Classical Field Theory
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Lecture – 10

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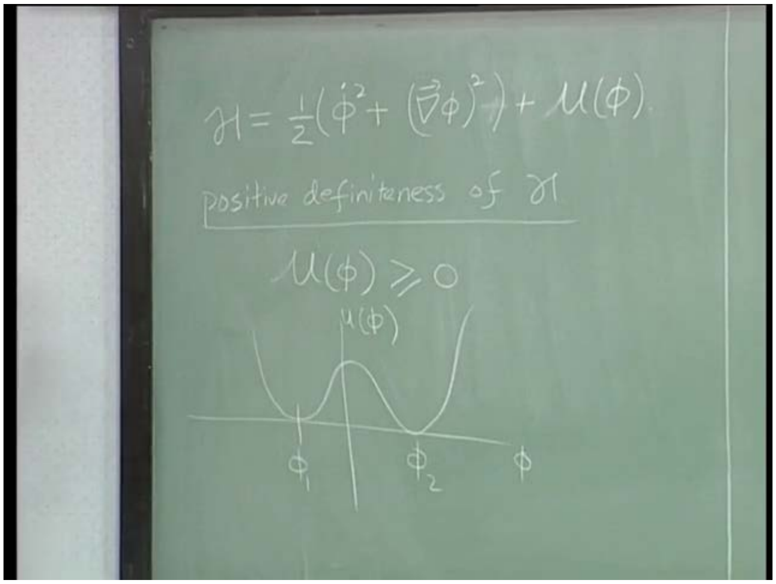

So, yesterday what we did was last lecture. That is we worked out the Hamiltonian density rather for and for positive definiteness of H or it is easy to see that, the first two terms are positive definiteness. So, it is really is a condition on the potential so, we need to write u of phi should be, greater than 0 or equal to 0. I think yesterday in the last lecture I put a modulus here, by mistake that is not correct and so we saw some examples of this.

So, suppose you are given a potential which has which is bounded from below with some other number you can always add a constant and shift it. So that this satisfy this condition at least in the in quantum field theory also either is nothing which fixes the 0 of the energy. You need to couple to gravity to see where there is a 0 point energy etcetera. So, we so this is the condition we will assume, any potential which is bounded from below can be satisfy this and so. Let us, consider a sort of a nice example of something like this, suppose this is phi u of phi. So, we have this I, am just choosing a very specific example, where there are 2 values for which it achieves some minimum which is 0.

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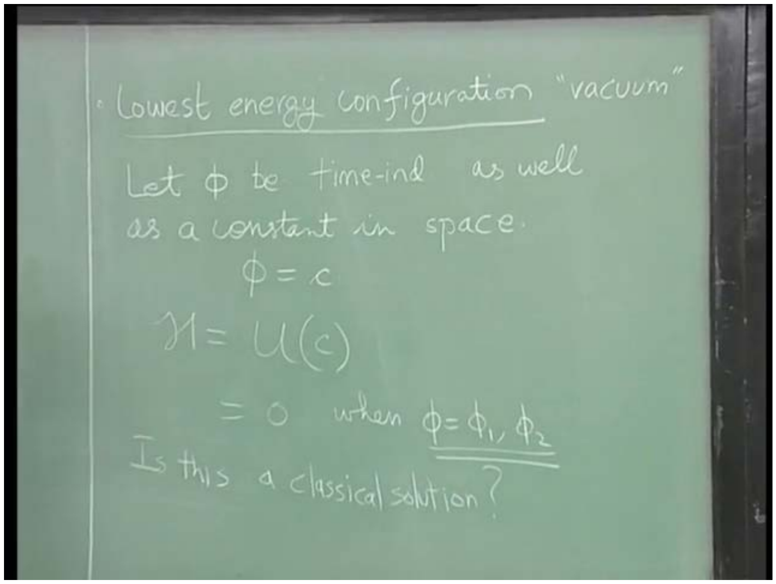

So, let us look at the lowest energy configuration for this so sometimes this is called the vacuum configuration. This terminology borrowed from quantum field theory but will just call it the vacuum. So, we can get these two terms to be 0 by choosing… So, let phi be prime independent as well as the constant in space. So basically, what I am doing here is that these two things go to 0. So, the Hamiltonian for such a configuration. So let us, call let us try that as phi equal to some constant, and so for such things the Hamiltonian density would be u of the value of constant.

So, clearly the lowest energy configurations happened. When u of phi takes it lowest value. So this is equal to 0 when? In this example more generally for all the… and phi 1 is equal to phi 1 everywhere or phi is equal to phi 2 everywhere. So, this is the lowest energy configuration, but now the question, is this a classical solution? By this I mean is it a solution of the equations of motion? And so, we have worked out the general equation of motion, but we have not worked out the equation of motion for the Lagrangian density. So, let us do that to complete the story.

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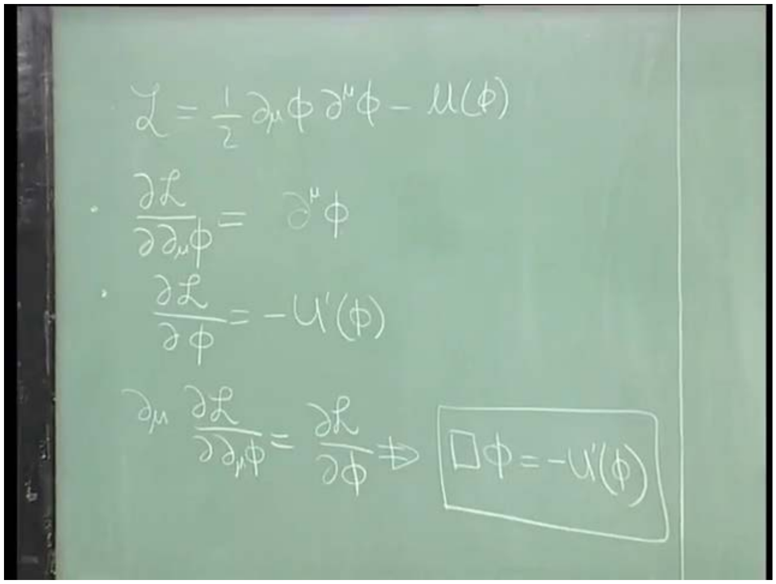

So, we need to evaluate this object. So, this will just be d mu of phi and of the field phi the derivative of this there only it comes only through the potential, and so this is equal to minus u prime of phi. So, the Euler Lagrangian equation which we derived earlier tells you that, equal to d L by d phi.

