Classical Field Theory
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Lecture – 11

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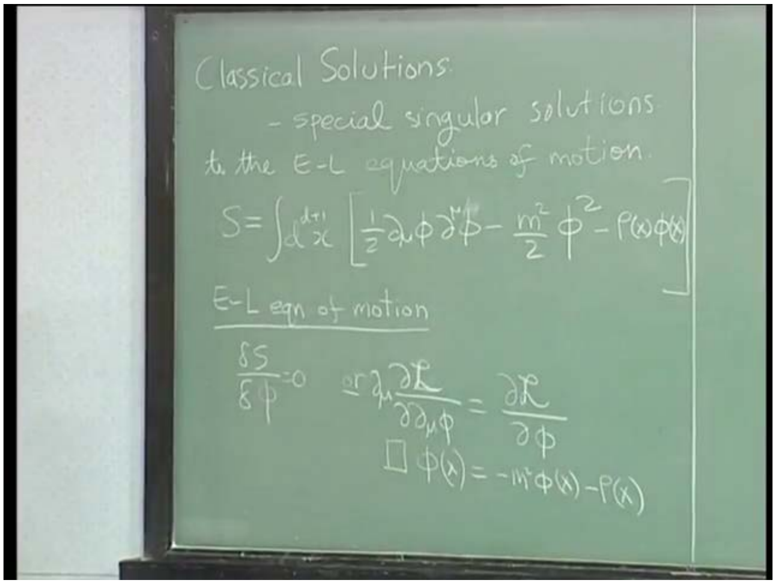

But, these are actually not usually called solutions, but we will still call them classical solutions. You will see why, we will just call them special singular solutions to the Euler Lagrangian equations of motion. So, again we will go back to our canonical example that of scalar field, where the action is just given as follows.

So, will only add a quadratic piece, this I call this a mass term because we want to preserve linearity in the Euler Lagrangian equation, which we will get. If you add a phi cube dot phi 4 kind of term, the equations become non-linear, but I will also add to this an extra term. A source term rho of x is not suppose to be a field, it is just something external, which you are putting in there and phi is of course a field. I am here, I have not written the arguments, but obviously all of them depend on x, so that. So, Euler Lagrangian equation of motion for this is basically.

So, we have already done this once, but let me just write it out. So, you can write it in this fashion or equivalently. So, this term will give you a box phi and, now we get we have to take a derivative of phi with respect to this. So, you get a term, which is minus m square phi of x times minus rho of x. So, this term as you can see is giving the term, which is independent of the field phi.

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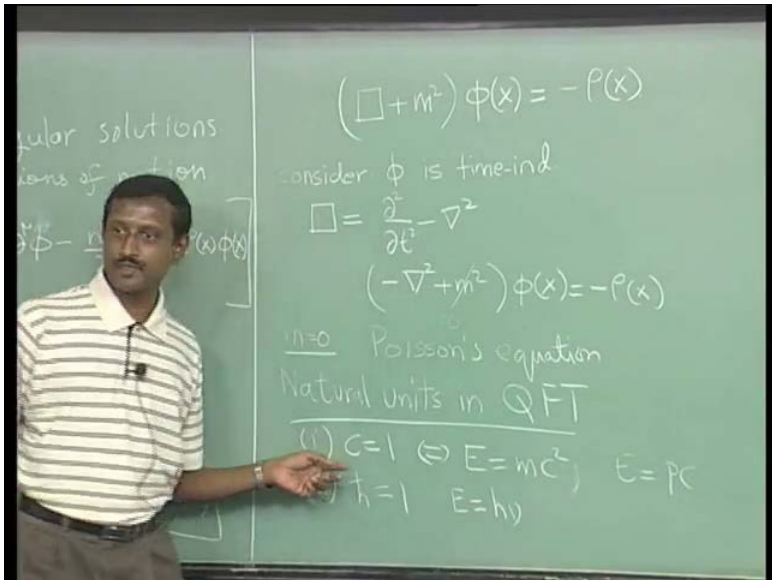

So, I will rearrange things to get, so as promised this is linear in phi, this is a linear equation phi, but there is a source term. So, let us just look at what this equation reduces too. So, for instance consider situation, where phi is time independent and when box, which basically corresponds to this following operator acting on phi. If it were phi was a time independent, then you end up getting this equation, becomes minus del square plus m square of phi of x equals to minus rho of x.

Of course, so to get a solution of this kind rho better also be time independent, otherwise I mean, you cannot satisfy this equation, but that is not I am just trying to show you that. So, you can see that, this is and let us also for a moment consider m equal to 0, then you see that just this equation becomes Poisson's equation. So, for m equal to 0 it reduces to Poisson's equation, this equation you would have solved in you are electro dynamics course for instance.

And I guess with the m square it is called the Elmore's equation, I am not good with names, but I think it is called the Elmore's equation. So, you can see that, this is a natural generalization of all these things; it just takes into account as invariance under special relativity. So, obviously, it tells you we got you have to put this. By the way, it always like to write a del square as minus del squared, the reason is that this is the operator, which is emission. If you did not put the minus sign then and, which is positive definite to be more important this is the positive definite operator.

