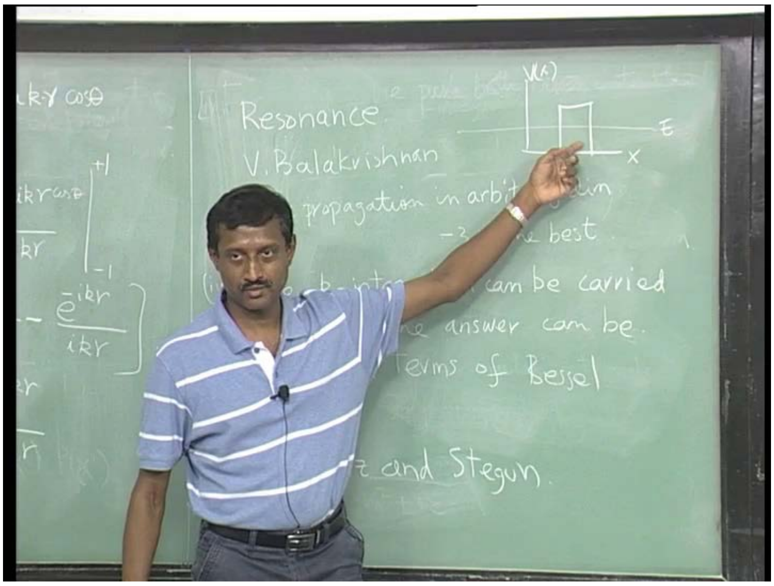


(Refer Slide Time: 42:20)



So coming back to the Feyman propagator, so what you will see in the Feyman propagator, is that you will get similar integrations, but we are interested in what happens, so we want to analyze what happens for space like separations. Out here of course, if you put t equal to t prime, things vanish, but there what you will see that only one a here, the way we got sign of omega k t was that both of the poles contributed. So it is very easy to see that what you will end up getting, is a not as sign, but one of the exponential only, and unlike here everything was real; that is pure imaginary.

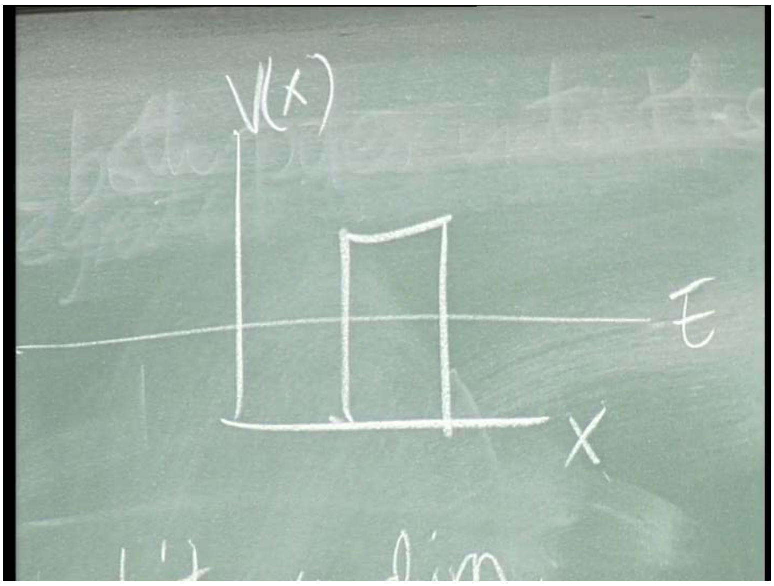
(Refer Slide Time: 46:29)



So that is also important to note and. So, you get an exponential, so the key point is if I put t equal to t prime; that will not vanish, so you are left to do an integral, which looks like this. So the integral has this form, since I am not actually going to work out all the various factors, I will drop various things, and only write out the k dependent pieces, this is not there, I get this, only these three pieces. This is what I get, so I need to do this integration, but again this can be written in terms of Bessels functions, but what we are interested in asking is, what happens when the separation is large, so what happens, and I will just write out the answer, which you can actually see from dimensional grounds out here. So what do we get, so, what you will find is that, you will. I will explain to you why I am saying that it comes from the dimensional grounds, or maybe you could do sadded point or whatever.

