Select/Special Topics in Atomic Physics
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Lecture - 8
Angular Momentum in Quantum Mechanics: Additional Theorem for Spherical Harmonics
Clebsch-Gordan Coefficients: – Coupling of Angular Momenta

Greetings, today we will discuss a theorem it is known as the Addition Theorem for Spherical Harmonics. And we will also get involved with Coupling of Angular Momentum.

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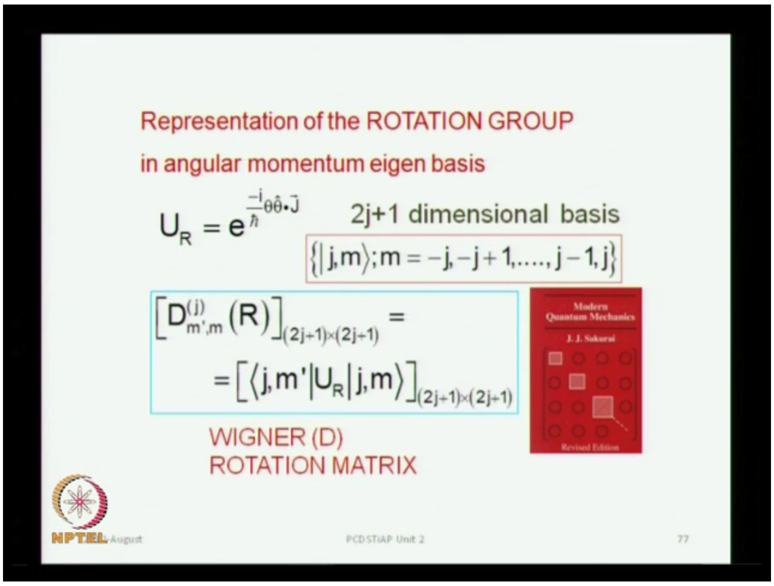

So, let us remind ourselves that we discussed the rotation group, we developed a representation for the rotation group in angular momentum Eigen basis. And this is the rotation operator, we have for any angular momentum j at 2 j plus 1 dimensional basis, and the matrix elements of the rotation operator, in this basis is what gives you the Wigner D matrices. So, these are the famous Wigner D rotation matrixes for different values of j and as j increases they become larger as one word expect.

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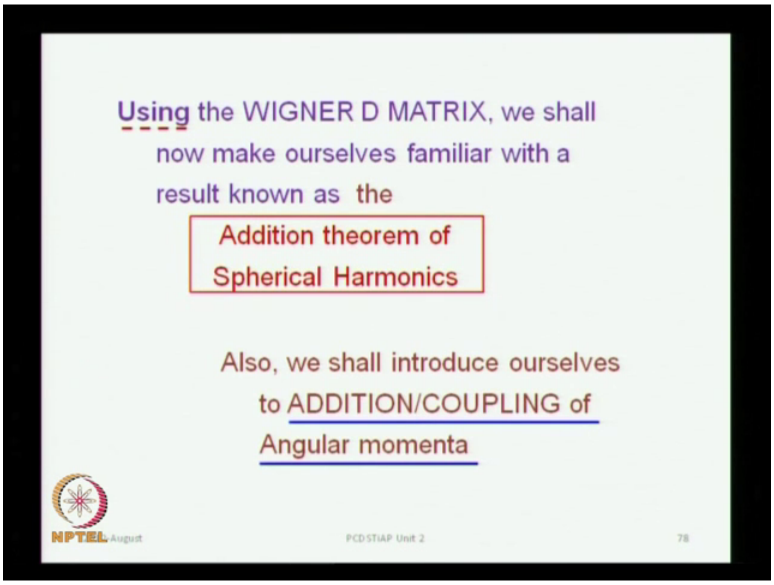

So, we will use the Wigner matrix to establish a theorem which is known as the addition theorem for spherical harmonics, it has a lot of application in atomic physics and also in nuclear physics. And then we will also talk adding angular momentum.