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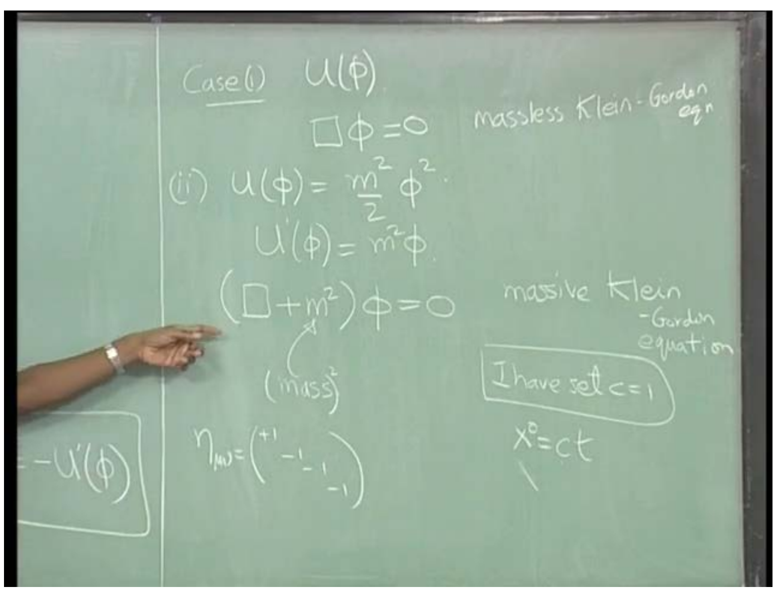

So we get d mu acting on d L of phi which is this. So, this implies so we can take a few cases before we come back to that thing no potential. So, this term is 0. So, this you get box phi or the dilambition phi equal to 0 and this equation is the massless Klein Gorden equation. The second case is to take u of phi equal to m square by 2 phi square. This satisfies this thing which I wrote is just. If you plot u of phi versus phi square versus phi you just have a parabola. So, if you rewrite this again we can work out what u prime of phi is it is just m square phi, and so what we get here? So now, this equation gets an a contribution out here, there is minus sign out here so I can bring it to the right hand side and I get box plus m square phi equal to 0.

So, this is the massive there were m is the mass at the mass square. I should I actually in all these things I have set will get back to this, I have set c equal to 1. We will discuss this little later. This is part of something called natural units. These are units which are natural to the problem like, we saw like we went from si units to Gaussian unit when we consider Maxwell's equations. So, that was natural to special theory of relativity and setting c equal to 1 sort of gets rid of this. So remember we said x 0 equal c t and xi where the. So, if you put c in coordinates where c is equal to 1.

Then there is no distinction between t c and t x 0 and t rather. So this is so you may have seen if you look at books there might be power of c going with this thing. That is one thing there might be you might find there some other books also have a minus sign out here. This again has to do with the sign convention that people follow. So, but at the end of the day this equation is the same equation depending on your notation.

So, whenever you take any book on classical field theory or even quantum mechanics discussing relative stick quantum mechanics. You need to check the signature of the matrix. So in other words, you n term mu in this course will be always plus 1 minus, but there are books which follow the reverse convention. So we follow this convention so these two equations are nice they are linear. So, you can see that if, I have a potential which is quadratic then I am guaranteed to have a linear equation, but we could try more general things.

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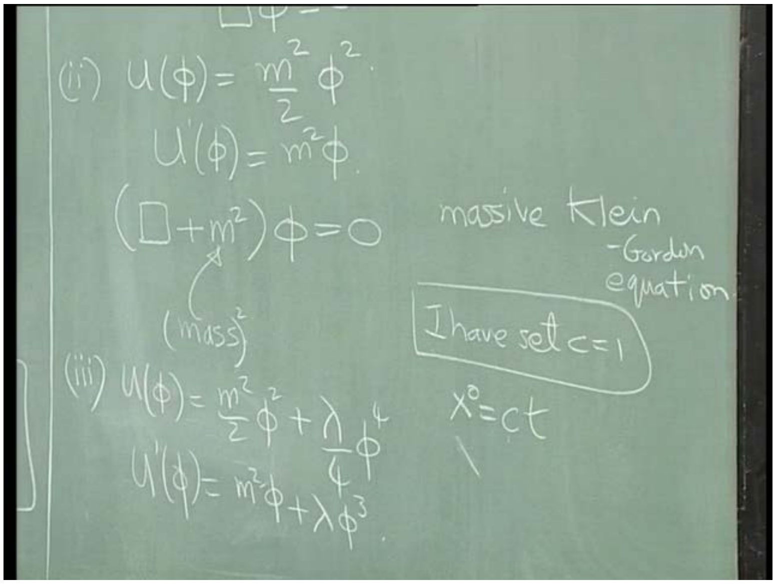

So suppose, I could add let us say we took u of phi to be m square upon 2 phi square plus some lambda by 4 phi power 4. Some quadratic interactions then you see that u prime of phi becomes m square phi, but now, you get a lambda phi cubed. So, the equation is whatever, is this equation is no longer linear in other word if you give me two solution super position will not work. So, now coming back to this equation, there is the most general equation and we asked the question.

If we choose phi to be time independent and a constant in space is this a classical solution. So let us, come back here and ask the following question you see, a box just consist of space and time derivatives so if phi is the constant this term vanishes. So, it will be a solution of the equation of motion where when u prime is equal to any of the external points out there, so what we see is the answer is.

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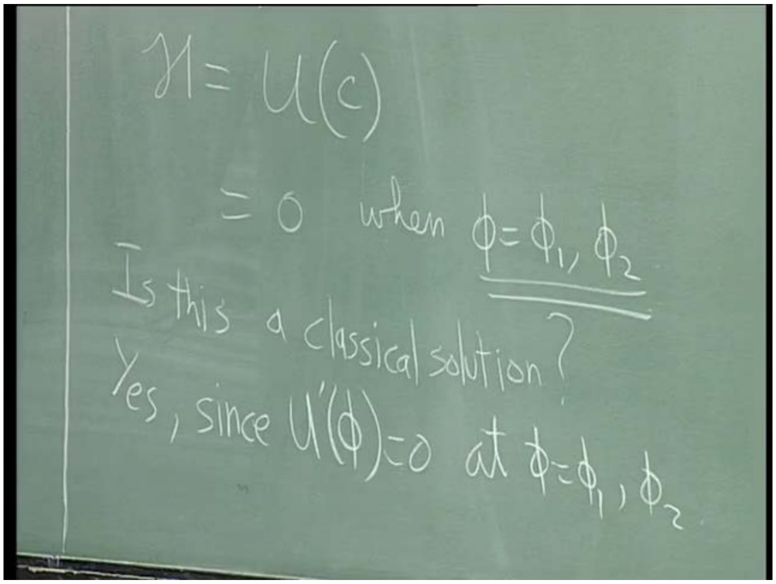

Yes, since u prime of phi is equal to 0 at phi equal to phi 1 or phi 2.