This minus del square as positive as again values, which are greater than or equal to 0. So, and come to think I mean, if you think about it even in quantum mechanics, this is exactly what happens. And I just want to again rewrite, what I said in last lecture? We will be always working in, what I call natural units, not just QFT in quantum field theory. So, the first one is to set to work in units, where C is equal to 1. Then time and in such unit time and space are measured in the same this thing.

And in this course we will not use quite a bit of this, but you can also set h bar equal to 1. So, this is an extra thing, so for instance in such situations you can see that energy. So, let us look at this first, if you look at your relation like, energy and mass you get E equal to mc square, dimensional just for dimensional ground. So, C is equal to 1, so E and m are measured in the same units.

So, you can say mass in G v or you could say energy in kilograms, but it is more common to work with mass etcetera in this thing. So, you can also see that, E dimensionally is same as P times C again momentum is also measured in terms of mass units or energy units. So, this is these are nice things, which happen similarly, if you put h bar equal to 1. So, let us look at few more things something, like H e equal to h mu or h cross h bar omega. So, you can see that energy is the same as 1 by time, but from since, we are putting C equal to 1 is like 1 by length.

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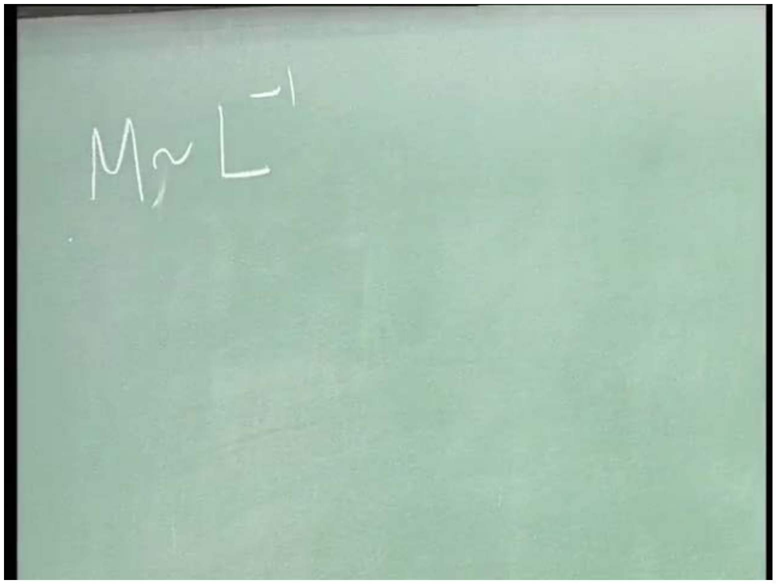

So, because of these things, what we will call the mass the physical dimensions usually where mass and L, but in these sort of things there was inverse relationship. And intuitively, this is very nice, this tells you that if you have high energy or high mass that is like, lower length scale. So, that is what they say that, when you are probing the quantum nature of matter you are probing at lower length scales or higher energies.

So, intuitively this is what is units, which are natural in quantum field theory, but these are the natural units in the general theory of relativity, because h bar it is a classical theory h bar is a constant that does not come in that theory. So, what is natural to general relativity would be fine to set C equal to 1, because s r sits inside that, but that is the Newton constant. So, you go ahead in set G Newton equal to 1.

So, you measure everything in terms of and, then there in you get something like, counter intention that M and L are actually the same. So, but in this course, we will be always setting, these two constants equal to 1 and that is what natural to this course. So, for the rest of this course, I will never, I mean, unless necessary I will not put these factors back and it is not very hard to put them back either. And h bar will not come except in isolated distances, whenever to make remarks, but otherwise C will be equal to 1. So,we need to solve this equation.

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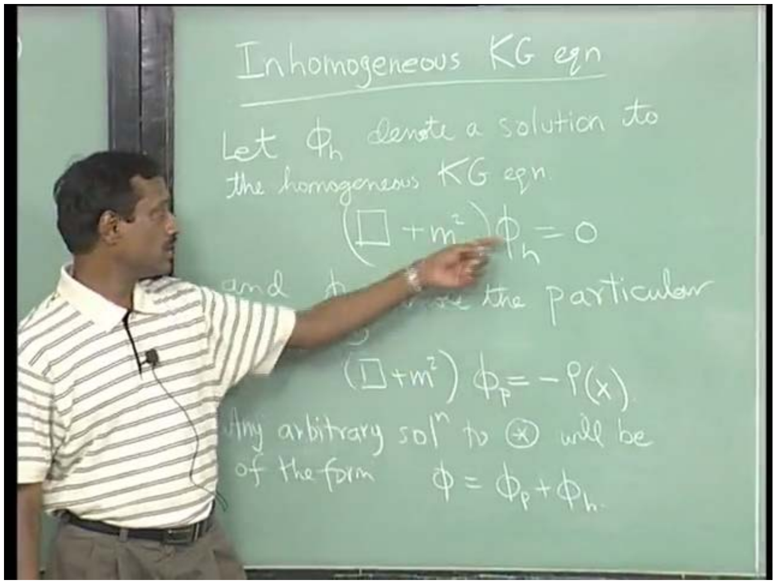

So, this equation is you know in homogeneous Klein Gordon equation, the inhomogeneous equation is when the source term is 0. So, if you have done a course in differential equations, you will know that there is something called a general solution and a particular solution. So, even out here, let us sort of denote phi h denote a solution to the homogeneous or Klein Gordon equation. In other words, it satisfies this equation with no sources, and phi P denote the particular solution to the inhomogeneous equation to, but because of the linearity of this equation any arbitrary solution.