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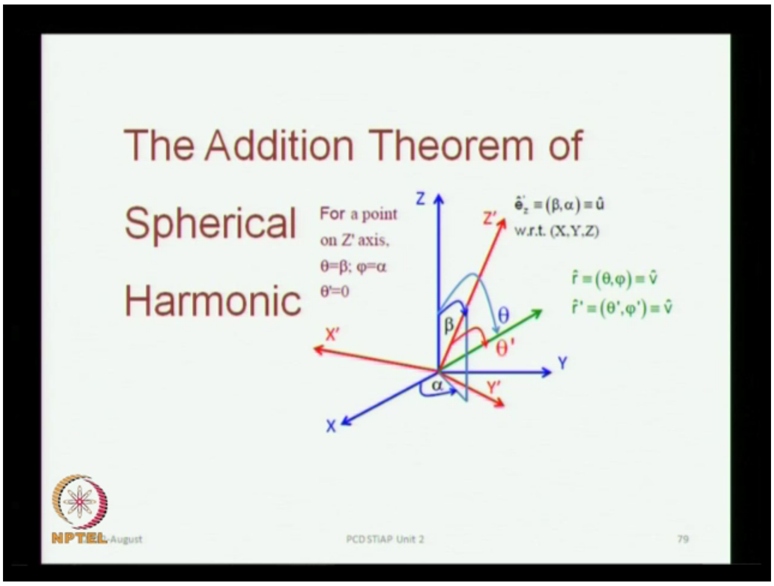

So, have a quick look at this figure, but I am going to develop this figure a little more systematically. So, you do not have to remember everything, but basically I am going to be using spherical polar coordinates, which all of you are familiar with, and I will reference spherical polar coordinates with reference to two different Z axis. One is a Z, which is a blue axis there are other a Z prime which is the red axis, and this one is tilted with respect to the previous one.

And the polar angle is always measured with respect to the Z axis or for the red frame it will be the Z prime axis that is the polar angle. And then there is also an azimuthal angle, which is measured with the Z X plane in the spherical polar coordinates, in the new coordinate it will be the Z prime X prime. So, just remember that and then we will develop this in some specific detail.

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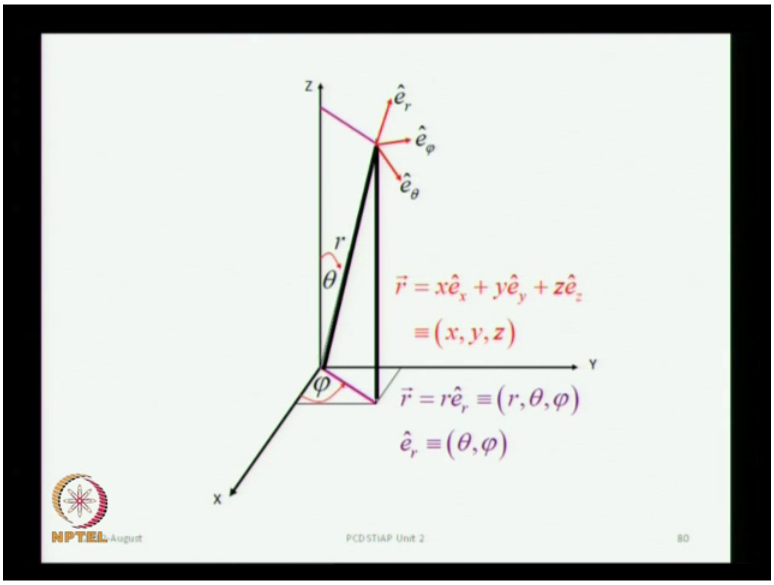

So, this is the summary of the spherical polar coordinates, and you have got the polar angle which is defined with reference to this Z axis. And if you draw a perpendicular from this point to the X, Y plane, the angle that this line would make with the Z, X plane, so this entire plane that you see over here, this makes plane an angle of phi with respect to the Z, X plane. So, that is the angle which is azimuthal angle, and any direction which is the direction of this unit vector r, this is specified by these two angles which are the polar angles. The polar angle and the azimuthal angle, so these are the angles which describe a direction in space, and any unit vector we specify the direction is thus equivalent to a polar angle and azimuthal angle.

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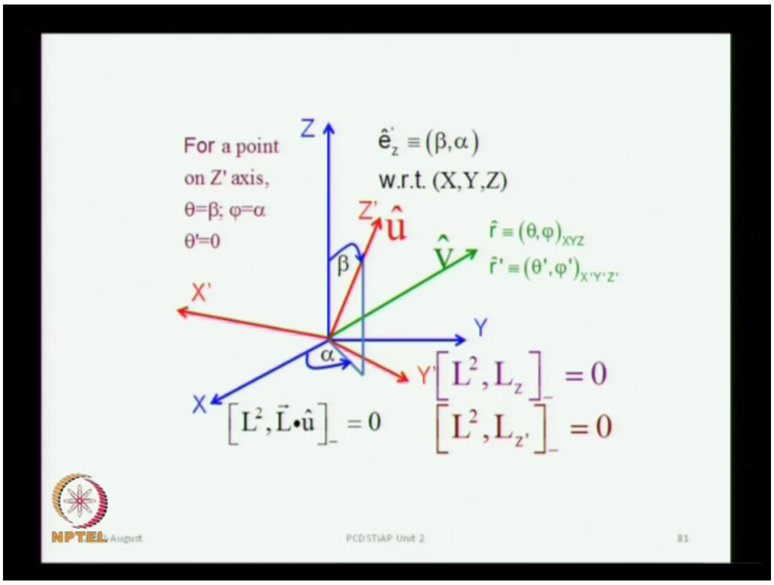