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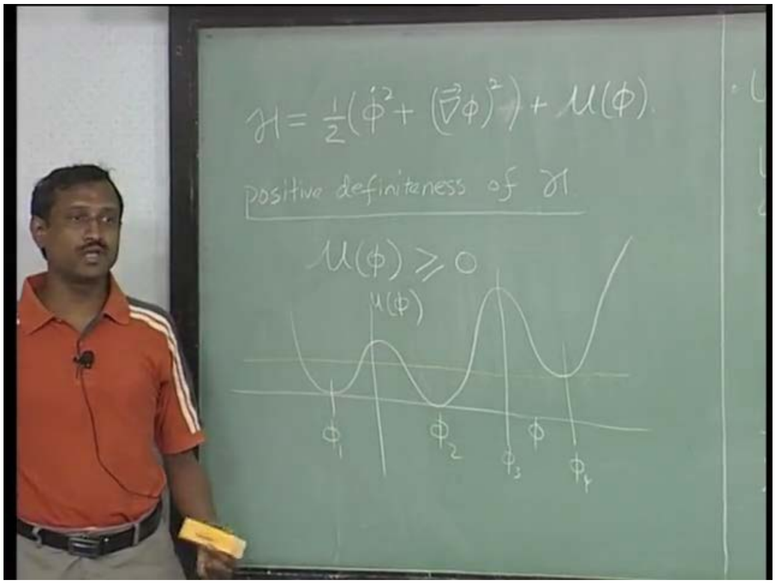

Suppose, we let us say that, let me be powers here. And choose some more call this phi 3 call this phi 4. So, clearly I can have classical solutions for instance like this, 1 phi 3 where u prime of phi equal to 0. So, there is definitely a solution which says phi equal to phi 3 constant which is phi 3 because it satisfy the equation of motion, but remember the action principle just tells you that an extreme. So, then you come back here and look at this and what is this tell you that the energy of such a configuration is infinite, which one this one is infinite because it is a constant but the value of this function is some finite number. So, the energy density is constant but that implies that the total energy which is the integral over space time blows up as the volume of space and similar thing rules out even this solution. So, it is important to note that these are indeed solutions of equation of motion is that clear?

Student: Supposing this should be a bit lower so in other state is zero and two states are negative values. So the management say some state quantum negative.

So, that is the point I said, earlier. Suppose, if there is no scale, nothing which sets the scale of energy. So, I can shift things up and down. So I could do something like, what you are suggesting is to take. Let us, say we choose this assure energy scale, but now then the lowest energy configuration is never the less still remains phi 1 and phi 2. But, that will be some negative infinity so, but you know that you want to measure things with respect to the ground state or the lowest energy so you shift things and go away so that you can always do.

Because nothing changes see adding a constant to this does not change the equations of motion at all but it does change. What you mean by 0 of energy for sure? And so, you I am using of that freedom to take care of this ambiguity. So, it is true that this exists? So I understand what you are saying. I mean the point here, is that you can use you can some idea of stability or whatever in this neighborhood. And you can may if you jiggle things that is the nearby solution which is probably lower energy or whatever. That is what one has in mind but, in that sense this would be this like a local minimum.

But it is worst because what this is the thing is that this is a Hamiltonian density not Hamiltonian or energy density. So, but the fact that you have more than two classical. So, here is the case where we have degeneracy, in the sense there are two different ground state solutions. And so degenerate by here degenerate I mean equal energy solutions two degenerate energy vacua, which is phi equal phi 1 and phi equal to phi 3 phi 2.

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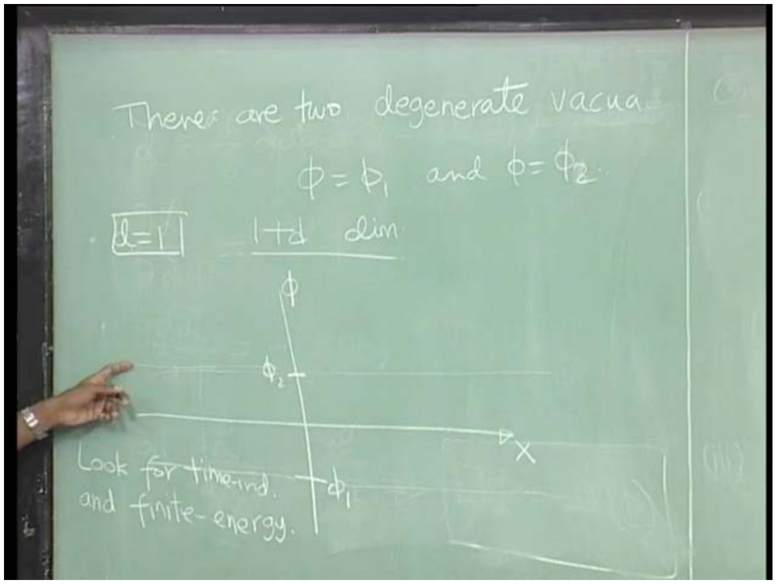

So now, let us say let us stick to special dimension being one. So, we are in 1 plus 1 dimension. So another notation whenever, I will write 1 plus 1 d dimension for pointing out that, there is one time like in d space. So usually, this is what people do this also is to remind you that there is a signature in the matrix. That some this is plus and this is all minus. So, it Remer agent of that, but if, you talking about Euclidian space, I would just say d plus 1 dimension when there is no signature. So will choose d to be 1 so, we are in 1 plus 1 dimension so space is just 1 coordinate.

And so now, I can ask again I will stick. I will still choose I look for solution which at time independent just to simplify things. So we will look for, and I will put in another condition and you will see that it is a strong condition and finite energy. So, finite remember finite energy is not a same as saying that you Hamiltonian is finite. We already saw examples Hamiltonian density is finite it does not mean anything.

So, what we want is actually I should not be calling this the Hamiltonian density. I should just call it energy density because, Hamiltonian is a function of the Conjugate moment well this I have written in terms of phi dot. It is only semantic so, I could so in this case phi is just phi dot I could write phi square. But, the arguments are important so, the correct statement is this is just energy density.