So, let us call this equation, star will be of the form phi arbitrary solution, will be some linear combination of a given particular solution and the homogeneous solution. So, if you give me two different solutions, another way of saying is that take two different solutions to this equation. You take their difference; it is easy to ensure that it satisfies, the other equation that is the in homogeneous Klein Gordon equation. So, what we will do is to try to solve for this equation, but will solve it through a method call the green function.

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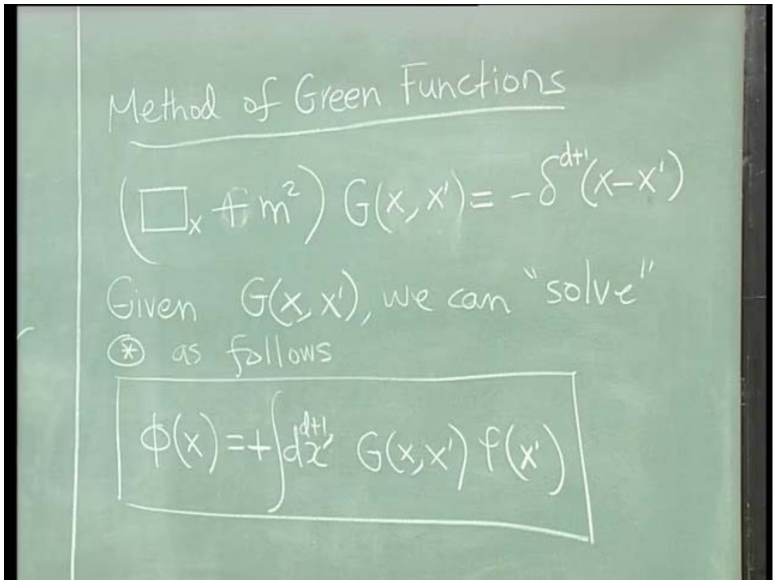

So, we use the method of green functions, this generalizes in a, so coming back to Poisson's equation, what we would do in electromagnetic, electrostatic is to solve. If you know the answer for a point charge, we can write out the answer for arbitrary charge distribution. What is a point charge? It is data function source. So, we will do the same thing out here, we will choose a delta function source and try to solve not that equation, but this equation and will call this as the green function books. I mean, I think it should be call the green function like, the color green. So, wherever I saw, this rho of x I just go ahead and solve for, so this is a d plus 1 dimensional green delta function. So, in other words, you have a source at some point in space and time not just space like, you would in this sort of equation, but in this equation.

So, suppose and you can see this object here, G of x, x prime as 2 arguments that is in part because this location is independent quantity. So, this is the equation, we would like to solve and it is very easy to see. Suppose, you gave me such a G, then I can give you a solution to this equation. So, given we can I will put solve in quotes because it is really not, I would not call it a solution yet, solve equation star as follows. So, you just write, consider the following thing phi of x equal to integral overall. So, let me just do one thing, let me put a minus sign out here. So, I saw the reason is I just want to put rho as some delta function. So, that would I just want to keep the minus sign elsewhere. So, it is not hard to see, that this satisfies equation star.

Student: (( ))

Integral is over x prime, but the key point is that... One more reason is to say that, I put quotes under solve is because, I mean the a differential equation is not, I mean it is not well posed, until you put boundary conditions etcetera. So, you need to put the boundary conditions and that comes back to this splitting. So, it looks like, you have the infinite set of solutions, but in fact that is not true. Once, you give a proper set of boundary conditions. It will uniquely fix what phi h should be you do not have any freedom. All the ambiguity that is there is gone.

So, coming back to this problem, we would like to ask what is the good boundary conditions etcetera. We have discussed a little bit equations of motions. The surface terms have to be vanishing, that usually is a good set of boundary conditions. And it is a very nice way of actually deriving things, what are the good boundary conditions. So, the first step is to actually go ahead and work out what G of x, x prime is, but before that let us try to understand some property.

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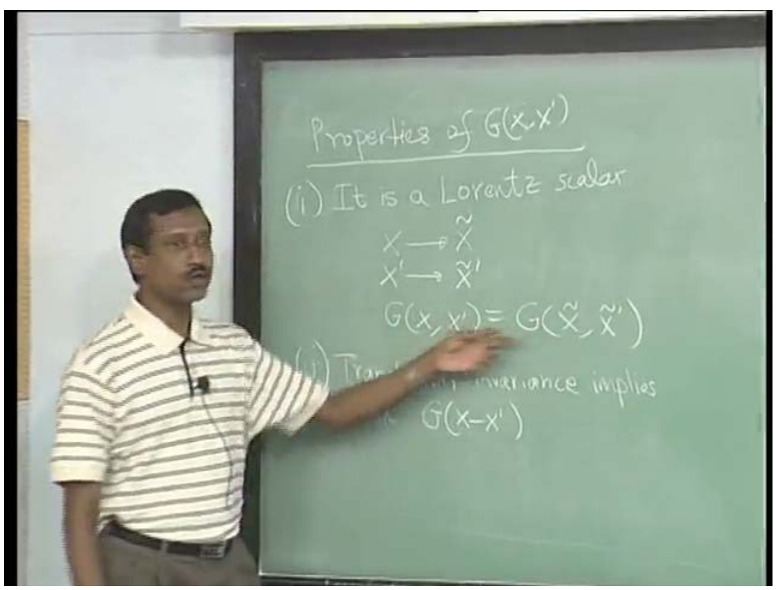

