Classical Field Theory

Prof. Suresh Govindarajan
Department of Physics
Indian Institute of Technology, Madras

Lecture - 1
What is Classical Field Theory

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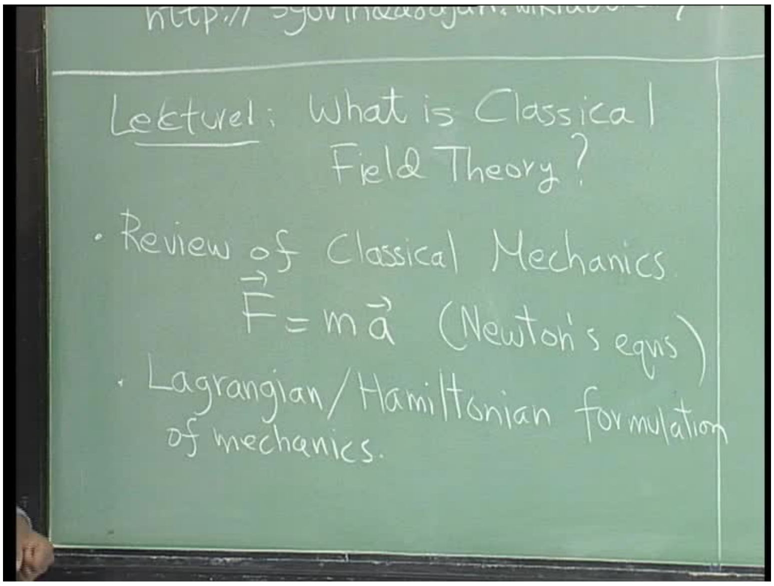

Good morning, and so the plan for the first lecture is to tell you what is classical field theory? So, what we will do is, we will start with the familiar hopefully all of you know what classical mechanics is, so let us just review classical mechanics.

So, you start in high school, where you learn, find we usually write things like F equals to m a and this is Newton's equations. But as you move ahead you realize that there are more sophisticated ways of handling things, for instance to learn about the Lagrangian and Hamiltonian formulation of mechanics. The advantage of this formalism is that, they are much more general than Newton's equations and sub some them. So in some sense, if you choose a particular Lagrangian or a Hamiltonian, you can recover Newton's equations.

And most of the forces in mechanics set once look at they are usually velocity independent. And of course, friction is an example of force, which is velocity dependent it is dissipative. But there is one more velocity dependent force, which is not dissipative its conservative, What is that?

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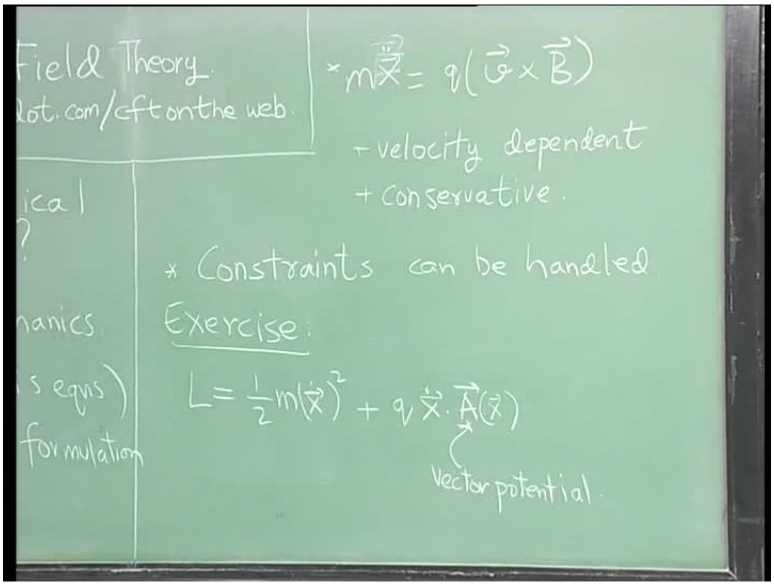

The electromagnetic force, especially the magnetic force; it is the force, which is unlike any other force, its velocity dependent but nevertheless it is conservative. And so this is very nice example, who is which can be completely dealt with in the Lagrangian formalism. Another set of a nice system, the reason to use Lagrangian is that you can handle constraints can be handle. Not all sort, all kinds of constrains but some classes of constraints can be handled in this formalism.

So, for instance a simple exercise for you and I will keep giving exercises, which you should go home and check is to take a Lagrangian for a three dimensional particle, which would be Lagrangian of this form, where A is the vector potential. So, now you will see we have a Lagrangian here is this were not there it is a free particle but, this is term which has the velocity dependent potential and you can go back and check that it you end up getting equations of motion which is not this but, f e so the equation would me m x double dot equal to q into v cross B.

So, go back and check this equation and so here is a case, where we can nicely handle this, and it can also be handle in the Hamiltonian formalism. So, the reason to look at general formulation is, that as you going to situations, where you are in normal instinct does not work, it is better to have abstract methods of handling things.

And, the nice about this particular formulation, we will see is that it is generalizes naturally to classical field theory. And you can incorporate a whole bunch of things as we will see.

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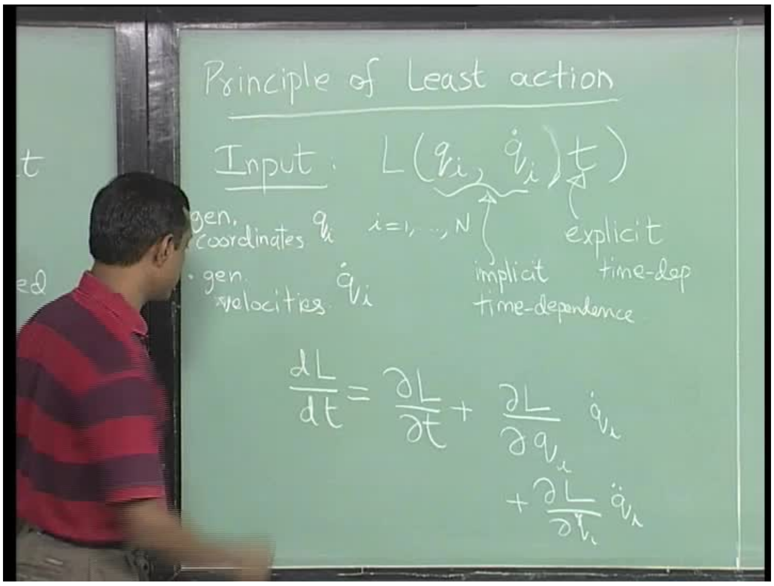

So, let us say sort of do a quick review of the Lagrangian formulation, and that is usually done, where something called the principle of least action. So, the input something called the Lagrangian, which we already discussed, which is the function of coordinates and velocities. So, now q i here, let us say there are n degree, so freedom in your system there will be that many generalize coordinates and as many generalized velocities..

And impressive well this could also, have an explicit dependent on time. Ok, so this is what we will call explicit time dependents. I suppose to implicit time dependents, because both coordinates and velocities can be can have time dependents, through their arguments, they can vary with time. So, this we will say our explicit time dependent, well this gives you implicit.

So, for instance if you want to know, how this object Lagrangian changes with time or you want to i take its time derivative, you do in multivariate calculates, you will compute something called a total derivative, which has several pieces; one is the explicit time dependents, which you write a partial derivative and then, you look at the various arguments into d q by d t, which I write as a q i dot. So, the first term come for the explicit time dependents, while the next two terms come from the implicit time dependents on q i gives you this term, while q i dot gives you this term. So, this is what is a statement of taking the total time derivative.