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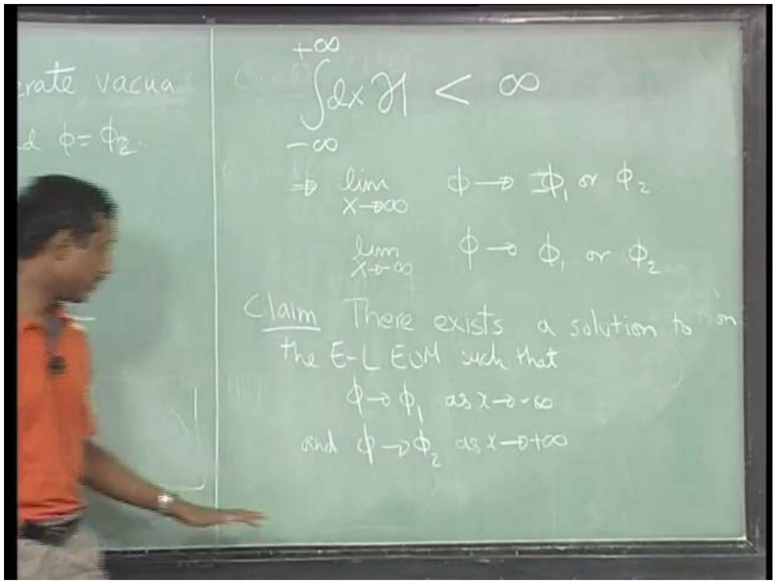

So, what we want is integral of the x from minus infinity to plus infinity of H must be. So, this is the finite energy condition. So, yesterday, we discussed fall off behavior and that tells you that the key point was that it should go to the so, let us mark we want to plot phi versus these thing. So, let us say this is phi 1 here, what phi 1 was negative and phi 2 is positive. So, let us respect that let us say this is phi 2. So, coming back to this, this term is 0. But, there is a gradient term and there is also au of phi.

So, lest first focus on u of phi. If u of phi has to gives have to give a finite contribution to the energy then this should go to 0. So but now, we can see let us look at x equal to plus infinity. We see that there are two possibilities u of the phi should go either go the asymptotically to phi 1 or phi 2. There are two possibilities similarly, out here there are again two possibilities at minus infinity which is again phi 1 and phi.

So let us, consider the case where both sides plus and minus infinity is phi 1 then the lowest energy configuration is nothing but the one which you already looked at that is the vacurect. Again if you take phi 2 at both ends that is again the same solution. Where now, there is a possibility of a solution which interpolates a solution which start from say phi 1 on that minus infinity and goes this ways as to have some shape of course. And we do not know the shape. But, so clearly this condition implies requires limit x tends to infinity phi should go to phi 1 or phi 2. And limit x tends to minus infinity phi should go I am just writing out.

What I just already said invert. So we will, have something like this but now, clearly gradient will not be 0 unlike the vacuum case. We need to because, it has to change somewhere. But, the rate of change has to be such that so, again this should have certain fall off behavior, but them the key point is even though we may not be able to solve the equations. Obviously there is some horrible nonlinear equation and in principle. We cannot solve for the Euler Lagrangian equations the box which I just erased out here. We cannot solve for the Euler Lagrangian equation but, I claim that there has to be a solution with this asymptotics.

So, these are claim there exist a solution to the equation of motion such that, phi tends to phi 1 as x goes to minus infinity and so the and interpolating solution and the reason is as follows. So, there is one way of understanding this. So this is like the analog of writing. The trajectory of your problem this is not electric space, but does not matter we are extremising the action. So I can keep wearing this thing. And somewhere, I have to hit the I have to when I am talking about, this number no that was knowledge different league 50 degree centigrade outside.

So, the claim is that there exists a solution to the Euler Lagrangian equation of motion. Such that phi tends to phi 1 as x tends to minus infinity. And phi tends to phi 2 as x tends to plus infinity. One thing you can say is that, by just flipping the access. If I can find the solution with this as asymptotic. I can definitely find the solution with asymptotic phi goes to phi 2 x, goes to minus infinity and the other one, but there is very nice way to see this first point is to since. You are fixed the boundary condition at these two and you really cannot temper with these guys.

But, what you could do is to keep changing that this thing jiggling it is such that you know the action is being extremis. So, there should be something but there is a much more important reason for this. And this has to do with the fact this solution caries some charge conserved charge which is different for this as compared to the vacuum solutions, which we looked at so let us, see how that goes. So since, we are in 1 plus 1 dimensions the (( )) dot answer is epsilon mu, nu.

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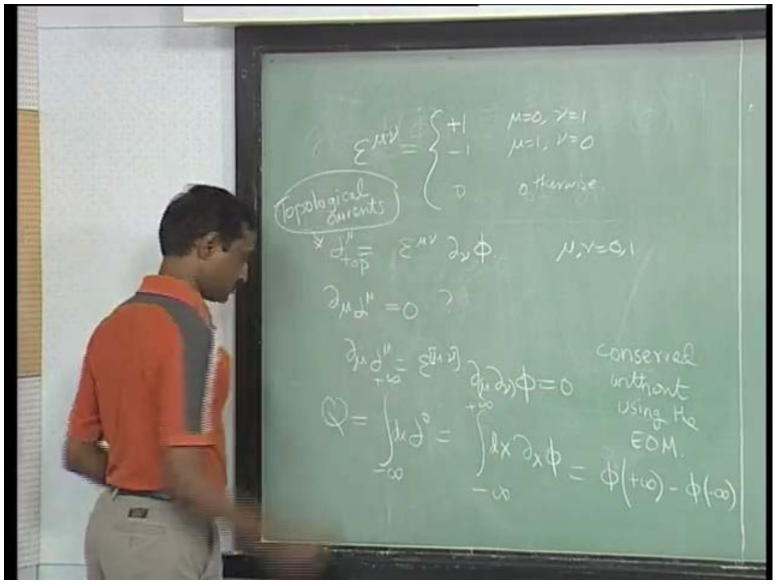

I just define it to be this is equal to plus 1 for mu equal to 0 nu equal to 1 minus 1 for mu equal to 1 nu equal to 0. For the other 1 so, I will define at two current which is so, last lecture we saw that if I give you a current which is conserved. Then from that we get some charge by integrating. So, first I have to show that this is conserved. So let us, look at d mu j mu so, in this mu and nu only run over 0 and 1 because we are in 1 plus 1 dimension d mu j mu equal to 0. So let us, see how that goes.