So, let us see what are the Properties of the green function? So, the first thing is the since, it is a green function for the scalar field, it is the we will take it to be the Lorentz scalar. If you change coordinates, if you go from coordinate X to some X tilde and similarly X prime to x tilde prime, then G should change as follows. Second bit is, you have translations invariance, you can see out here.

If you shift both x and x prime simultaneously by a constant it the sources or I mean, it depends only on the difference. So similarly, translation invariance implies that G should only be a function of the differences. So, you may wonder, where I am going to use this we will see a little later, that we can use all these symmetries and it is enough to check. For instance if you want some properties of G, suppose we say, that G should vanish in some region. Then you might be able to check, it is might be easy to check for some parts of the region. And by the rest, we will just say that, because it is Lorentz scalar it follows, but you can go back. If you do not believe, that you can actually do it the hard way and prove, that it vanishes in other regions.

So, let us see where it should vanish. So, let us look at this, so as it stands this is kind of non-local look here, x and x prime are two different space time points, then they need not be even causally connected. So, what you are saying is that, if this is one way of understanding, this if you have, if you create some privation or something and x prime, it is affecting something at some other space time point. Now, what would causality tell you, that x should be in the past in the sorry, future of x prime, otherwise if such things does not influence things. Then this would get causality violation suppose, x prime was in the future that means, you this field already knows about something in the future.

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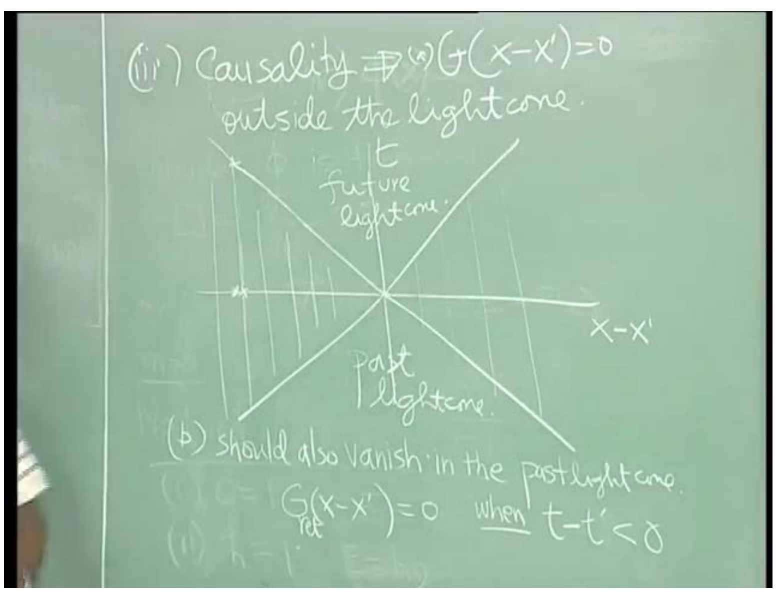

So, for instance so let us see what causality tells you so that is a third thing. So,I will write something outside the light cone first. So, let me remind you what the light cone is. So, I will do draw it for one space and one time, but you can extend it to arbitrary space dimensions. So, since we are working in unitswhere C is equal to 1 a ray of light can move at and will move at the trajectory of the light would be. Suppose, you emitted light a photo on here it will move around this line this at 45 degrees, which slope 1 or minus 1 depends on which direction it is moving minus x. So, dx by dt is 1, so this point here the thing is all events which can be influenced in the future by point here have to align inside the light cone because nothing can go faster than the speed of light.

So, these regions cannot be affected by point here and in some dramatic cases someone is sitting here and someone is sitting in the nearest in alpha century or something like that at the same time. There is no way this person could influence this you have to wait at least you send a light ray you have to extend this. So, let put it here it will take some time to come even a light ray would come much later in the future, but we are talking about this space time point. So, this space time point is causally is disconnected to this thing. So what we say is the points which can be affected so this is called the future light cone.

The reason it is called the cone is suppose, you have a y direction, let me add that coming out of the board. Then you can spin this thing and you get an object which looks like a cone in higher dimensions, you call it some hyper cone or something nobody where call it hyper something. You just call it a cone just to remind you of what it is. Here in this picture it is only like sort of edge the edge become’s a cone. So, point which lie in the future of this point of the future light cone and points which lie in the past of it would be called the past light cone. Causally the first thing we see anything outside the light cone, so here let us actually plot x minus x prime because, it is a difference of things.