So, let us begin with the first coordinate system this is a Cartesian coordinate system, and let us have a new Z axis, which is referred to as the primed frame. So, the Z prime is tilted with respect to the Z axis, a longer direction which is specified by the unit vector u. Now, the angle that this unit vector u makes with respect to the Z axis, this is the polar angle of this direction with respect to the blue frame.

So, this polar angle which would have as theta this is a specific angle, which is beta and the corresponding azimuthal angle, which you get by dropping a perpendicular to the X, Y plane. And then connecting this to the origin, and then measuring this angle, so this is the alpha, so beta alpha give you the direction of the unit vector u, which is along the Z prime axis, which is the new Z axis, which is the tilted axis. And we are doing to refer to spherical harmonics with reference to the blue frame, and then also with reference to the red frame, and then see how they relate to each other.

So, this is the Z prime axis, so this referred to as the e Z prime, this is the unit vector this is described by the two angles beta and alpha. And now you have an X prime which is perpendicular to the Z prime axis, and then a Y prime as well, so that X prime Y prime Z prime, constitute a right handed frame of reference. So, this is your new frame of reference, and if you have any arbitrary direction in space like this, this will have different polar angles theta and phi with respect to the blue frame.

And different polar and azimuthal angle, which we call as theta prime and phi prime with respect to the Z frame Z prime frame right with respect to the prime frame. So, this is some arbitrary direction in space, which is shown with a vector from the origin, this has got a direction V caret. So, this is the unit vector and this has the angle theta and phi with respect to the X Y Z frame, and theta prime phi prime with respect to the X prime Y prime Z prime frame it is quite simple as such.

Now, having define this geometry let us consider a point of the Z prime axis, any point on the Z prime axis, you take this point whose polar angle theta is equal to beta. So, that is what we have written here, and whose azimuthal angle with respect to the Z X plane phi is equal to alpha. The polar angle theta prime, with respect to the Z prime axis, with the respect to the primed frame of reference is; however, 0 because it is a long it is on the z prime axis, so that is the geometry.

Now, we know that you can simultaneously diagonalize the square of angular momentum, and also one of it is components it does not matter which component. But, one component can be diagonalize, and if that component is along the unit vector u then you write the commutation relation that L square and L dot u commute. This could be the Z axis, in which case you have the commutation L square comma L Z equal to 0, you can also write L square comma L z prime equal to 0.

Because, you can always simultaneously measure L square and one of it is components it does not matter which component, you cannot measure two components. But, L square you can measure with any component, and there are infinite of them, so in L square you can always diagonalize with infinite unit vectors L Z different directions in space that comes from the isotropy of space.

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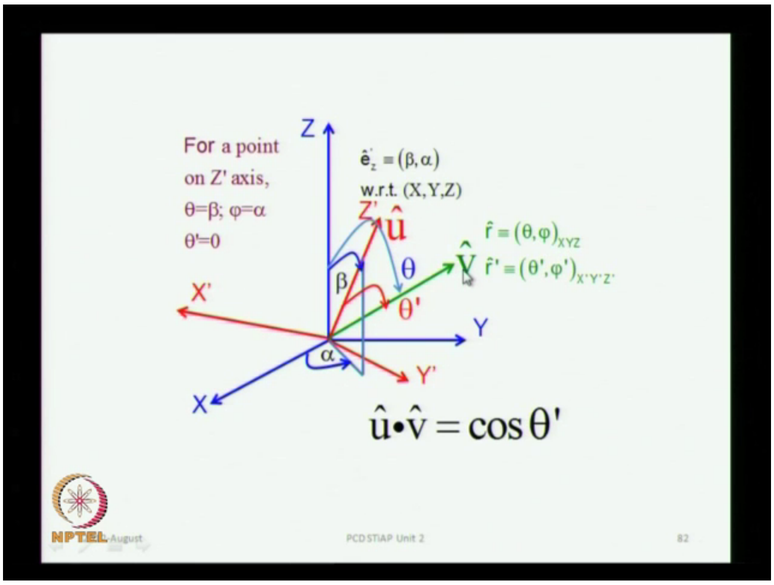

Now, you now have these two directions one is the direction u caret, this is a unit vector along which you have taken the Z prime axis, the other direction is V which is some arbitrary direction, whose unit vector reference to X Y Z is this r cater. And whose direction reference to X prime Y prime Z prime is this r prime caret, and these are specified by their respective polar and azimuthal angles, which are namely theta and phi in the X Y Z frame. And theta prime and phi prime in the prime frame of reference which is the X prime Y prime Z prime frame.