Now this is anti-symmetric well this is symmetric. So this has to vanish so the amazing thing here, is that I did not use any equation of motion or anything. So, this is conserved, the another current which I mention last lecture. Will be conserved modulo equations of motions that is something we will see. We have not derived it, but this is easy to see because, this is anti symmetric. And this thing is anti-symmetric and while this is symmetric. So, if you contract something symmetric with anti-symmetric with something symmetric it has to vanish.

So now, the need stuff is let us ask what is the charge? So charge will tell you, it should be q of j 0 integrated over space, but what is j0? It is epsilon 0 then this can be only 1 d1 which is dx of phi, but what is this? Is just the value of phi at plus infinity and minus infinity, but those are your boundary conditions. So, you can see that the vacuum solutions correspond to having charge 0, while the this particular solution has q equal to phi 2 minus phi 1, while the other like, I said remember I told you the flip solution will have charge phi 1 minus phi 2.

So, the key point is that there is so, is that since this is a conserved charge, but it translates here into the statement that you will the asymptotic are never changed. So, whatever you do jiggling out here. You should find a solution so there is all I am saying it is exactly not a great statement that there, exist a solution in this class in this with q equal to phi 2 minus phi 1 with finite energy.

So even though we do not know, the explicit answer because it might be a horrible differential equation a difficult differential equation to solve. We know that such, a solution can exist, but there are special potential for which there exist a solution. I mean you can write out in analytic expression and that is something, which will come for you in one of your assignments. Probably the next one but, you will exactly go back and check. So, these currents like this, which are conserved without using equation of motion are called topological currents. So, they are not associated with any symmetry or anything else.

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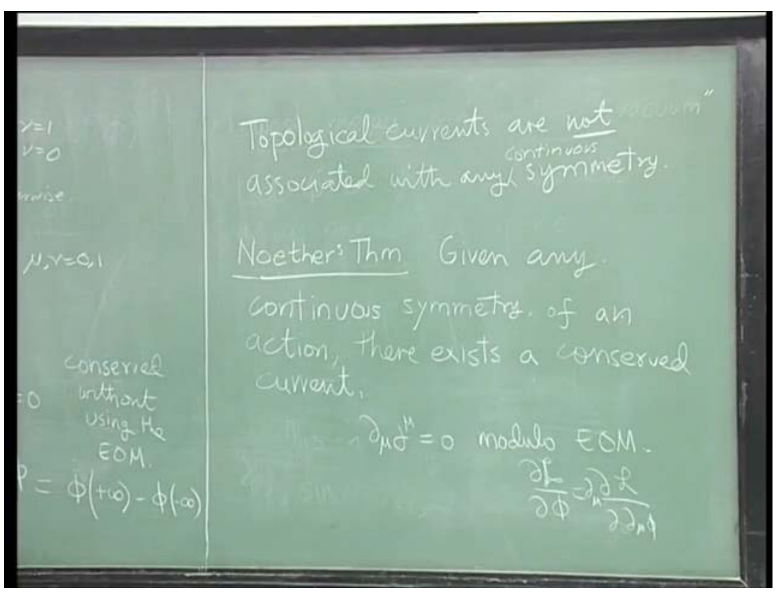

So, actually the precise statement is that there is symmetry should be continuous otherwise, you do not expect symmetry. So, where you have some parameter that you can vary, so I will, just make the statement of Noether's theorem. And we will discuss this, given any continuous symmetry of a Lagrangian of an action there exists a conserved current. Modulo equations of motion that is why.

So, in other words so you if you write out the current and you want to check that it is equal to 0. You should get the equation of motion you said that, is equal to 0. So, what I mean by that is some equation of motion or something like this d L by d phi modular this. So this is something which we will prove in increasing generality in this course, but right now, we will just use this to prove that there, exist a current for a for translation in space and time. Which will be call energy and momenta would be the conserved charges. So, let us see how so before that let us understand what we mean by this.

So first let us, understand what we mean by a continuous symmetry. So we already, seen in groups that there are discrete groups which are finite number of elements. There are also continuous groups like, the rotation groups. So what so the key point is if you have a group like, say zn there is no notion of smallness. You just have a bunch of elements or a neighborhood of an element.

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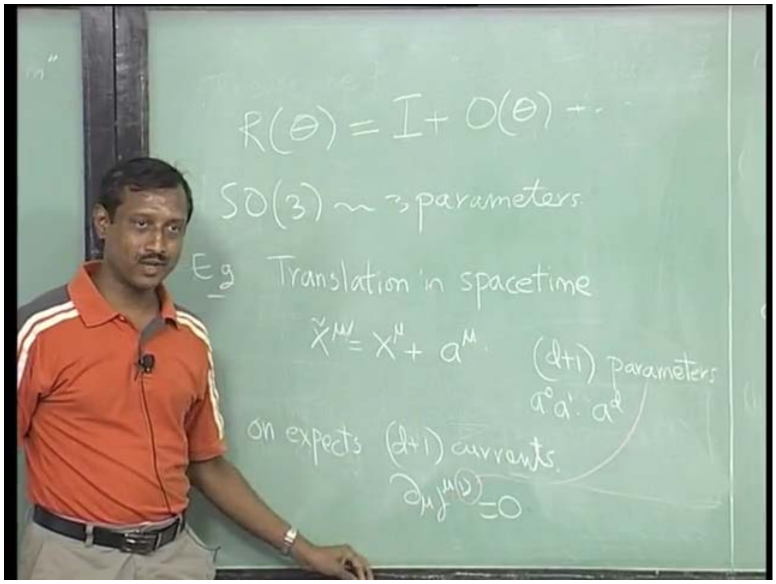

But, if you have a rotation by some angle theta there is indeed a notion of small theta. So you would write R of theta is equal to identify. So when you in other words if, I set this parameter to 0 it is close to identity and so I get plus some order theta term etcetera infinite series. So, there is this notion of smallness of some parameter etcetera. And you will have a bunch of parameters so if you have SO 3 you will have 3 parameters.

So, I can actually if, I start from the identity matrix I can go in 3 different directions roughly speaking, I can turn on any one of these parameters. The Euler angles I get 3 parameters. What this tells you is that for every such parameter there is a conserved current. So, if you have a rotation of SO 3 kind of invariance, it tells you that you will have 3 conserved currents and conserved charges. If you have n parameters if in your which parameterizes your continuous symmetries then you will have n independent currents.

So, the example which you will look at now is that of translations in space and time or let us put it tilde here. So, what do we mean by this? That so a mu now we are not no longer in one dimension that was just for our example. We are coming back to the d dimensions. So we see that there, are d plus 1 parameters. That is a 0 up to a d. So, what you what we would expect, so 1 expects d plus 1 currents.