So, when x equals to x prime this that would be origin out here. So, we are asking what are all the points affecting it can affect only the future light cone, and it cannot affect things in the past light cone. So, this tells you that G, G here should vanish outside the light cone number 1, it should also vanish in the past light cone. So, by that I mean G of x minus x prime equal to 0 when t minus t prime is less than 0.

So, it should only vanish in one of these things and we will just call such a object, we will say that we will call it a retarded RET, for retarded green function. It takes a retardation effects into, but sometimes you are solving an inverse problem you are solving something, where you know the answer. Now, but you want to know what was the object in the previous which was the thing which created this today then you would look at something, which does the opposite. That is called the advanced thing that will vanish when t prime less than t is 0 the other combination.

So, the question is how do we go about solving for this? But, already we have when you are solving classic equation of Poisson equation the natural thing is to take the fourier transform. So, here, we will do exactly that we will take the d plus 1 dimensional fourier transform. What it does for you is to convert all differential equations into algebraic equations.

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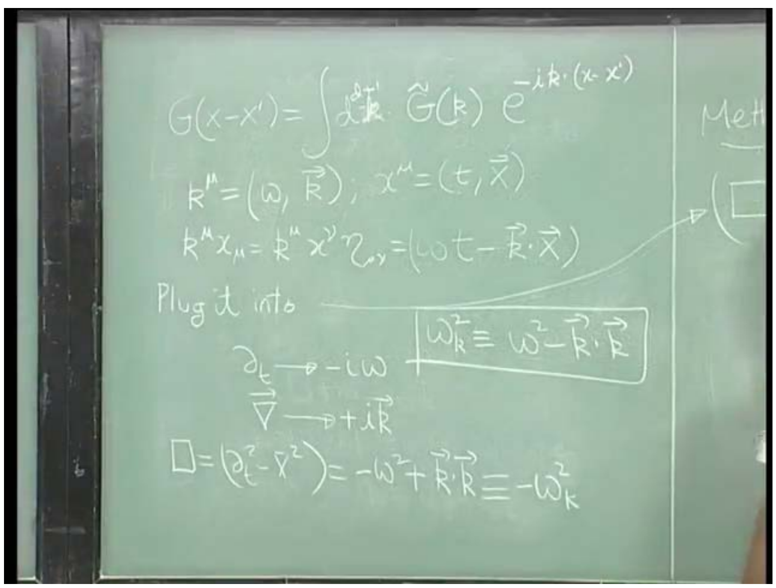

So, let us do that let us define k here is a 4 vector which I will write as omega and some special vector k. So, k mu x mu this would be omega t where again the usual thing the x coordinate is called t and special in some vector. So, this vector will be as many dimensions special dimensions as you have and this minus sign is the signature, which we will follow

Student: (( ))

D, d plus 1 okay thank you. So, now we have to plug this into the equation. So, we will plug it into equation star, which I have erased, but hopefully you have it in your books and the formulae are very easy in the following sense that d t square. So, we just have to use d t acting on this will pull down should be replaced by what? Minus i omega and d at the del operator, the nebula should get replaced by plus. Remember it was acting on the first guy x, x prime is was just therefore to the right. So, we can see that the box operator which was d t square minus del square plus becomes minus omega square.

And so this is not bad, but we do not want to plug it into star actually we want to plug into this equation right so it is not star, but into this particular equation into this equation so luckily I have it on the board. So, now we can see that we can just go ahead and let me define this particular combination will keep coming again and again and we let us define this to be minus omega k square. So, in other words omega k square is defined to be or we also need to use one more thing we need a representation of the delta function, but that is not so hard it is just it is the transform of one and that is 2 phi power d plus 1.

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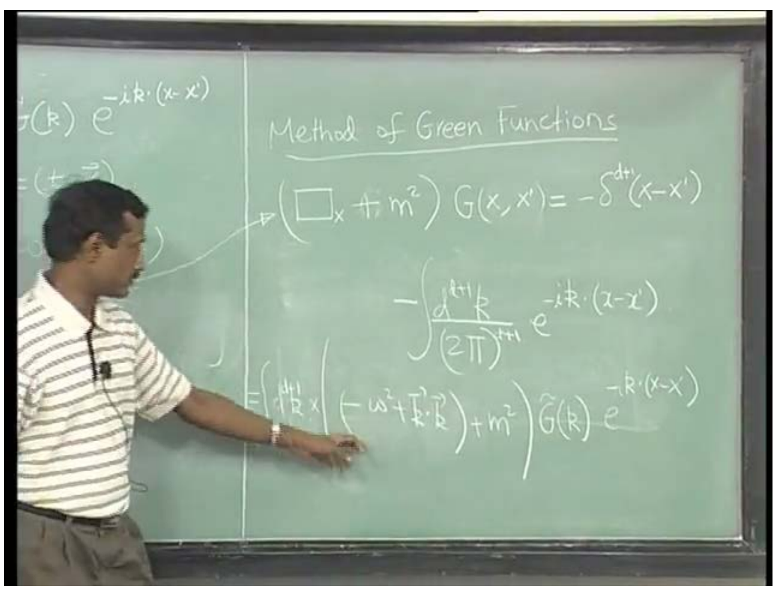

So, delta function in case you have forgotten there it has a 2 phi times e power i minus i k dot x minus x prime. So, now we have written everything in terms of this guy. So, what it tells you is that this side will just be minus 1 in the algebraic equation while, this will be this particular various operator. So, let me just plan put that in so what we will get is an integral, so let us just stick to this.