The angle between these two directions u and V is given by this cosine of theta prime because theta prime is measure with respect to the Z prime axis. So, theta prime is the angle between the Z axis, it is the angle between the unit vector u and the unit vector V alright.

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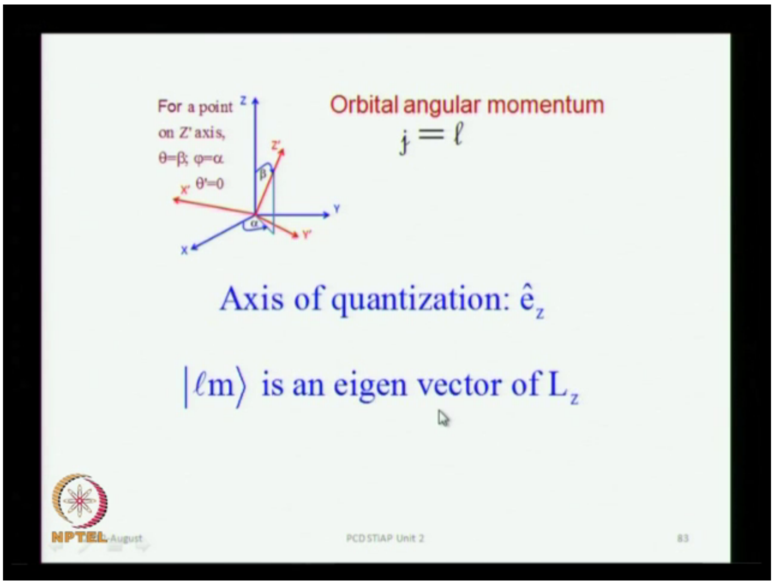

Now, let us consider the axis of quantization to be Z and you have an angular momentum Eigen state, which is an Eigen state of L Z.

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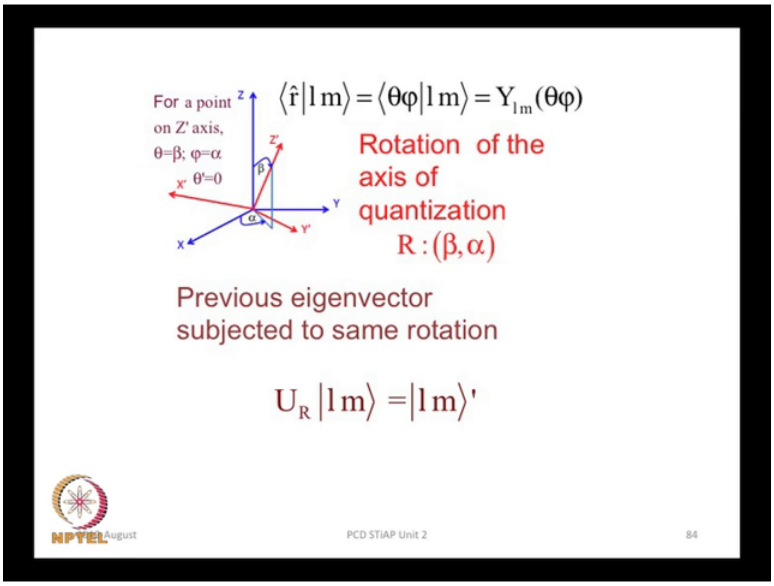

And what you are going to do, is to subject this angular momentum state to the same rotation that you have subjected your coordinate frame to. So, you tilted the coordinate frame from the blue frame you went to the red frame, and now you subject the angular momentum state to the same rotation. And this rotation is affected, through the angles beta and alpha as we have already found, so you take the previous Eigen vector, which is the Eigen state l m it is coordinate representation in terms of the angles theta and phi is what gives you the spherical harmonic right.

So, this is just the corresponding you know notation in the de Broglie notation you would right this as a spherical harmonic, in the direct notation you would write this as the coordinate representation of the state vector l m. Now, you subject this l m to the same rotation and you get a new angular momentum state.