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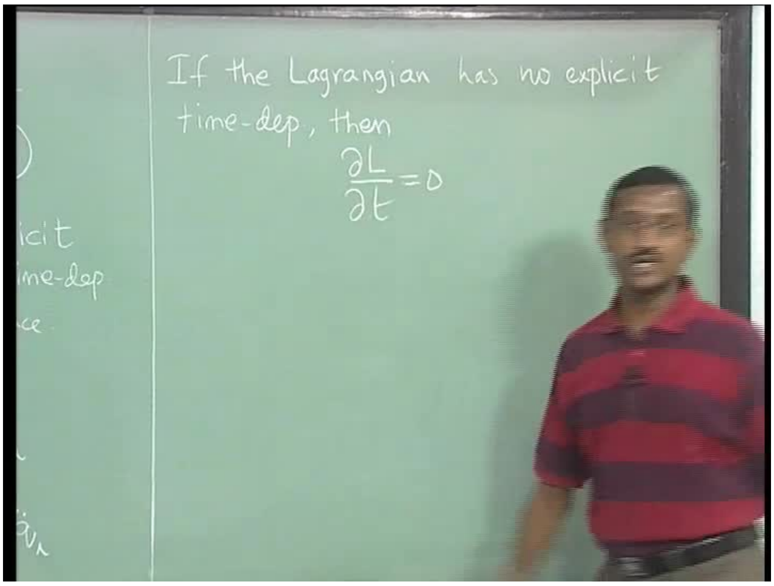

So, if you say that, the so if the Lagrangian has no explicit time dependents, then what would you say? Then you would say then, the mathematical way of stating that is d l by d t is equal to 0, the partial derivatives with respect to time. But of course, if you look at the total time derivative it can change.

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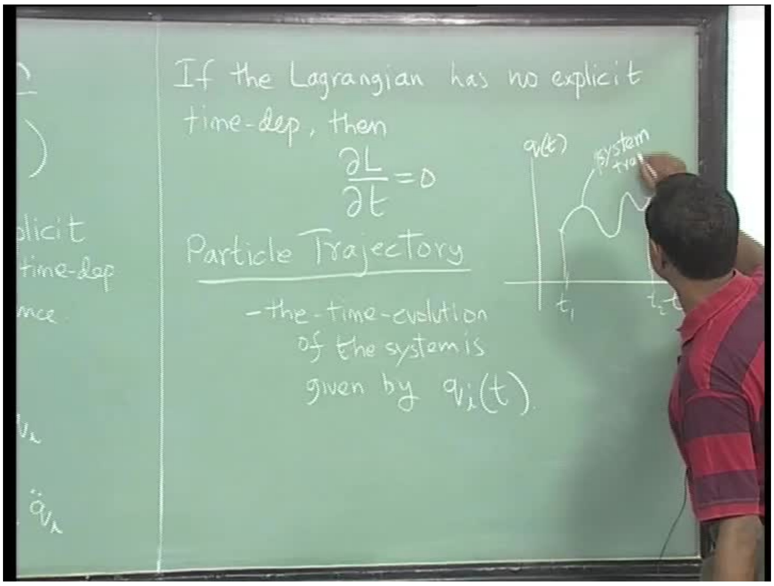

And, then there is an ocean of a particle trajectory, which is very simple; it just tells you, if you give me a system with some generalize coordinates, you tell me how q i varies as a function of time. Ok, so this is just the time evolution of the system, is given by some function q i as a function of t. So, if you have a one if have one degree of freedom, then I can draw a nice picture. So, let us do that q of t versus t and let us say, you have initial time t 1 and at some later time t 2, let us have you only observed, during this range particle started somewhere, here and did something and ended here.

So, this would be an example of a particle or a system trajectory, let us call it a system, because it may not be a system of particle. Even though, I called it particle here but, it is or just simply call it trajectory. And, the key point here is that the Lagrangian tells you, how the what a trajectory of the system would be gives you the dynamics of the system.

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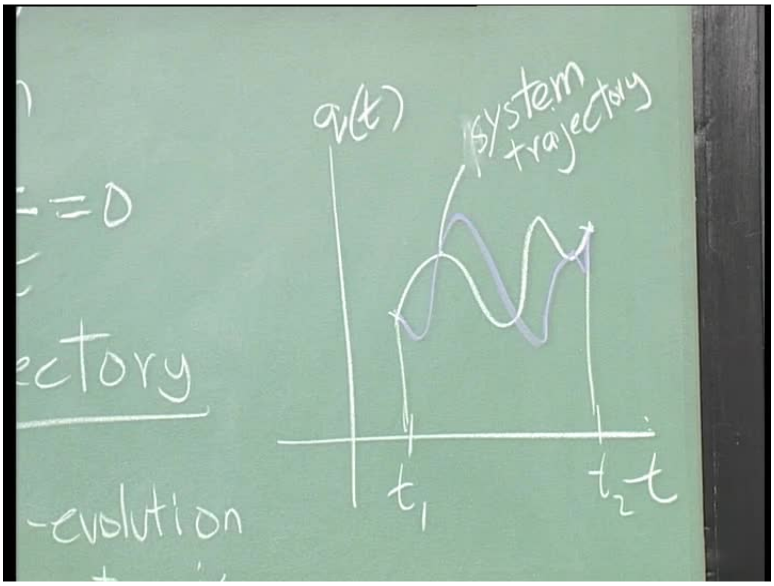

So, in principle I can keep drawing different kind of trajectories, let us change color for a moment here is and another one. But the question is, what is the trajectory followed by the system? The actual trajectory and that is determine through the Lagrangian by other principle of least action. So let us see, how it is done, so it should tell you among the various trajectory.

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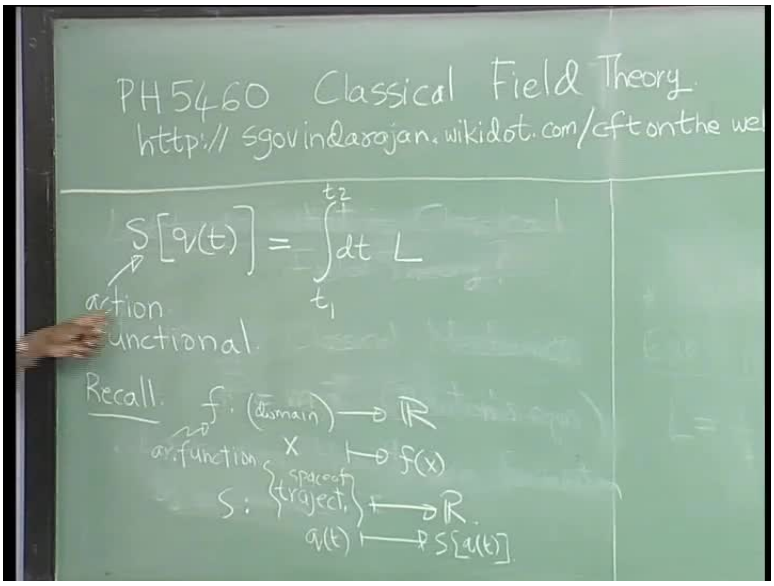

So, first let us define something call the action, the action takes as is input. Now, for the rest of the lecture, I will for simplicity, I will just choose one degree of freedom, you take so it is something, which takes as input the trajectory or the of the system and gives you output a number; and that number, so let us say so this is the action. So, this is similar to a function except, that as its argument its taking its taking a trajectory, which is another function or many function depending on, how many degrees of freedom, so this is an exa this is an example, is call the action functional. So, just recall, that a function is a map from some domain, so it is some from function some domain to reels.

So, if x belongs to this thing, it will give you f of x, which is the number but, a functional so, this is a function, while a functional like s, it takes as input trajectory; all trajectories, so this should be space of trajectories but to again to some real number. So, in that sense, it is exactly similar to this, you give as input a trajectory, output is a number. And that is what, and this is a functional, so example is exactly this, you take q of t and then u get s of q of t. So, now somebody comes and gives you a function, you would like to ask generally its sort of first course in calculus, you are asked to find out its extrema in other word point, which are maximum, where the function takes a maximum value, local maximum or local minimum.