So, now the current should carry an extra level. Which is the parameter level, so what we will do is, so we will write something like this, d mu j mu nu equal to 0. So here this nu is actually refers to these where d plus 1 parameters it is a little bit confusing in the following sense, that this index both of them run over the same thing. So, what you get is conventionally called t mu nu not j mu nu.

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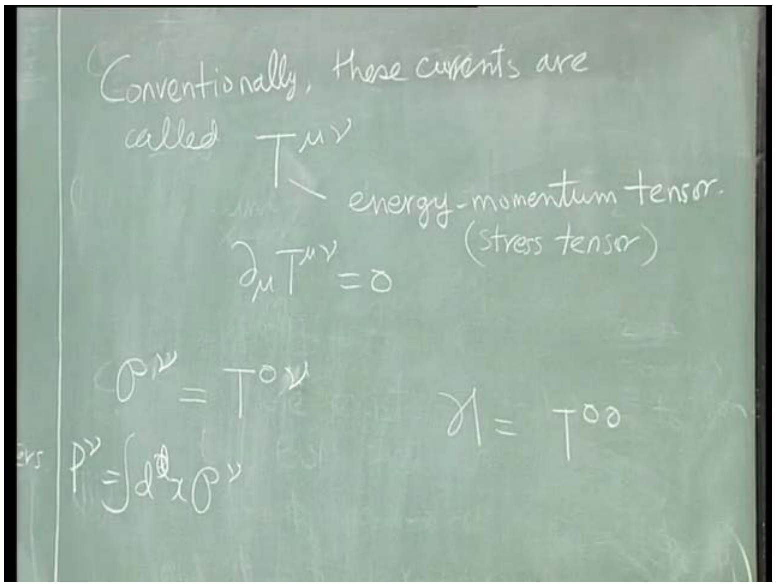

Conventionally these currents are called t mu nu and it is called the energy momentum and as things stand mu and nu are actually distinct in this things have no relation. So, all the size is that d mu nu of t mu nu equal to 0 this is the statement. If you just take that Noether's theorem and it tells you, this thing it does not tell you anything about, the new index it is turn out. That will find in at least the first example which we look at, but not generally true that this will be symmetry under these two things, but as it stands there is no property.

So, I have to actually is no notation I have to tell you, which is my current index and which is my parameter index. So I have just, chosen a convention which says that the first index is the current index and the other one is this thing. So now, we can ask so what are the charges so you will find the charges. I will just call them p mu I can just define it to be the 0 part of the current. This thing this what you would call it, let me use the same index which I used there and if, you integrate over d dx of p nu. You will get the conserved charges which I will just call p nu.

This lot of notation but it is important because it is confusing at first that the same index structure arises from these two. So now, so we need to now at least work out Noether's theorem for this particular symmetry. Which is what we will do now next. So obviously, you know that from classical mechanics in normal systems. You know that if, you have translation variance momentum is conserved linear momentum and this thing.

So but in a complicated system this the conserved charges, which are associated with translation symmetry are called energy and momentum of the system so this is the definition. So, what I wrote as h is nothing but t 0 0 because it is associated it is the thing associated with the parameter for time translation, special translation would just be the other guys. So, this is that is why this is called the energy momentum tensor sometimes called the stress tensor also this is from the continuum mechanics these thing. So let us now, try to prove this so the question is, when does one have translation invariance?

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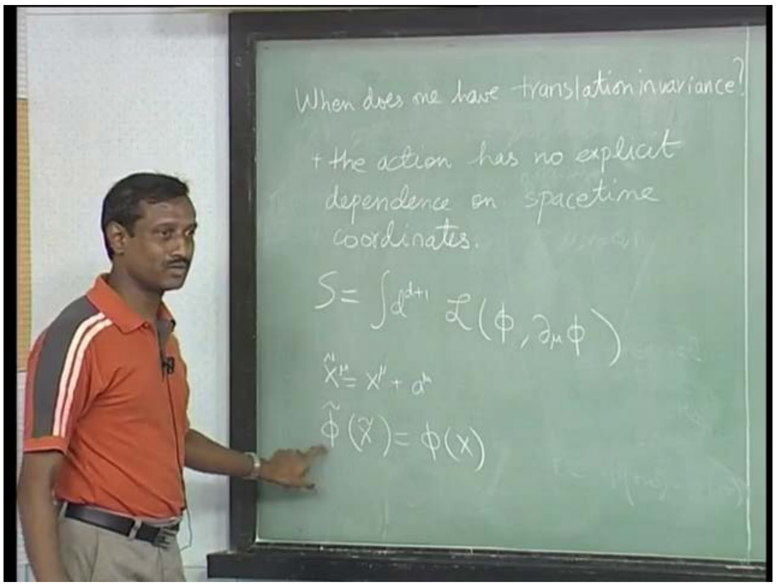

So that we have already seen the argument that there, should be no explicit dependents. So the action on space time coordinates, right? Then we expect so in other words for so the once which we have been looking at so far. The action we never had any a dependents on coordinates. We only had dependents of fields and their derivatives. So, now let us, go back to these symmetries and ask, what happens?

So let us, say that x still the mu is x mu plus a mu. So what does that imply for the field phi? So if, you make such a coordinate transformation it is a scalar field. So, you just define phi tildes of x tilde should be nothing but the field phi of x. So, let us just take a simple example to understand. What we mean by this? So this tilde is just the coordinate change but this tilde here, indicates that the functional function has changed.

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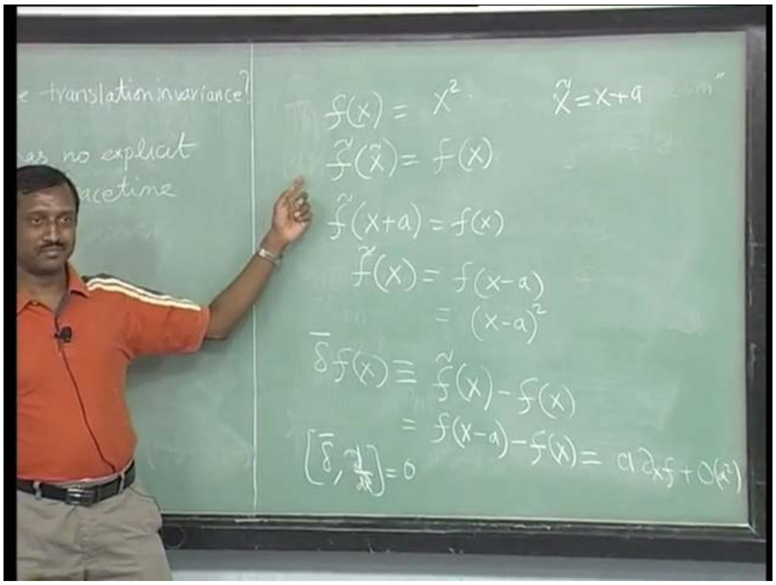

So suppose, you are just a simple function f of x in 1 dimension and you change it argument let us say, it is x square and you change its argument then you get a new function so saying that this is equal to f of x what does that mean for this function f tilde so where x equal x tilde. So, you go back and plug this in here says it is f tilde is equal to x plus f tilde of x plus a equal to f of x or equivalently f tilde of x is equal 2. So now, we can go back here.