So, that comes from this plus m square into G tilde of k into e power. So, that is a left hand side of this equation and the right hand side is just this. So, I will just what I will do now is to erase this because, I am running out of space and say minus of this is equal to this. So, this equation all I have done is put G of x I have substituted this here and then, I have written what the operator gives. So, now the key point here is that this holds for every k, so you get an algebraic equation which is just the coefficient of e power minus i k dot x into x prime you write equation for algebraic for G tilde of k which I will write it over here.

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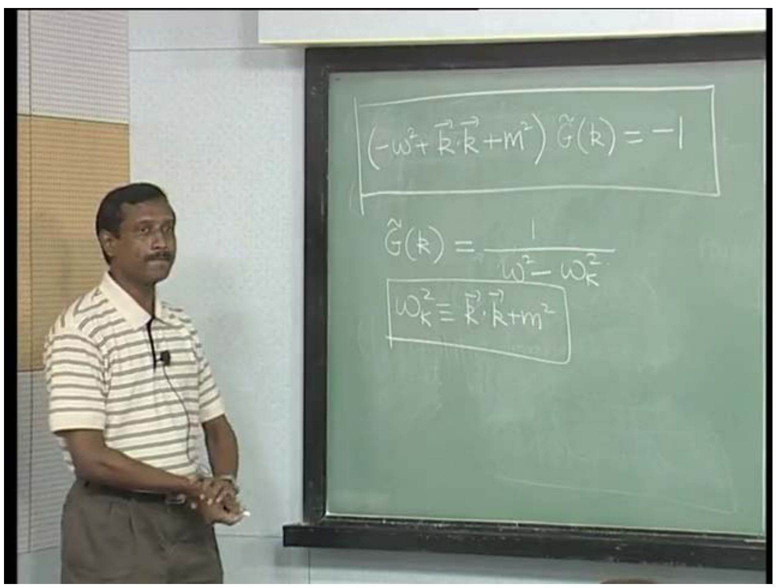

I want this combination to be m square omega k square. So, I will just do this first. So, all we have done really is by going through your transform you transform this equation, which is differential equation to this equation which is an algebraic equation and in k for G tilde of k. And if you are able to solve for G tilde of k, then we are we have we can do the inverse for your transform and get back G of x prime.

So, this will just tell you that G tilde of k is 1 by this is what I wanted to define that is omega k square. Sometimes one it has jumping steps in once mind, so this is what you get. So, you get something very simple is this clear? So, just remember that omega k is only a function of the 3 vectors and this m is a constant. So, we are almost done, but we need to be so if you are done, so since we have already solved now for G tilde of k. Remember every solved it as a coordinate because, we are really not putting any boundary conditions etcetera and we will see how to do it.

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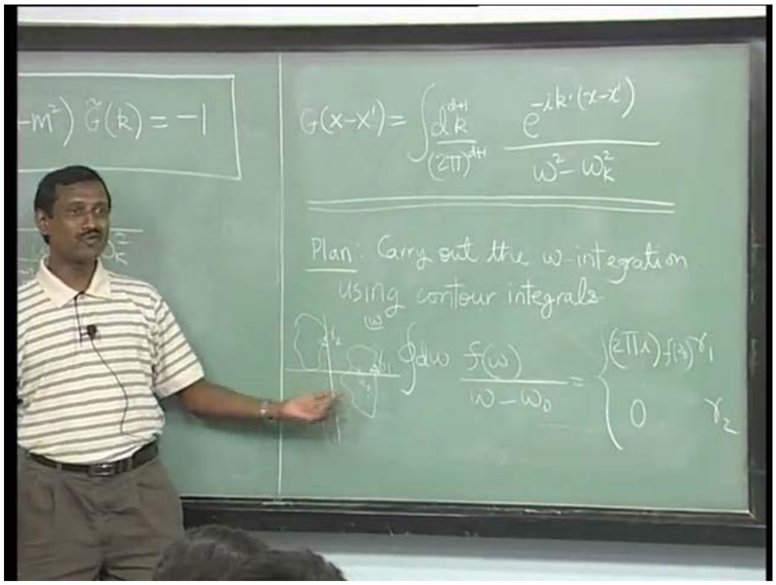

So, now as we got this guy, we can plug things back into the other equation and so now we have a what we call formal solution. There is some powers of 2 phi, right? So, let me do this let me put the 2 phi d plus 1 here also. So, that I do not they get cancelled away otherwise you have to keep carrying this annoying powers of 2 phi. It is just because my definition of G tilde, so this has a simple form. So, now we need to go ahead and do integration, but what we will do here is not to do the complete integration at least I will not do it, but I may give it as exercise for you. So, I will do the omega integration and we will see that I will incorporate for instance issues as causality right away.