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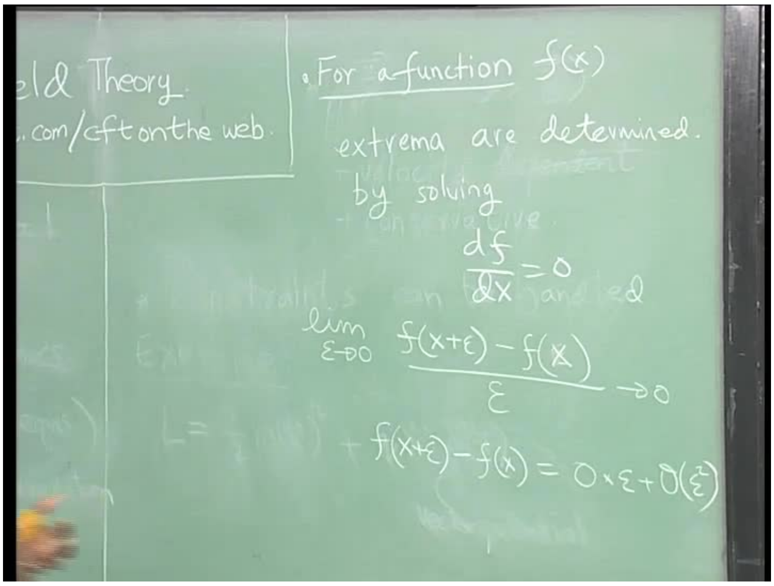

So, if you have a function, so the way you determine for a function extrema, at f x are determined by solving. And then, you used the second derivative, to determine whether it is a minimum, local minimum or a local maximum, global will require you to compare across this things.

So, now so that is another way of understanding this things, so derivative always requires you to compare values to nearby values, and so you could think of it as asking this going back to the discrete k l limit, which is like limit some epsilon tends to 0 f of x plus epsilon minus f of x divided by epsilon should tend to 0. But, you can do it in a slightly different manner in a formal manner think of epsilon as a small number and what you doing is, if you did f of x plus epsilon minus f of x, you know there is a full tailor series but, the key point here is, it is 0 times epsilon plus order epsilon square.

This is another way of stating it. You just compare to neighboring values of function and the every times, it hits a 0 to first order in epsilon, that is equal into doing this. It is a same statement; it is not anything different, because in as much as you can tailor expend the function the coefficient of epsilon is indeed, the first derivative, first smooth function. So, we will assume that, the function is smooth.

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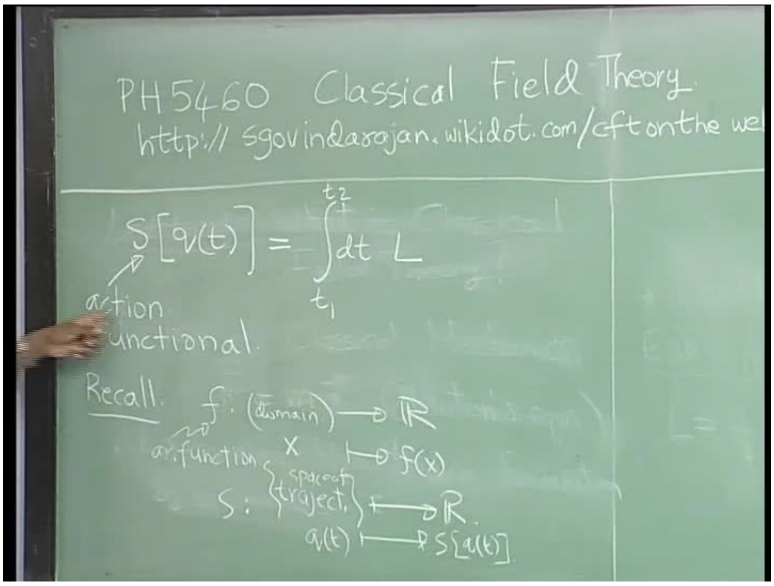

So, the point here is that the action, the principle of least action tells you, that you should choose the trajectory such that, the action is takes a minimum value. So, the first step to solving that is to ask, what it should take an (( )) value.

So, the problem is exactly identical to this except, here what did we do to find the thing, that you keep changing that value of x and ask, where this is true.

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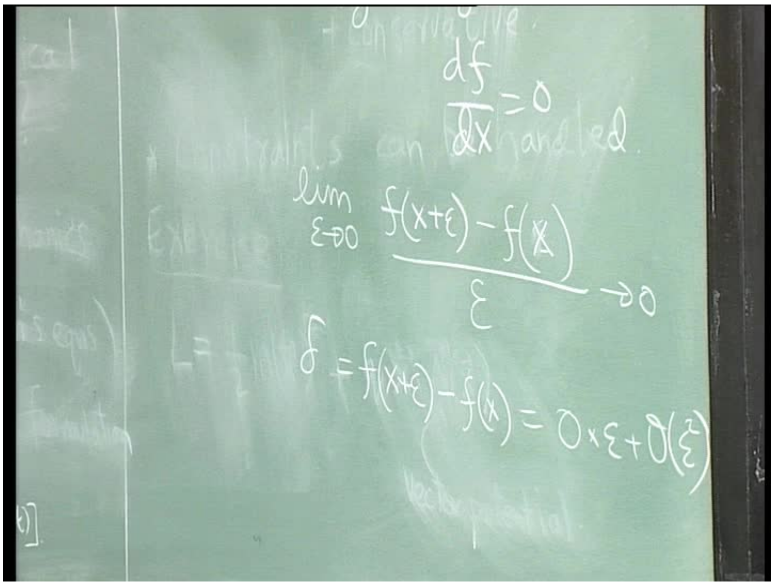

So, a similar question to do out, here is so the analog of x, here it is a trajectory, so you have to keep changing trajectories and we need to ask, when this the variation. So, we just call this delta f is 0 to first order that is all.

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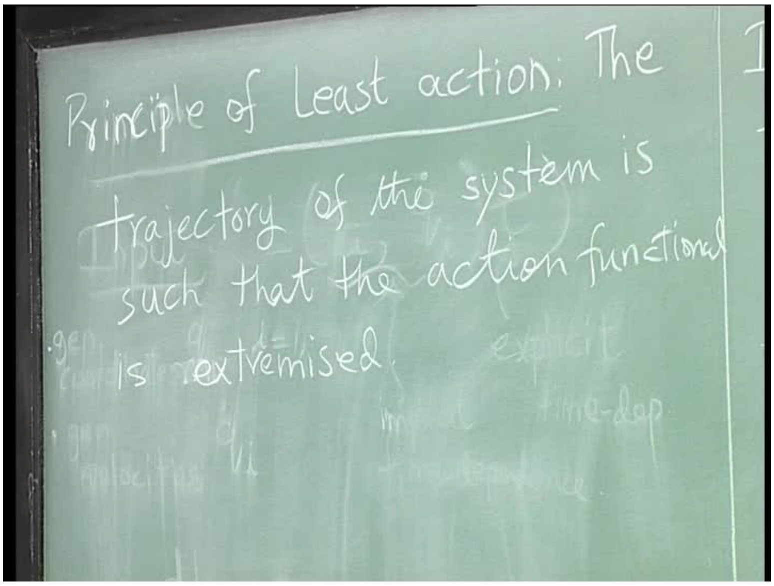

So, let us see how that goes, so the trajectory of the system is such that, action functional is minimized, I would say extremised because …

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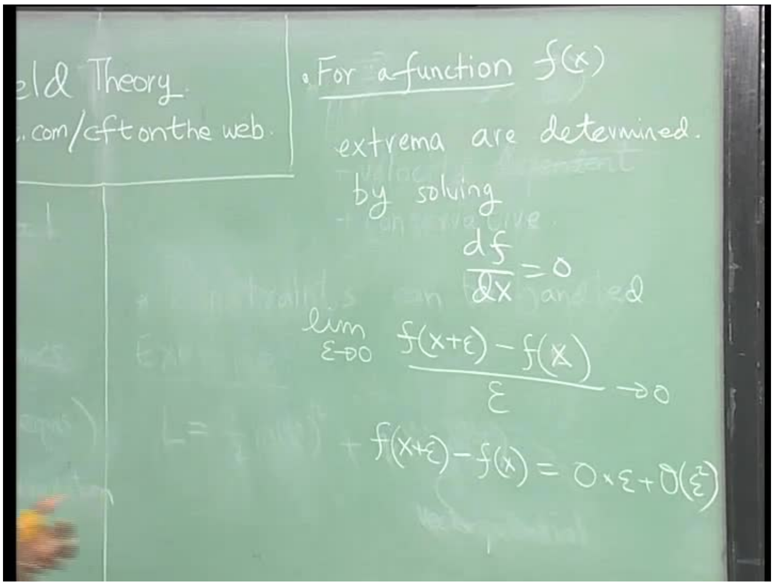

So, now that what we will do is to do the same thing, which we did here but, we will work it out in this particular form. So let us see how that goes.