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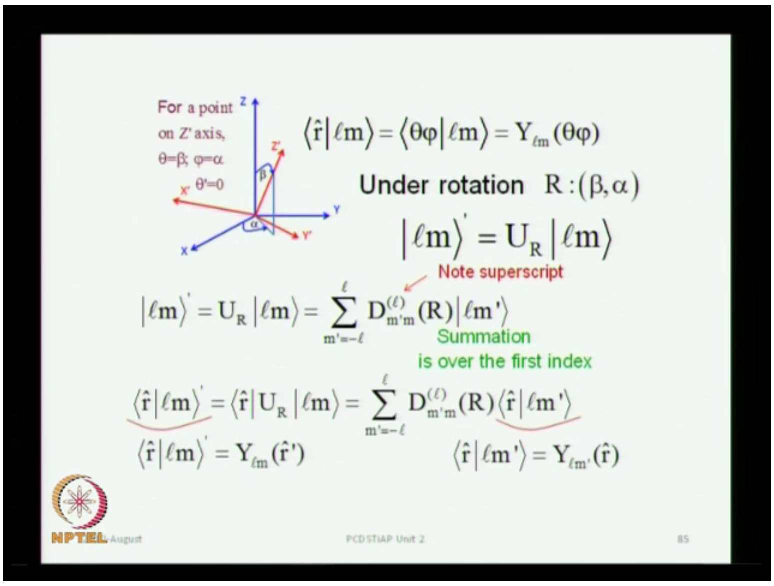

So, this is what you have got you get l m prime as a new vector, which you get by subjecting l m to a rotation, we know what this rotation is this rotation corresponds to the beta and alpha. And this would be expressible as a linear super proposition of the angular momentum states, in which m prime is the dummy index, it can take 2 l plus 1 values, those values going from minus l to plus l. And the matrix elements will be the matrix elements of the rotation operator, in standard matrix representation formalism.

And these are the matrix elements of the Wigner D matrixes, corresponding to the angular momentum l, so there is a super script l which must show up in this D matrix. So, now let us you know here I have used the summation index which is the first index, the first index is the dummy index, now you take the coordinate representation on both sides that is all you have done in this system, you have take the coordinate representation on both sides, so that you can write this expression in terms of the corresponding spherical harmonics.

And this is the spherical harmonics with the quantum numbers l and m prime corresponding to the direction r, and this one is the spherical harmonic with reference to the direction r prime because now you have rotate it.

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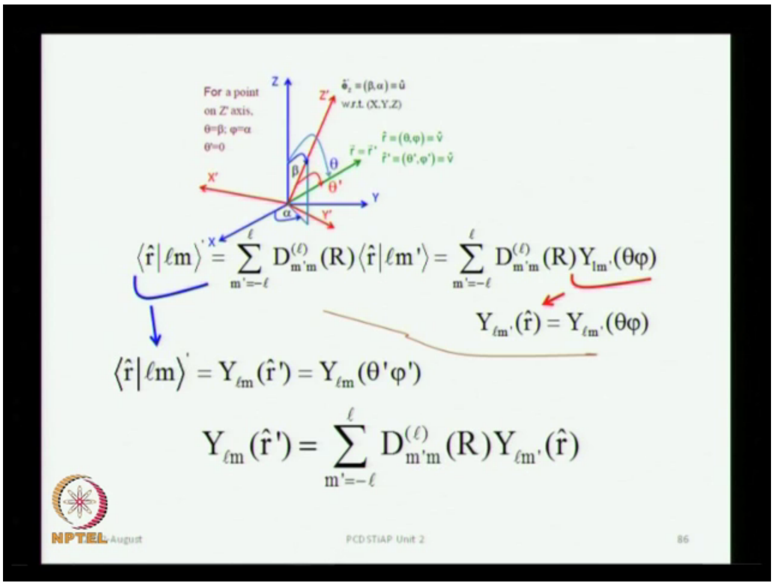

So, you got this relation now the same relation I have written over here, there is nothing new on the slide, so far. And you have got the spherical harmonic on one side, with the primed arguments, and here you have got the spherical harmonics with the unprimed arguments. So, this is in a nutshell the result that you get, the matrix elements in this expansion are the Wigner D matrix elements, as you can see that this whole proof is really very simple.