So mathematical speaking x square and the function x minus a whole square are not the same function. So that is what we are indicating here, and this is just that simple statement that is what this. Let us, you derive what this is and so now, I had defined yesterday something called delta bar, which was the change in the functional form. So what is delta bar of f of x of f of x that should be how? That should be defined to be f tilde of x.

So, let us, plug this let us look at this so according to this is just f of x minus a. And let us, expand this thing to first order in a neglecting term of order a square. So as I pointed out yesterday delta bar as the advantage of delta bar commutes with taking derivatives. In this case taking, the derivative d by dx. So, in fact the confusing part is that normally you would just blindly you say. Let me take, delta of f of x to be f tilde of x tilde minus f of x that, is what you would have first written. But, that is 0 in this example because you are changing the argument and the function. So you are doing this double stuff, so but that delta will never…

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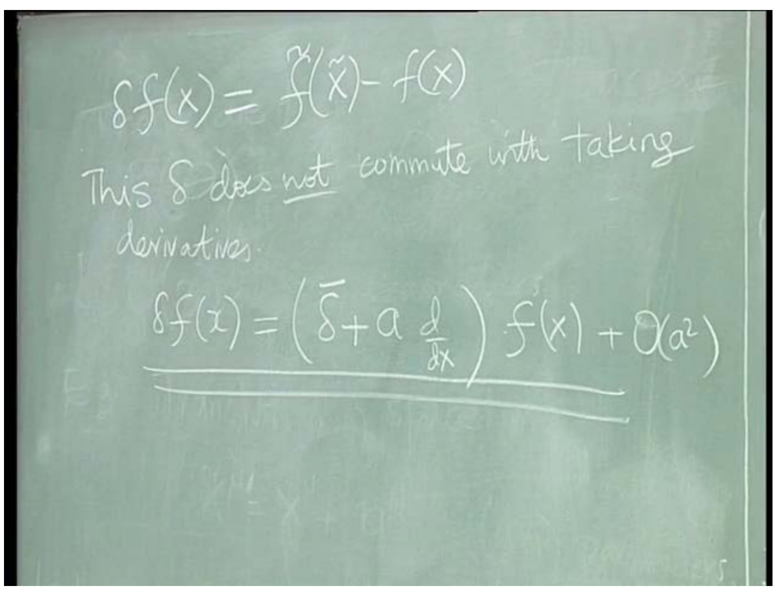

So, you could define I mean let us forget that it is 0 for this for a moment, but the point here is this object this will not commute with derivative because if you are doing a shift in x, I mean it has actually a two arguments implicitly. So it messes I mean so this, delta with derivatives, so in fact you may say why should we even look at such a thing like this. But you will see that it is natural in some instances you will quickly write something like this, but you should remember that whatever, is this the delta it will not commute with this thing. But, we can actually in this example we can even go back and say, how was to at least we know things to first order in a. Let us, ask can we ask what delta should be so let us worked out. So delta of f of x is equal to 0. But, I will show we can see that this can be written also has there, is some minus sign, right? So this is what you get tells you and it shows you that there is some derivatives acting so it matters ordering matters for this guy.

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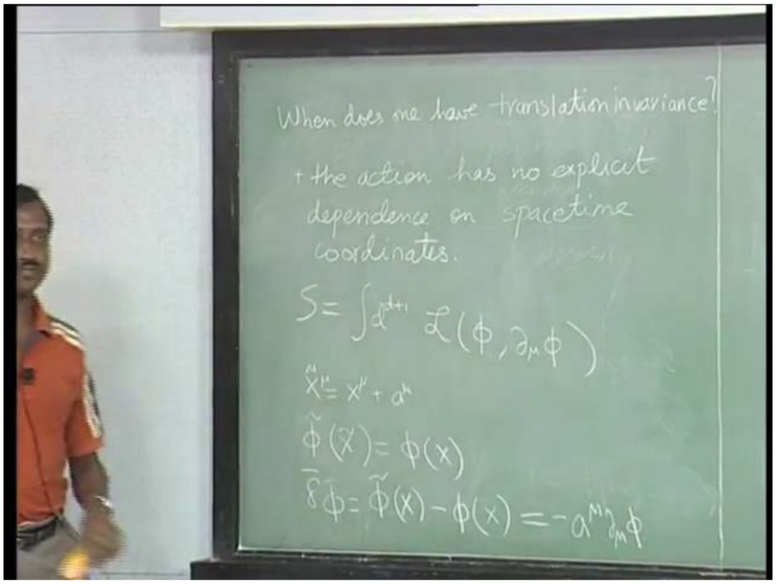

So now, coming back to this thing this example it is a same story out there, we can work out what delta bar of phi would be, but that should be just by definition phi tilde. This is just definition the functional form, how it has changed and this is following a similar arguments it is just there, are many more derivatives. So, this would be minus, but even the Lagrangian density is scalar, so I can actually think of what happens to the same thing out here.

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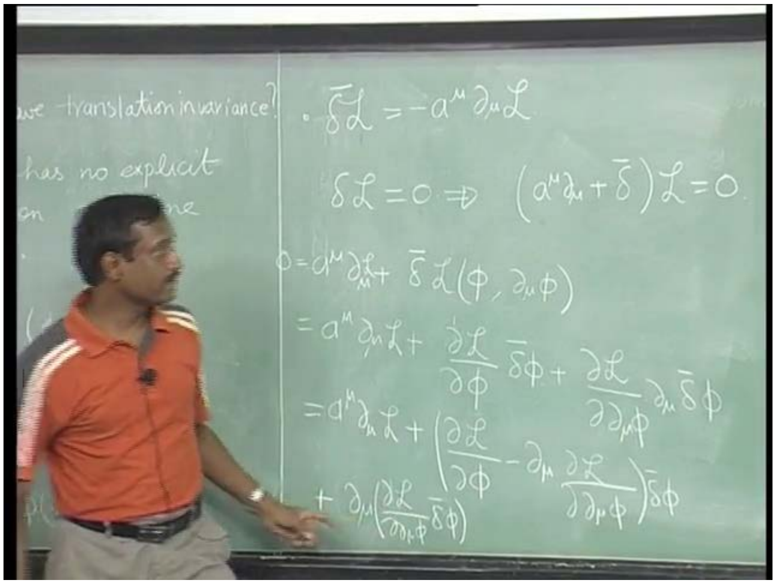

So, for Lagrangian density also I can define delta bar of L and following similar arguments. You will see that delta bar of L is nothing but minus a mu in a mu of phi of L. So the statement that it is the symmetric can be written if you wish as saying that delta L is equal to 0. And we will use that kind of formula, but this is equal to so, they just tells you that the action the this is it. So, let us see what this say we can expand this out and see what this means. So, this has bunch of terms into delta bar of plus. So, it has two arguments so the reason for working with delta bar is here, that it commutes with d mu.