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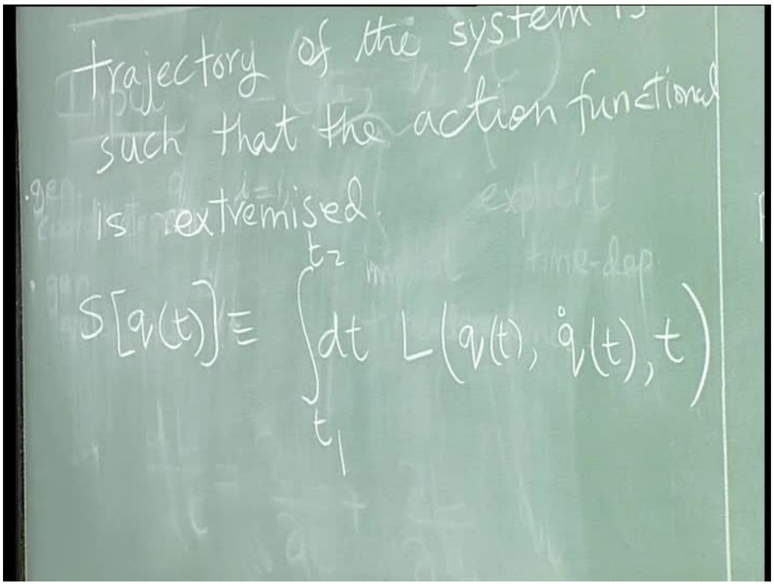

So, let us take a one degree of freedom, from some initial time to some final time d t of l of q of t, for once I will put all the arguments may be, it has even explicit time dependents that is. So, now what is it we should do, we should take to neighboring trajectories.

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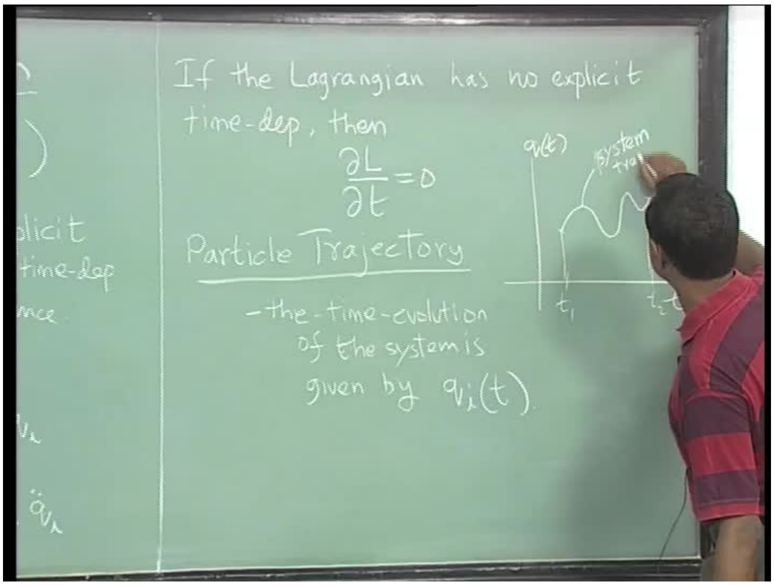

So, the question is you know, what is it mean for two trajectories to be near by. So, when we talked a function, we know what it means, there is notion distance and but, here what we are doing is we are talking about the space of trajectories. Is there a natural distance to them?

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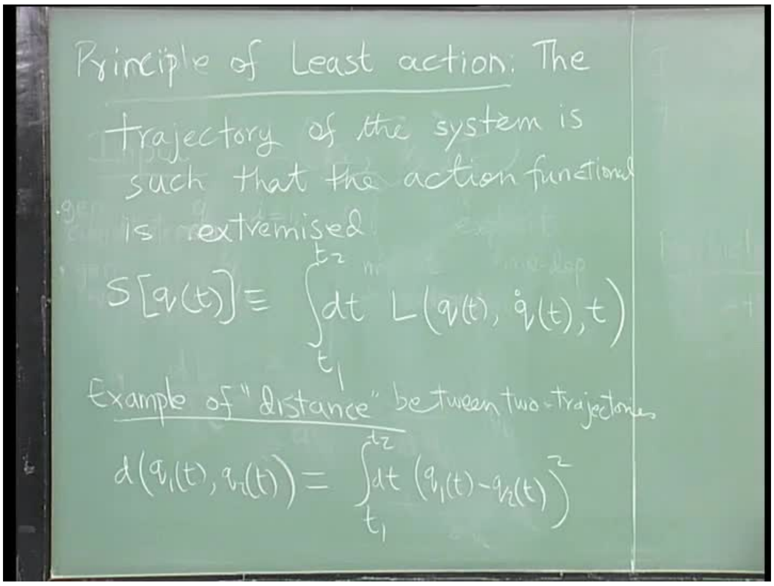

These are questions, where to which; I think the general answer is not known but, you can think of think of a distance by saying, let me let us consider, so an example of a distance of two trajectories, so d, so let us say q 1 of t q 2 of t are two trajectories, which are nearby. So, let me for instance define them to be something like this square.

This is I am not saying, this should be the distance but, here at least I have some notion, when two when the two trajectories coincide this becomes 0 and intuitively at least this just capture something like that. But, what we will do is, what is called formal stuff, where we will just look at the coefficient of the guy, which is we will write some formal parameter epsilon. The statement formal here means, that we do not worry too much about issue of conversion, which I must tell you, it is important but, not for this purpose. So, formal means that, we do not worry about conversion issues.

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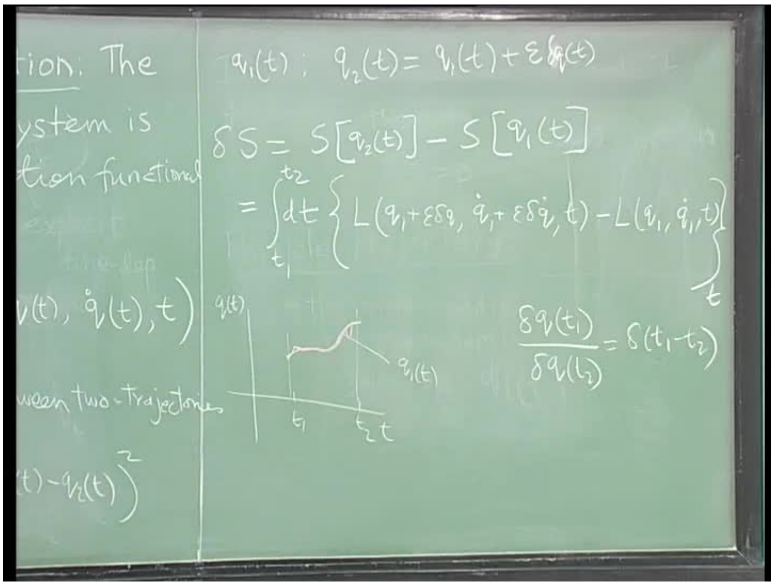

So, what we are going to do is we will take two trajectories; one will called the q 1 of t and the second one, which I will call q 2 of t, which I will write as q 1 of t plus epsilon some other function of t, so let us just call it g of t or even better I just call this delta q. The important thing here is epsilon is my book keeping parameter and I need to expand thing to first order in epsilon. So, what I will do is define delta s to be s, you valuated for the trajectory q 2 of t minus s q 1 of t. So, as it is stranded, it is just a difference of two numbers.

My next step, what I will do is, to substitute for q 2 in this form and use the explicit formula that I have. So, all I have done is to take the definition out here. Now, you may wonder is this is delta q dot something independent; the answer is no, I will just assume this is smooth function, I take q 2 dot is just q 1 dot plus epsilon delta q dot.

Now, the important point here to notices, that the object in the square bracket in this curly bracket in curly braces is at sometime t, this is the number but, the argument here, if you is at some particular t between t 1 and t 2. So, it is also a very useful device to take time and make it discrete, I will not do that but, you make it discrete so that this becomes integral becomes a summation. So, you will you will see that this object in brackets, there is one term for every time step.