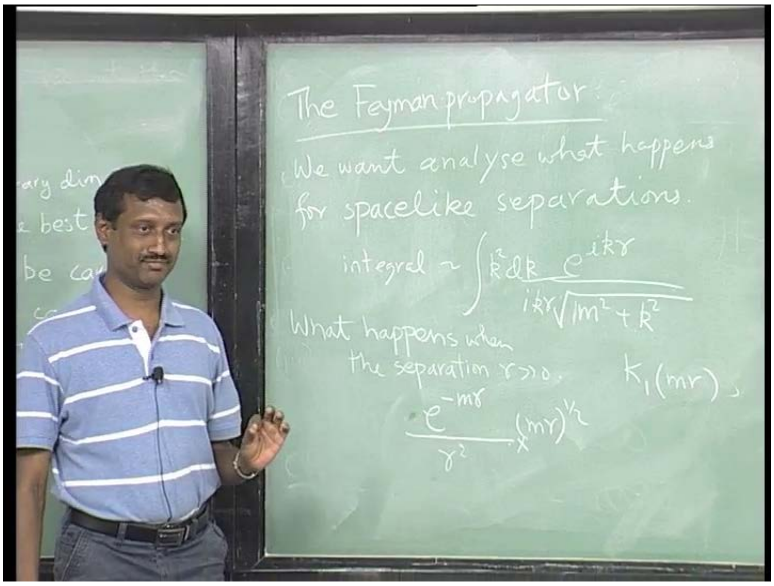
So you look out here, and you can see that, the dominant contribution is coming when k square equal to m square, now you think k as a complex number, but you can do it from writing out the Bessel function, so you get I think k 1 of m r is what you get, the final answer. This is the Bessel function, which is for large R has this e power minus m r behavior, but what I am showing you is that actually you could do it, without even looking at these things. So dimensionally this d k and this d k goes off, so there is a k square going around, this compensate, for that by putting a r square, there is a k r out here, wherever you see a k, you put m r. Is this correct, this should be power half. So I just put these things, but the key point here, is that it is exponentially suppressed, so it does not vanish. The (( )) which I take away from here, with that the Feynman propagator does not vanish, but this is something happening in quantum field theory. This is not something unusual I mean, we know that that can be classically forbidden reasons in quantum mechanics.

(Refer Slide Time: 46:34)



Suppose you have, a potential of this kind, and you have energies, so this is v of x versus x. So if you choose the one dimensional particle with equal energy, which is less than that of the barrier, you know that that, if you look at the wave function. The wave function is zero out here, its exponential is suppressed, it is similar to that.

(Refer Slide Time: 46:58)



So what you thought were classically were zero; quantum mechanically. I mean there is a reflex it does, it has some non zero answer, but never the less small in some sense, in as much as h bar is small. So in that sense, so this is, it should not, it is not supposed meant to be a shock or anything that it happens in a quantum field theory. We have seen it already in quantum mechanics in some sense, and so, but its pleasant pleasing to see that it actually dies off very fast. You may ask what happens from (( )) particles, I think I will leave that for you to think about a little later, but for now I think I stop here, and so I think there are all these exercises, I would strongly recommend that, you should carry out this last integration. I did not want to bore you people with the discussing special functions, but I think we should be at a level where we, think of the Bessel functions should on par on with the sin and the co sin functions.

So when you are in high school sin functions and co sin functions are nice functions, you know everything about it, but if you take these Bessels functions, now we are at a level, where you should think of them on par. Of course, there are properties of the Bessels functions, which are not that obvious as that of the functions. So whenever when does the sin function vanish, whenever its argument is to, is some n phi, you know that, but if you want the same story where the zeros of the Bessels function, that is not, I mean you need to look at the tables. Again these books have fantastic you wants to know where are, what is the first zero of j 1, or whatever, you can see it out here. But, there are certain properties which you should be aware of, the fact that how it goes to, its behavior near zero argument, and it behavior at large. So large things, the j is become, auxilitory while the k is have exponential, it could be. So these are things which you should know, and become familiar with, if you are not its, but you there is a chance, for you to play with these things. In principle we will not be using a lot of it in this course, but I think we have to move to the next level, where we think of Bessel function, or something normal.

Thank you.