If I had try to work with delta it will be a big mess. So, that is the key the fact that delta bars commute with this thing is very important. So, now I can just go ahead and exchange this two guys and so we can just, use plus so I can exchange these 2. So, I will jump one step and write this as d mu of delta bar of phi and so this will be equal to and I can integrate by parts plus the all the piece. Which I have left out, which is there is total derivative which have the reason is the is that this is the equation of motion. So I should say 0 equal to all this. So what I get? So I still have to put in delta bar of phi which is minus a mu d mu phi.

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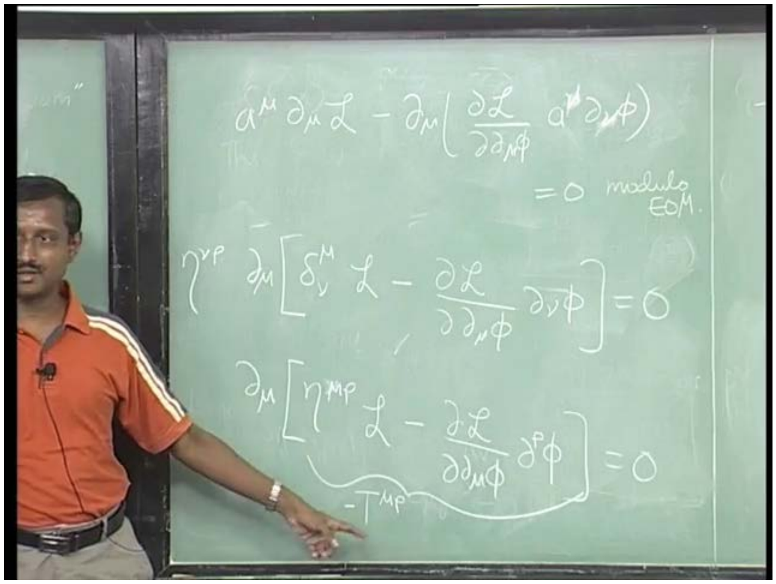

So, I get and delta bar phi is modulo equation of motion. So we started here, this is just a statement of it being a symmetry, and then I convert it to this statement and I am always working to order a square I do not need to a to be very large. So, this what I get so now, we look at this if you so I have to take care see I have used the index mu out here. This is a nu so, I can just pull out a nu and ask what remains, but now I have a problem in the sense this has a index mu.

So, there is some minus sign so let us see, let me redefine this to be nu and let us, call this mu just a this is I can pull things out. So, this is becomes a mu times actually I want to do the opposite to match what I had earlier let it remain as this is. So, and let pull out a nu so this is what I have to do. So this has to hold for any a. So, that will just imply that this object is different is true.

So we just define whatever, we get here and then there is a slight problem in the sense. I have a nu in the denominator. So, I can multiply this whole thing by say eta nu rho and raise convert basically, I want to raise all indices out here so I get the d mu of eta nu eta mu rho L minus dL. So, this object I will call t mu rho with a minus sign convention has to do with this. So I chose put a minus sign out here. So, we get a nice formula.

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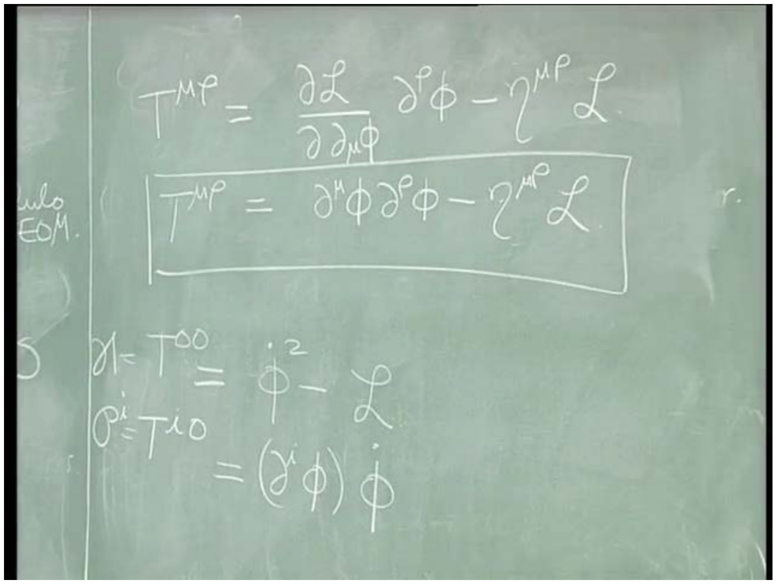

Now, in this here what has happened is that the rho is the parameter rho variable for the parameter and mu is the current part so that is what we have here. So, this is equal to because, I am taking care of the minus sign now. And let us evaluate this for the action that we have it is just a. So, this is what we get, but now you can see here. What we have here, is very nice this complete symmetry between mu rho.

This is trivially true eta mu rho it is same as eta rho nu rho mu. But, these 2 are also the same, but I just want to tell you we got lucky. But, it turns out that we can always make it so even if we did not get lucky as we will see in some examples. And let us, just check what is 0 is eta 0 is 1 for us, this is exactly what we wrote yesterday but while we are at it we can even ask what are the other guys. So, let us look at ti 0, ti 0 will be just di phi into phi dot I just stop here.

So, these would be the charges conserved charges. So this is what I would call pi. We will continue next time we will look, at this a little bit more. We will come back to more detail derivation of you know the equation in much more generality will derive a master formula, but that is the down the road. What I wanted to get for you is show you that there is a method given a symmetry you can get. So, this is the correct definition for what you will mean in a by energy and momentum in any field theory. You get the conserved current and then the conserved charges are what are called energy.