And for every time step, there is a delta q and each for each time step, the delta q are independent of each other because I could do the following thing, coming back to this picture and let us say that, we chose we choose this is the trajectory q 1 of t and let us say, I want to do q 2 of t but, I could do things like, let us say that I make q 2 only different in two different neighborhoods just for, so it is the same as here but, may be somewhere, here it is slightly off then it comes back to this follows this some other space it goes off and comes back

And I can adjust it, in whichever way I want and you can see that, I can make this peak, I can look, so the point here is the this delta q is completely independent of this. More generally, the delta q s at any two time instances are independent of each other. And, the mathematical way to state that, is that delta if it delta q of t 1 and delta q of t 2 are actually independent of each other, and except; when they are the same, they will be it will be 1 but, the when you discrete it would be 1 but, when you are in the continuum, you would replace it by the direct delta function, so this is roughly this is exactly, what I just told you.

Since, that any two instances delta q those things are independent of each other. So, this is in some sense the most important formula, which captures this import this sort of trivial but, important statement. So, now we can go ahead and look at this thing and expand it. So, now we look at what we are doing is, we are looking at, what is inside the square inside the braces and that is at given time and we going to compare these two numbers.

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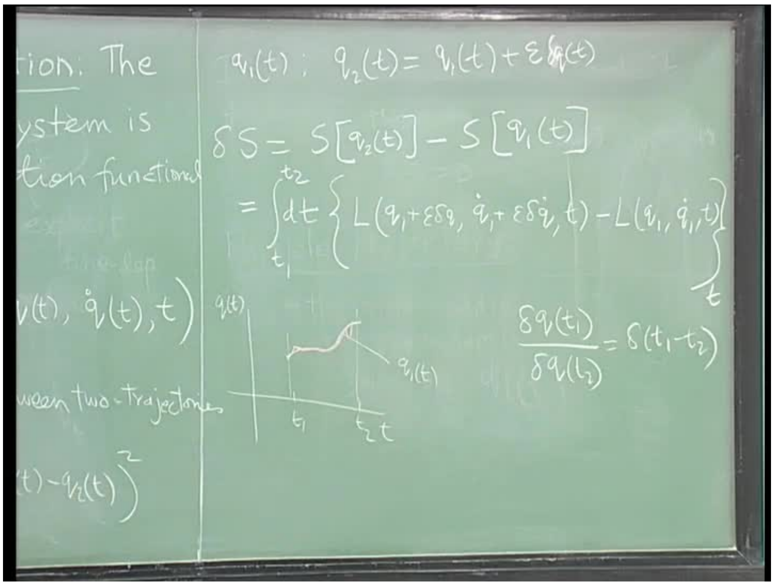

So delta s, so now you can see that in these things, there are two terms, which will contribute to the change in l and so, we can just use normal you think of l is a function of q 1 and q 1 dot two independent variables.

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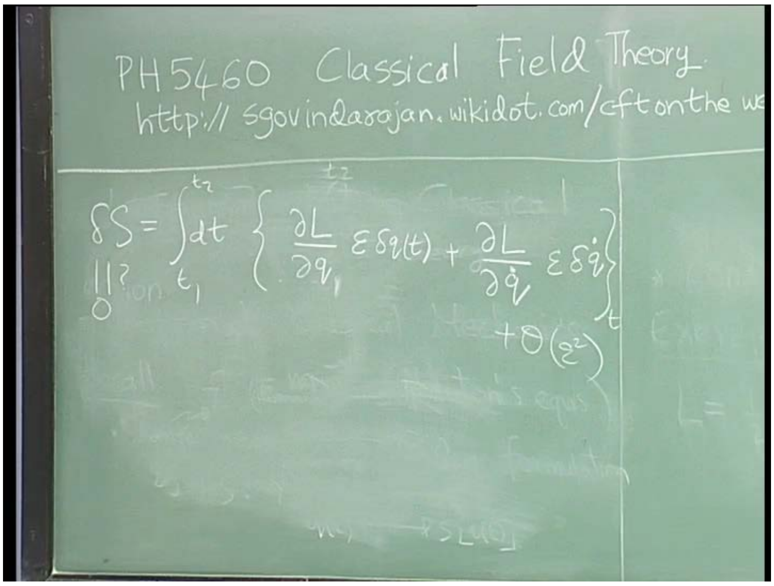

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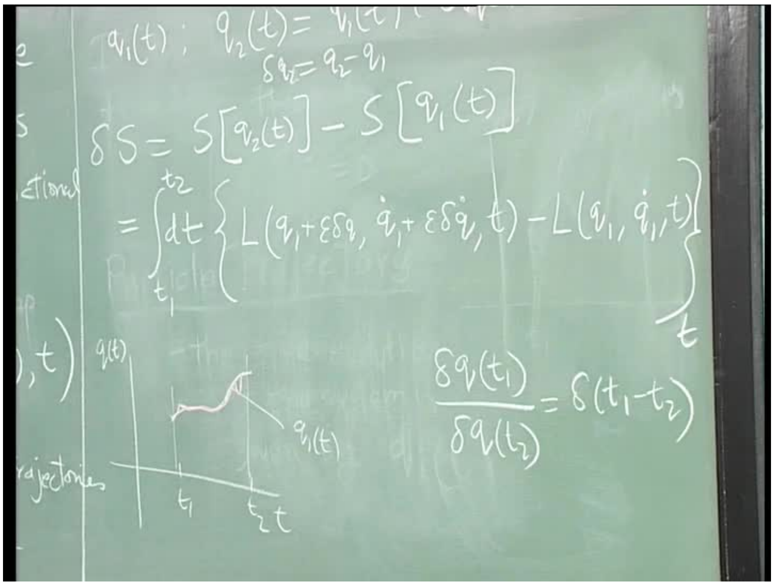

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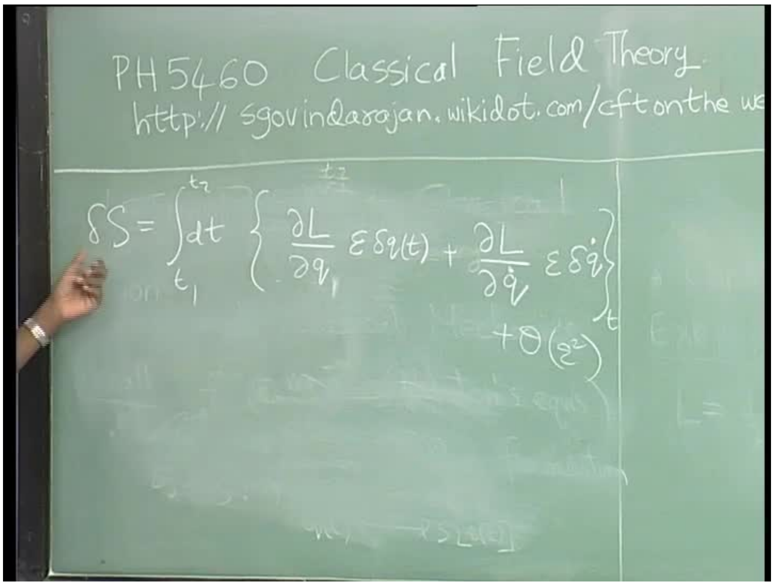

And, so the 0 thought of p is cancels, so I will chose, I will only write out one times delta 2 sorry, d l by d 2 times the change, so that is q 2 so delta q 2 delta q is nothing but, q 2 minus q 1, so that is epsilon. I am just keeping these reminding you, that it is for independent for every time. Of course plus order epsilon square, we are not bother about that at this point.