So, first thing to do is so the plan is just carry out the omega integration and I will use the complex method, so using contour integrals. For this I just need only one formula, which says that if you have a so let us just write d omega some f of some omega, omeganon. And let us say we have some close contour, so omega think of some omega complex number. So, I have this omega is complex. Here of course omega is real, but what we will do is we will think of omega as a complex number and that helps you suppose omega non is something out here. So, the point is that we will pick some contours there can be a contour like this and choose the orientation to be counter clock wise. So change the orientation by minus of those things. So, let us say gamma like this, let us call it gamma 1.

Let us say we have another contour which does not enclose this. So, there were two possibilities, so let us look at gamma 1 and gamma 2. Standard results from complex analysis tell you that if the contour encloses the pole this pole, then, the answer is 2 pie i and otherwise it is 0. For gamma 1 and 0 for gamma 2, so that is a 2 pie i.

Student: (( ))

Did I forget something?

Student: 2 pie I f of omega naught.

Call the residue of the pole and if you have gamma the same contour going the other way. You would write you could think of it as minus gamma 1 and the answer will be minus of it. You could have contours, which wind n times with positive this thing then you will get n times that answer. But, it follows from basic property of integral is that if you can always break up the integral into separate sums. So, if you have something which winds around this n times that is equal to n times one going around one. And n equal to minus 1 is a special case.

So, we want to do that so we will just carry out the integration, but we have several issues, first thing is that so the omega integration runs from minus infinity to plus infinity on the real omega axis, and this has two poles. One equal to omega plus omega k and minus omega k. And so we can use the method of partial fractions to split this.

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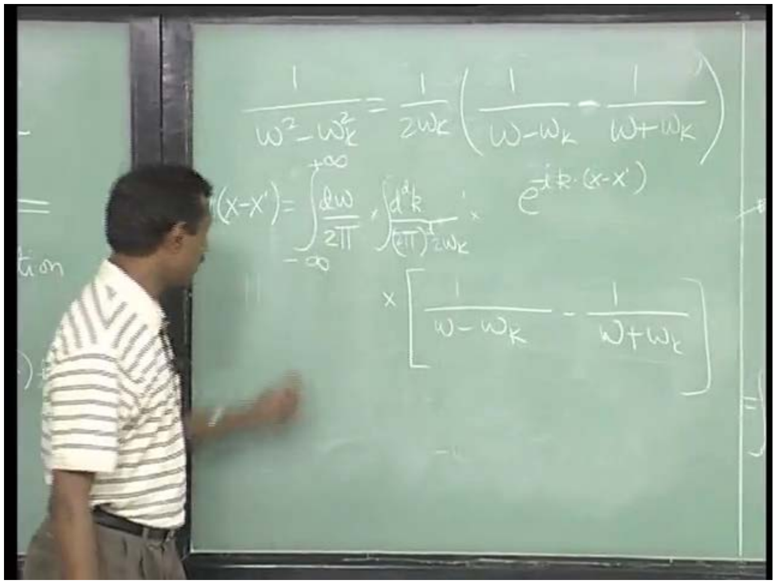

So, this is just fancy term for this the simple elementary formula. So, now you can see that if I focus on the omega integration of 2 pie i into of course, so now you can see that an omega integration goes from minus infinity to plus infinity. So, now you can see that at least I have wrote into some form like this, f of omega, but f of omega is just this e power I the omega t into t minus prime. That is what is the only the omega dependent part or you call this f of omega. And you get it as sum of two terms.

So, we have to do two separate contour integrals, but we do not have a contour. What we have is a integration which is a line. A contour enclosure, so what we will do is to look at this structure we have out here.

Student: (( ))

And there is a minus here even in the partial fraction even in the partial fraction there is a minus sign,yes this minus sign. Yes all the signs are very important. And I keep dropping signs obviously. Now, we just need to do this and the trick is very nice. It is more than a trick, it is an important method holding this or doing this integral.

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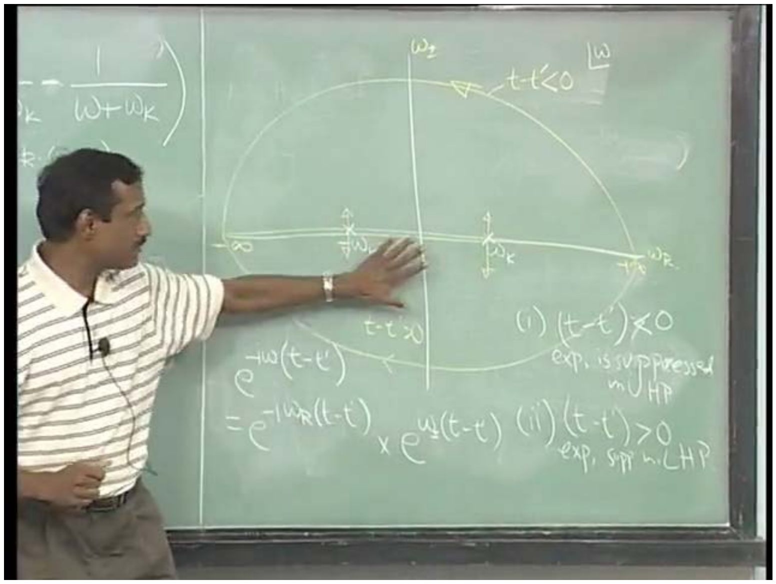

Since, this is important I will draw this thing. So, what we have is let me use a different colour here, so we have a contour, which runs like this. Which is minus infinity and plus infinity, this is omega real and omega imaginary. But, first thing is we already have a problem, we have a problem that our contour runs through these poles.