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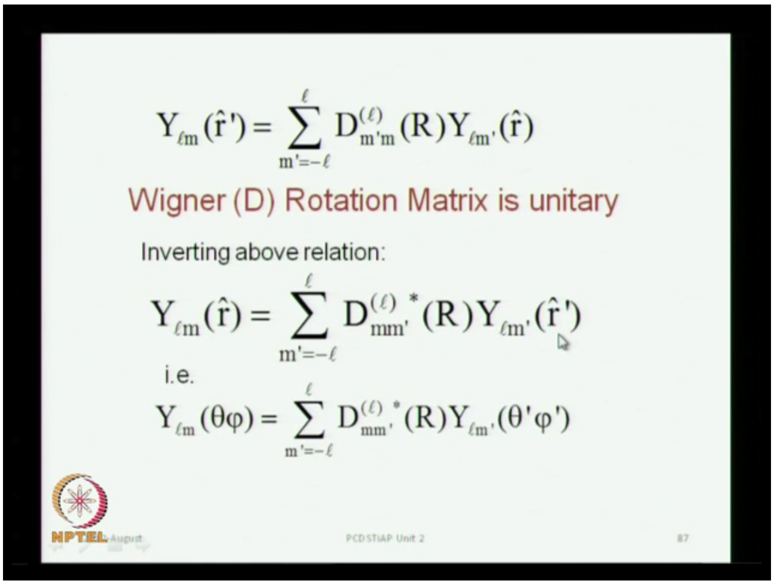

And you have got this relationship now, and what we will do is to invert to this relation, you had written Y l m of r prime in terms Y l m's of r as a linear super position of that. Now, let us carry out the inverse transformation and you know how to go from one ortho normal basis to another, and how to carry out the inverse transformation. So, I will not work out the details for you, and the invert relation will be this Y l m of r will be expressible in terms of the Y l m's of r prime, with these matrix elements.

But, you will have the complex conjugation and transposition over here right because these are unitary transformation. So, now, the summation index is the second and there is a complex conjugation which is shown by this asterisk, there is a star over there which tells that there is a complex conjugation. So, now we can write this relationship also in terms of the angles, so these angels are theta phi, which are completely equivalent to the direction r caret. And these angles theta prime phi prime are completely equivalent to the direction r prime.

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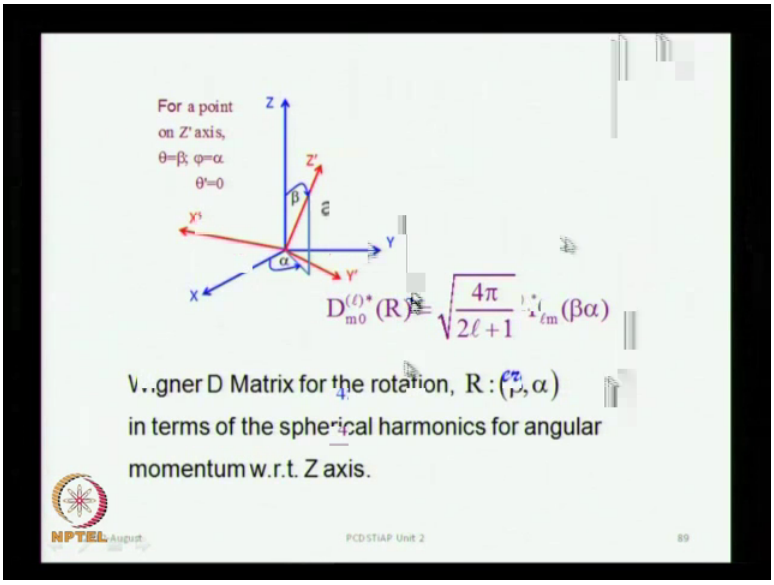

What have you got, we now take a particular case consider a point on the Z prime axis we have done this earlier as well. Now, for this point we know that theta is equal to beta right, for a point on the Z prime axis the polar angle theta, which is with respect to the Z axis of the original frame that polar angle is beta, phi the azimuthal angle is alpha. But, the polar angle with respect to the new frame, which is the red frame will be; obviously, be 0 because it is the point on that axis.

So, the theta prime is 0, so consider this special case and insert the value theta prime equal to 0 in this spherical harmonic over here. So, you have rewritten this relation over here once again, but given specific values to theta and phi, theta and phi are now beta and alpha. And theta prime is 0, now this is a spherical harmonic corresponding to the polar angle, which is 0 now no matter what the value of l and m prime is from properties of spherical harmonics, you know that it is always equal to the square root of 2 l plus 1 over 4 pi.

And this is the value it has only if and only if m prime is equal to 0 otherwise it vanishes, so these are well known of spherical harmonics, which you would have met earlier. So, we will use it and now we can carry out the summation over m prime and then because of this conical delta you will get a term only one term will survive corresponding to m prime equal to 0, which is this. And now you will get an expression for the matrix element of the Wigner D matrix in terms of the spherical harmonics, because you have expressed the spherical harmonics in terms of the matrix element of the Wigner D. So, you can just invert that relation and what you have is an expression for the Wigner matrix elements, Wigner D matrix elements which are given by essentially they are spherical harmonics as you can see.