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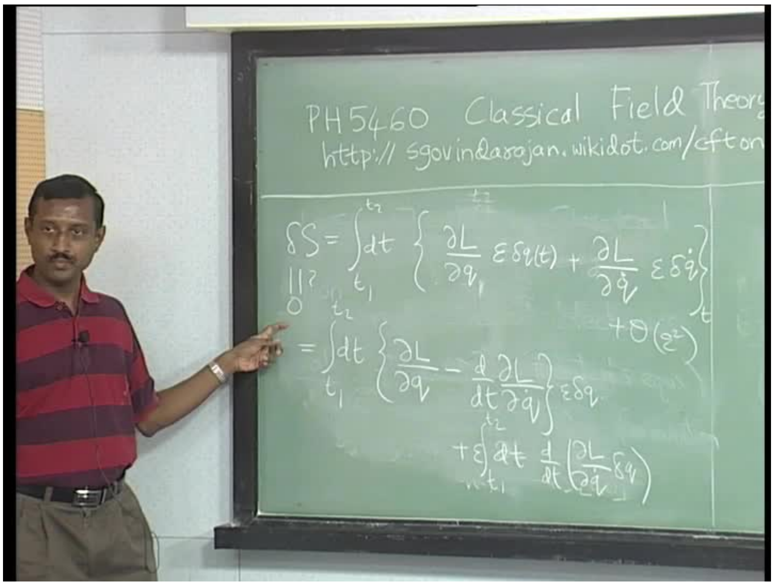

So, now you can see that, since each since this has to hold for every time step, this thing so we want so we want to find the extrema, that word in other words, we need to solve the analog of delta f equal to 0. So, we need to check whether what we want is whether when is delta s is equal to 0; this is the question we would like to answer. But, as a pointed out for each time step, they are different so this can happen only, when every term vanishes separately for each time. So curly brackets should be 0 by itself.

But, we will do a little bit we will do one more step before, we do that and we use the fact that delta q dot is nothing but, d by d t of delta q and we will integrate by parts and re write these thing. So I am jumping some steps, because I am sure all of you are seen these before and there is surface term, which is a total derivative piece that is. So, we have two terms one is the order epsilon, oh there is the epsilon even here. And, we want thing to be 0 and since a total derivative, I can integrate this will only get contribution from the two end point.

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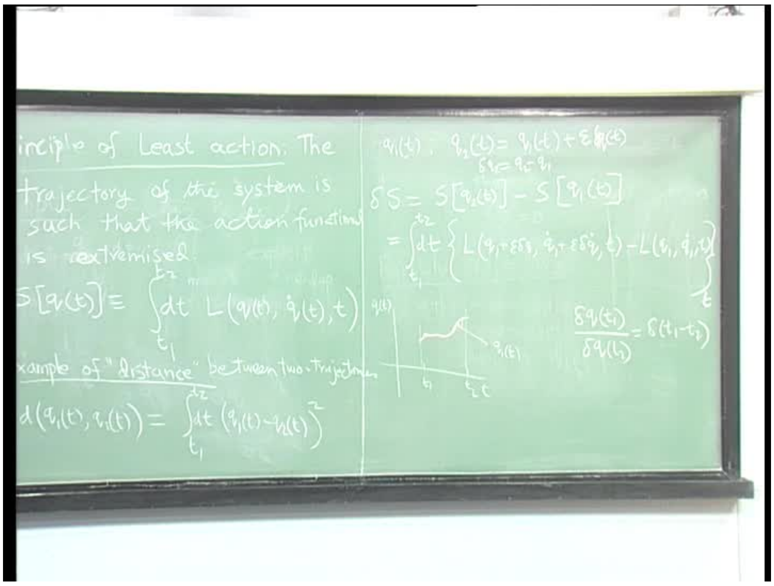

So, let us look at when t is not either t 1 or t 2, then this term does not contribute; so only these term contribute and you can see, that you get d l delta s equal to 0 implies d l or d q. But, what about t equal to t 1 and t 2 that has, so there what we would do is, if you want we will choose we choose delta q at t 1 equal to 0 and delta q at t 2 equal to 0. So, this is what we will call a boundary condition. In other words, we do not look at arbitrary trajectories, we look at space of trajectories such that the initial and the final points are given.

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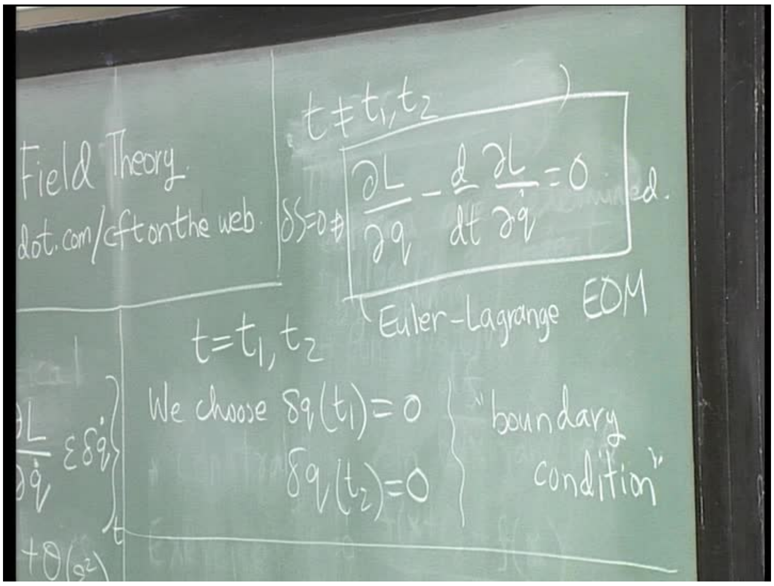

And once, we do that all we need to do is to solve this equation; this is the Euler Lagrange equation of motion equation of motion EOM is something, I will keep using and so this is how the principle of least action works. Are there any questions? The advantage of this picture is that, the transition to quantum mechanics can be seen in a very easy manner. So, what about quantum mechanics, actually one is not really done with this, so would look for solution of this but, now the question is like any function, that could me many extrema some of them will be minima, some of them will be maxima.

So, now the so in physical systems there has notion of stability, if you wish and so, that is the same saying, that you need to pick the situation, where the energy is minimized or whatever. So, that is something which, you need to check you need to check if your solution, which I mean this is only the statement of the first derivatives, so there is an analog of checking for the second derivative. And there will be instances, in which we will consider different solutions classes of solutions for, whatever reason we will see we will consider them. And they will be in equal it in a particular sense that, if you started out with a configuration, it could never go into the other. There is some sort of topological quantum number or something, which will stop it from going.

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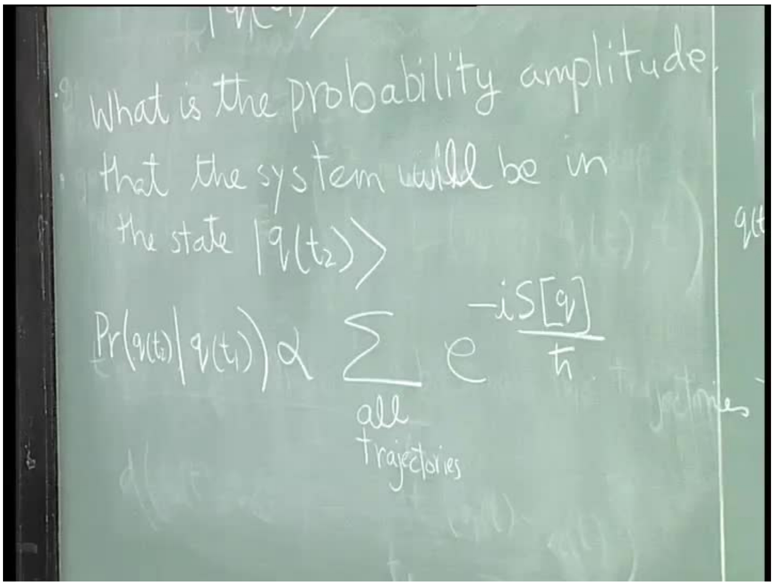

So, Feynmans description of quantum mechanics is that follows, he says that suppose your system is at some state at initial time it set its in its localize in space at this point q of t 1. And, you want to know what is the probability amplitude, that the system will in the state in some other state some value.

So, let us say ya q a t 1 in some other value at time q a 2. So q a t 1 and q a t 2 are numbers, which is exactly like the initial and your final condition it is localized out there, and the answer is so usually that is given by some sort of an amplitude; so the probability amplitude not probability of being at q at t 2 given that it was a q at t 1 is up to some normalization proportional to the some work all trajectories with a weight factor which is s, so the statement here is that actually all all trajectories contribute to the...So, you may ask what is so special about this guy about solution of this.