So, this is at omega equal to plus omega k and it has two poles another at minus omega k. But let us do things one at a time. Let, us first convert it to close contour. So, let us look at the omega dependent part of this, so it is e power minus omega into I omega into t minus t prime. And we have taken omega to be complex. So, what we will do here is try to add to it a semi circle by a semi circle, I mean first think of this as a contour which has a finite radius and then, take r to infinity.

So, I think of a infinity infinite radius semi circle, which closes either up or below, but one nice thing you can observe about here is that as, so if you write this out in terms of real and imaginary you will see that this is minus i omega real into t minus t prime times e power. So, the imaginary part will give you omega into t minus t prime, omega imaginary. So, if you look out here in the upper half plain omega imaginary is positive, while in the lower half plain it is negative.

So, now you can see that it if t minus t prime were negative then this is a exponential suppress term in the upper half plain while. So, let us write this out, if t minus t prime is great less than 0, this exponential so exponential part is suppressed in the upper half plain which I will write as u h. And the second bit is if t minus t prime is greater than 0, there is exponential suppression in the lower half plain.

Now, you can see that on this contour as the radius is becoming larger and larger, these terms do not contribute they get suppressed they actually go to as 0. So, now we know the rule it says that if t minus t prime is less than 0 you close the contour in the upper half plain. You are adding 0 if you wish, and if t minus t prime is greater than 0, you close the loop in the upper half, lower half plane.

So, now we have solved the first problem which is to convert our open contour into a close contour. So, let me draw this and explain it into pictures. I will draw it as a finite thing, but you have to think of this as an infinite measure dimensional radius. So, this goes like this. This we can do only when and again orientation remembers this is counter clockwise. So, this is this positive orientation, but we otherwise we will close it downstairs, this we can do when t minus t prime is greater than 0. So, now we have close contours, but if you close inthe lower half plain the orientation is opposite. It is clockwise. This is a clock wise contour, so if you have any pole out here.So, the question is the how do we go about doing the integration? Because, we still have to handle one more issue, which is a fact that there are two poles on the contour.

So, the trick here is to sort of jingle these things up or down. So, I could do this, I could work, I could take this guy and move this contour. So, let me use a completely different color chalk. I have four possibilities in this thing. I could move this contour slightly up. I could move it slightly down. I could do the same over here. There are four possibilities, but the question is which one of these four is relevant to us, because now you can see something nice. So, if you look here it is t minus t prime less than 0 that means t less than t prime.

So, if you want the retardation effect to happen what is it we would like we would have like t minus t prime to be greater than 0 for the G to not vanish. So, now you can see that you can determine when these things will have to contribute. So, what you have to do is to decide whether this false contribute or not you go back and look at your physical input, which tells you that G advance has to vanish somewhere, or in some other case it has to be non-vanishing. So, that will determine, how you move this so what we will find is either moving both of them up or both of them down. Simultaneously, we will give you either advance or the retarded one, but it leaves you two other things that you look I am not interested in the stuff, but life is not so simple.

It turns out in quantum field theory one of the other choices come you know when one goes up and other goes down. And you get I mean, you might think of I have got something a causes etcetera, but there is it is very interesting to see how that works. So, we will discuss that also even though course in classical field theory like dimension, but there will be certain issue which we can discuss so what you will find is, what is called refinement propagator responds to choosing the other. So even that does show up.

So, in some sense these you may think what is all these ambiguities out here. I am able to move it at will and get answer but that is not the point. The point is that this corresponds to choosing a boundary conditions. Remember in the beginning of this lecture. I told you that you get a particular homogeneous things the same things holds out here, what you are doing is these you can prove if you wish, it is not so hard if you give two different and take. For instance one is advanced and the retarded and you take their difference that will satisfy the homogeneous equation.

So, this so these they all differ in the choice of the homogeneous equation and that is determined by whatever boundary conditions that you are putting in. So, we still have to complete this problem, we will do it in next lecture we will carry out this integration we will see certain properties. And now the thing is you may say that you know, it might be true in one frame I have done this in some other frame this may not hold. And this is where I will appeal to the first point I may say that green function in Lorentz vector. So, if it vanishes outside the light cone at 1 point you can actually play around with it and you can ensure that it vanishes everywhere.

So, basically what we will show the differences this is if it vanishes in one curve, one line in the outside the light cone it has to vanish that curve can be made into any other point in the light cone by changing things. So, that is in some sense it is enough for us to do this. I will discuss this little bit in detail in next lecture. So, next lecture we will complete this story, we will also see what this refinement propagator is and we will. So, then we can carry out this integral, but we still have to do these the special integrals, and some of it can be done some of it cannot be done.

Well I cannot is wrong you could write it in terms of better functions and keeps like that, but it is little bit involved but you can carry out the integration. I am done.