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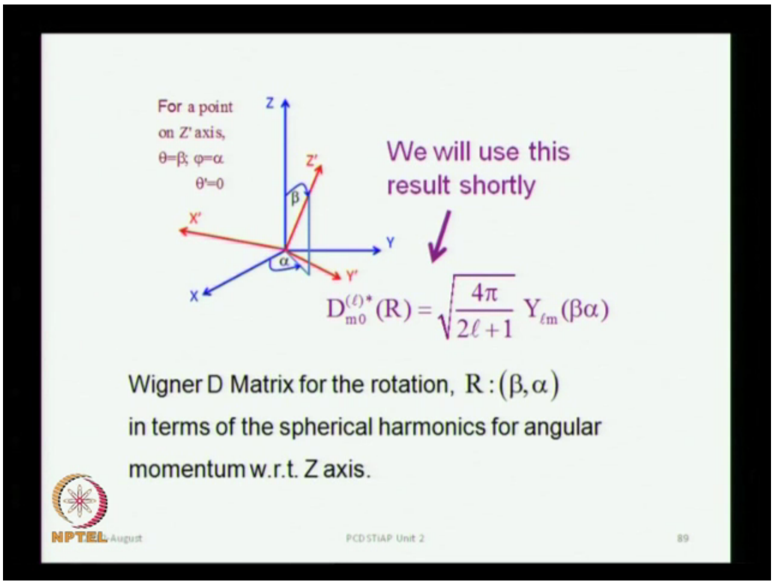

Now, this is a general result which you can use, so we have got this result for the Wigner D matrix, in terms of the spherical harmonics. And we are going to use this to establish the theorem, which is known as the addition theorem for spherical harmonics, so we will use this result. Now, you do not have to write down all of these in your notes because all these files are uploaded on the course warp page, so you know all the relationships are there, so you do not have to copy anything.

But, just keep track of the logical development of the topic and that saves us a lot of time, it saves me a lot of time to this write on the board it saves you the time to write it in your note books. But, all the information is available and the we can focus on the discussion, so that is the idea.

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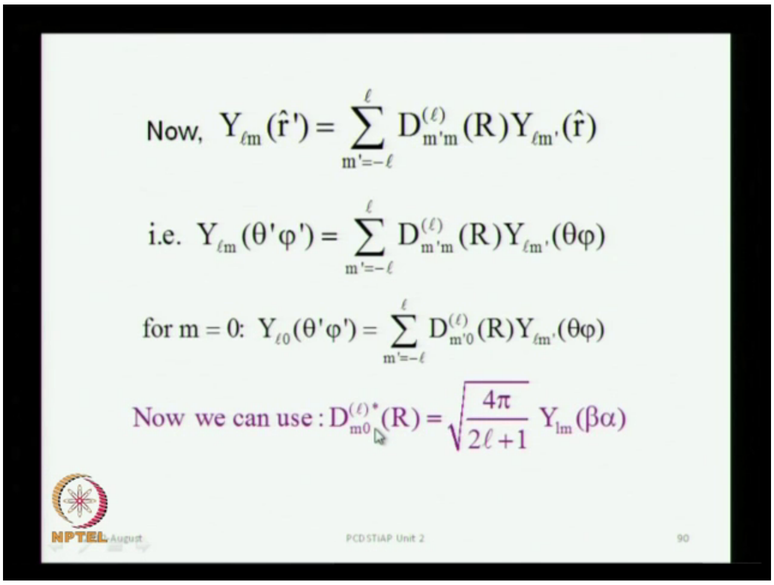

So, I begin with the earlier expression for spherical harmonics with reference to the primed angles expressed in terms of the unprimed angles, which we already had earlier. And in this, if you now put m equal to 0 you specialize this relation, which is valid for every value of l and for every value of m, and I take a particular case mainly the case for which m is equal to 0. So, I take this spherical harmonic with m equal to 0, so that what is I write over here, so this is the m equal to 0.

And at the right hand side, I have this m equal to 0, but I know what it is value is because we just found that a little while ago. So, we can plug it in that was in terms of spherical harmonics right, so we are going to use this relation which we had obtained earlier, keep track of the complex conjugation of course.