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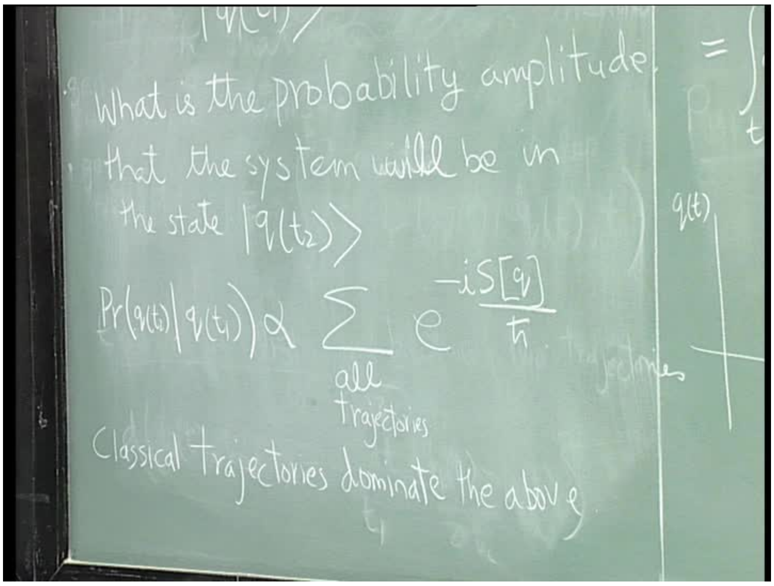

So, roughly speaking this gives the maximum contribution, so the classical trajectories dominate dominate the above and the way to see, that is roughly speaking the classical limit is taking h bar going to 0. So, in other words this the denominate is going to 0. So, now unless this was set it lowest value one point; is that I can always adjust the action such that it is lowest value is 0; that we can just mode out by some constant patience. And so, then this the dominant; it will go to 1 only, when this is set it is lowest value, which we have set to 0 but, for anything else it will be fluctuating wildly and so, his argument is that they cancel away.

But, if h bar if you want to know h bar correction, when you have to take the contribution of all the trajectories. So, in principle this is a very complicated calculation to do for an arbitrary action but, notionally it gives you I mean for simple example, you can actually carry this out exactly and so that it gives you answers, which are identical to what is you would have done using say the so dinger of the haze picture in your quantum mechanics course. So, we are sort of now ready to sort of understand what classical field theory is about. So, in usual classical mechanics, usually you have only a certain number of finite number of degrees of freedom countable or whatever you say.

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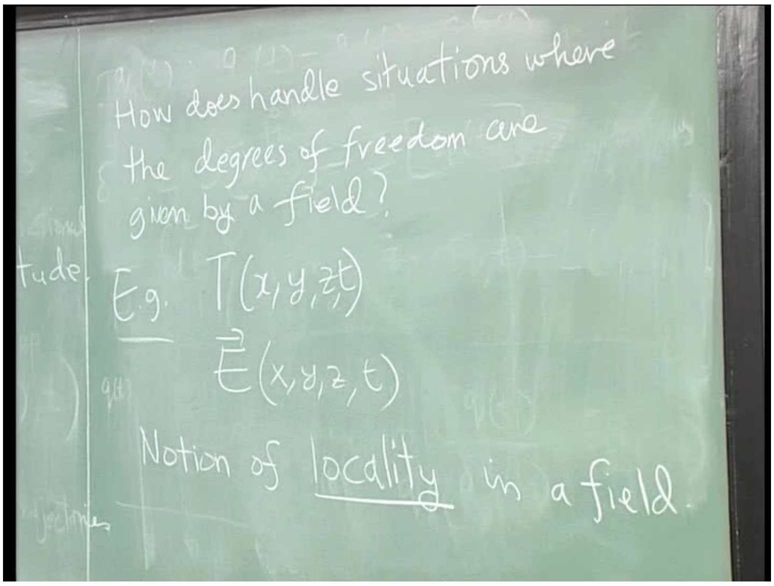

And so, the thing is how do, you how does one handle situations, where the degrees of freedom classically itself are given by a field. So, let us loot at example of a field, example of the field for instance, would be the temperature in this so, so at every point in this thing there is a temperature. So, if I want to discuss the change in the of the temperature in this room, I need to handle the fact; that it could the temperature may not be a constant here, it could be one value near the a c and one value very very far for the a c, so this could change with this thing of course, it is also function of time.

Similarly the electric field. It has roughly three times, the degrees of freedom as this, because at every special point this thing had a scalar degree of freedom, while this has a vector degree of freedom, so now the point here is that, we have infinite degrees of freedom like this. But, even if you take a guidal gas notionally; it has an avogadro number, which is 6 into 10 power 23 molecule, so three times that it has that many degrees of freedom.

But, there is difference between that setup and this; and that is the notion of two neighboring points, here we understand two degrees of freedom being close to each other. Well, if you have an ideal gas there, you have many different particles and there is no notion of two particles being close to each other or two degrees of freedom being close there is nothing like that. So, the difference here at there is really extra data, which is coming out here, the there is some physical distance that is there between the degrees of freedom. So, you need so that is a notion of what you will call locality in a field.

In particular, you know that if I want to discuss the electric field in this room, I do not really need to worry about things, which are very very far away, in the next galaxy I do not need to solve that problem. So, in other words that there is some decupling so what you expect is object, which are close to each other only interact not something, which is so we do not want to have things, where we have action at a distance. Somehow apparently saturn can affect our lives that, what people say but, that is action at a distance that is something unscientific and we do not put that in.

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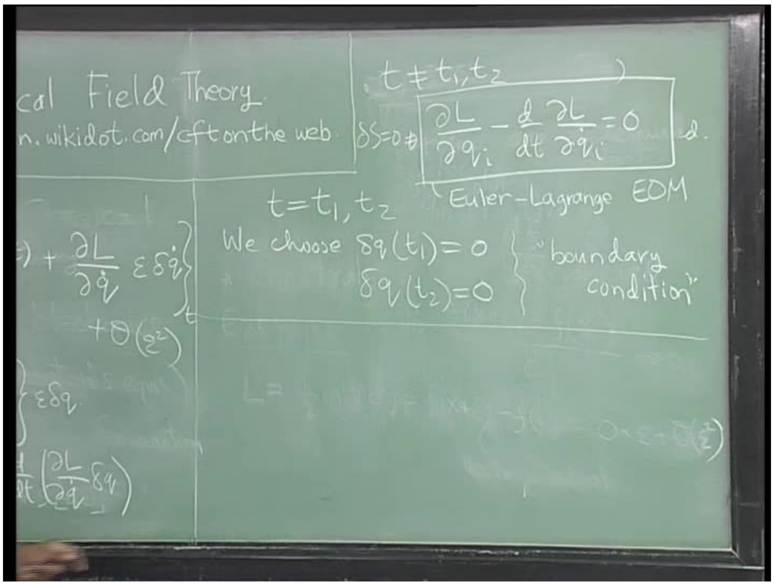

So, we will this is something which, we have to incorporate; so it is not just a question of saying, that I just go back and if you remember, I said if there are many of them, you will something like this put an index and you said I let it be infinite, that will really not be not capture the issue of locality.

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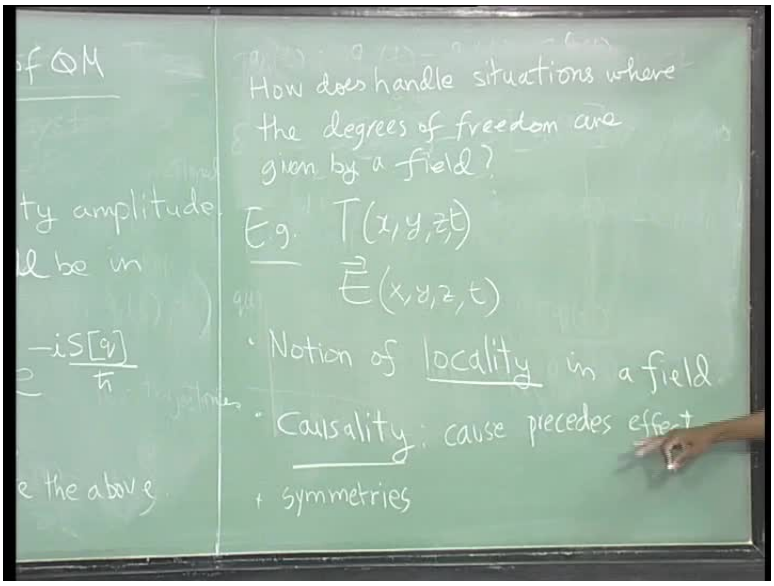

So, we need to somehow put this into the setup, and this is really, the one of the key another thing, which is need is a notion of causality; Causality is that you want cause to precede effect, so the thing is that, if you did something. So, let us say you have a system and let us say we take you know another continuum system, it is like a bucket of water and then, you and there the velocity field, if you wish is a field is again like this but, every point you have this thing and you go ahead and do a local disturbance somewhere, you know that that takes some times to propagate.

So, it is somehow we should also be able to incorporate local causality; that is cause precedes effect. And there will be more things there will be things, like symmetries in the problem, this is something that you will understand later, we need to incorporate some of them. So, you can see that it is not just question of taking the number of degrees of freedom to become infinite.

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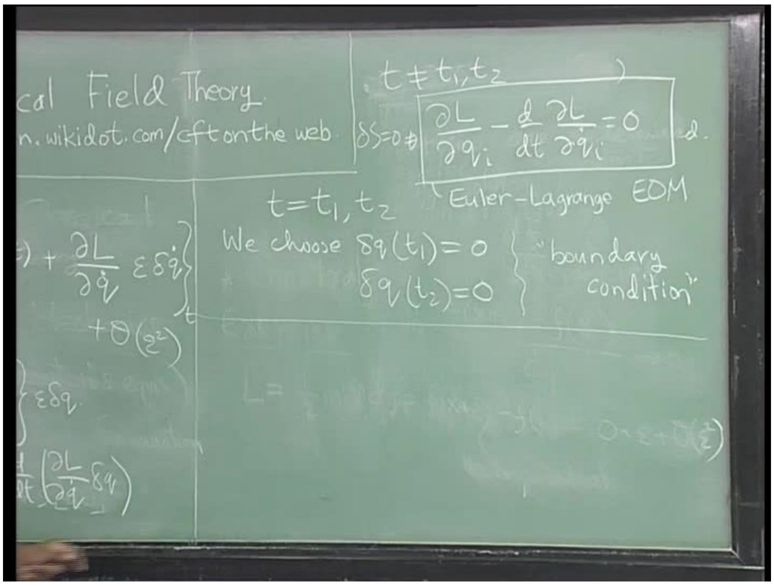

But, rather what you want to do is to somehow be able to incorporate this things, and this is something, which we will learn how to do? But right now, we are sort of ready to draw blocks. So, classical field theory is just the classical mechanics of fields that is all.

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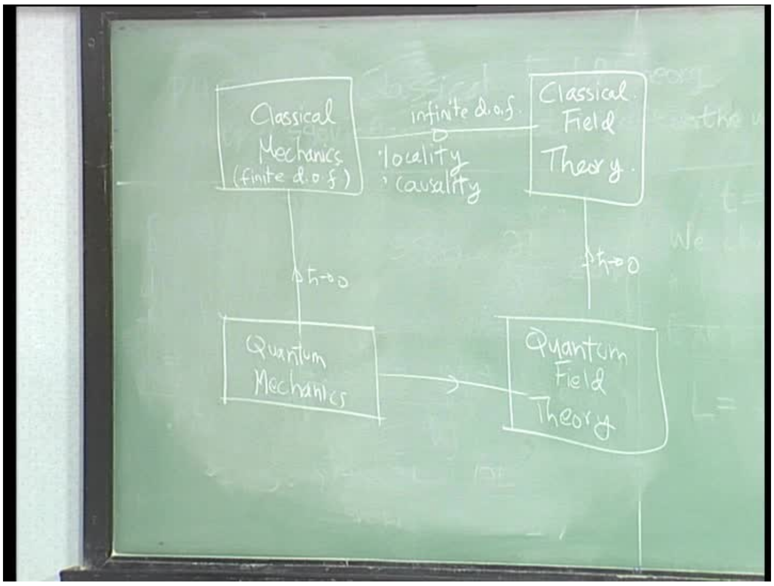

So, let us draw some boxes of, we saw classical mechanics and so, there we saw there was an action and then the extrema of the action gave you, the classical trajectory of the system. And in quantum mechanics trajectory, sampled so you get something like this, which we call quantum mechanics and we sort of flee h bar tending to 0, gives you this the arrow this way. And, now so this is just finite number of, so you go to infinite degrees of freedom and then, we need to incorporate things like locality, we do not know how to do this things yet causality so on. So forth but, you get the classical mechanics of fields and this is something, which we will call classical field theory.

This will more or less conclude my lecture but, obviously there is a box missing, because I could do the same thing out here, and so you could do a similar thing and this box is call quantum field theory. So, again this is similar arrow to this again, this would be h bar tending to 0. And in this course of course, we will not be doing quantum field theory. But, on and off I will keep referring to a quantum effects, just to show you, that I mean there is quite a bit and this is very rich subject and requires lot of study but, what we will be doing is doing on the classical field theory and may be in your next course you would be doing quantum field theory.

So this a good point for me to stop and ask you a questions. Are there any questions? Ok. So, in the next few minutes I am going to sort of discuss symmetries and we will do that in the classical mechanic setting but, in the next lecture we will start formulating, what is the meaning of the symmetry etc.

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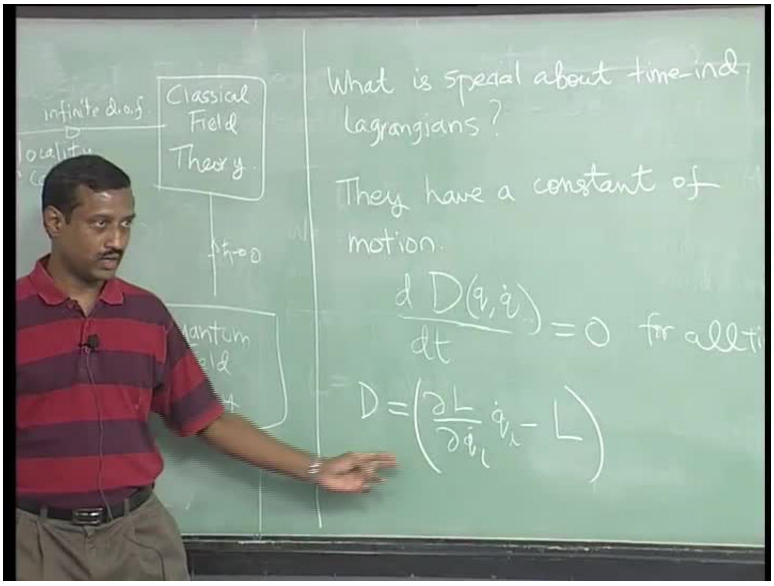

But, there is already something, which you know what is special about time independent Lagrangian; that means there is no explicit time dependent. Does anybody know what is special about time independent Lagrangian which do not have explicit time dependent?

They have a constant of motion?. That means, let d be some quantity, it is derivative, so it some function of q q dot or whatever, which is 0 for all time. So, let me ask again, what is the constant of motion? What would you call something, which does not change with the trajectories and it this is your Lagrangian is time dependent, time independent, there is no explicit time dependent. It is usually call the energy, so in this case d you can show with p q dot. This con this is the energy.

This is not the Hamiltonian for the simple reason; that the Hamiltonian is a function of two different variables. So, d here is only a function of q and q dot. And just, one more comment that, what we will be doing is that, we will use whenever an index is repeated with some over I do not write summation over i. So, here this is Einstein summation convention, so this I some overall i. So, there is an example of a symmetry and the symmetry here, is time translation and variance. Does not matter, when you do your measurement you action, if you translate would give you the same number.

If there was explicit time dependents, it matters the when you when you measured the action, what was the limits the t 1 and t 2 is not a, you cannot shift things away and you get. So, you do get a symmetry, this is an example of a symmetry. So, what we will see here in this course, is that there is an intimate relation between symmetries and conserve quantities. And so, we will have to go through a procedure of asking, what is the symmetry? How, what is the best way to define it? Etc. and that is what we are going to do in the next lecture. And, this will take about two to three lectures, and really the formulation of quant of classical field theory will happen only after three or four lectures. So, you have to be little bit patience, we will make a small mathematical the tour and come back to them.