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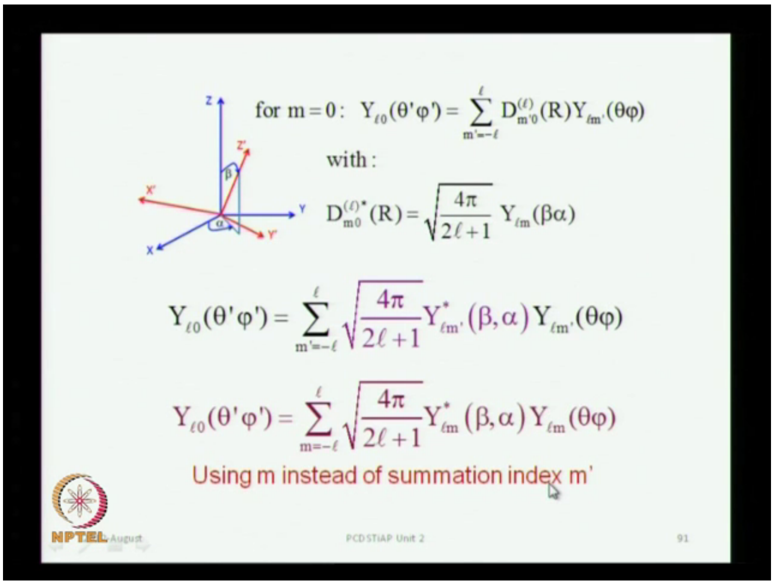

And using that this relation that we had obtained earlier, we get an expression for this spherical harmonic on the left hand side for m equal to 0. So, this m equal to 0 comes over here, the arguments here are the primed angles theta prime and phi prime, these are expressed in terms of the spherical harmonics with angles theta and phi, which are the unprimed angles. And the coefficients, which are coming from the matrix elements of the Wigner D matrix, are again spherical harmonics with respect to a specific angles beta and alpha.

We know what those angels are and then of course, there is the square root of 4 pi over 2 l plus 1 and because there was a complex conjugation over here, we now have a complex conjugation over here. So, you must always be careful that you do not lose any information, while substituting these terms, now the right side is a summation over m prime, and m prime it makes sense to use m prime to distinguish it from some unprimed m. But, since there is none in this relationship we might as well drop the prime altogether, it is just a dummy level which is summed over any way, easier to write it without the prime. So, I have rewritten this relationship with m used instead of m prime.

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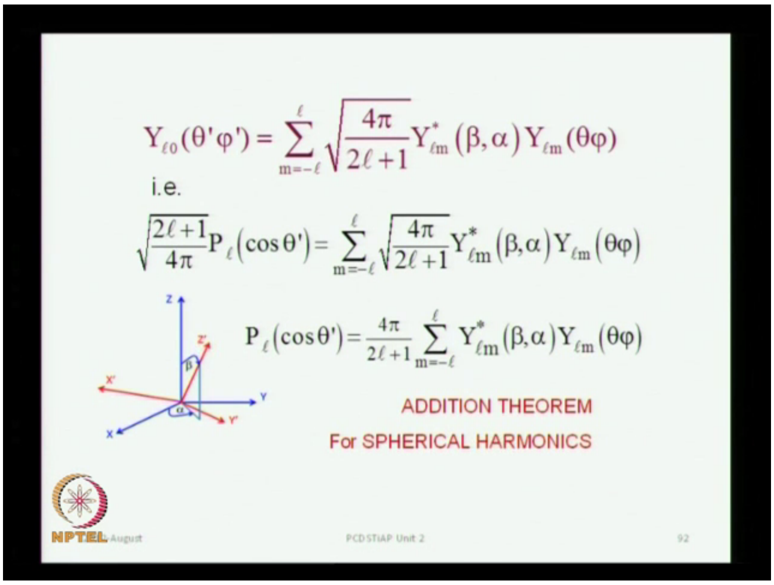

So, that is what we have got summation over m this spherical harmonic for m equal to 0 is nothing, but this polynomial function there is another polynomial, which you have met right for m equal to 0. So, the left hand side these are well known properties of the spherical harmonics, and then you get a relationship for P l cos theta in terms of the spherical harmonics, which is this. So, you take this factor root of 2 L plus 1 over 4 pi on the right. So, you get this 4 pi over 2 l plus one on the right.

And then you have got a summation over a product of spherical harmonics done, this is a theorem which is known as the addition theorem for spherical harmonics. And as you can see it is proof is really quite straight forward, but you will find that it is extremely powerful theorem, and you will find very many applications of this and there are a lot of angular momentum algebra that you will be doing. You will need to plug in this result every now and then so you would really need to have a good handle of this theorem.

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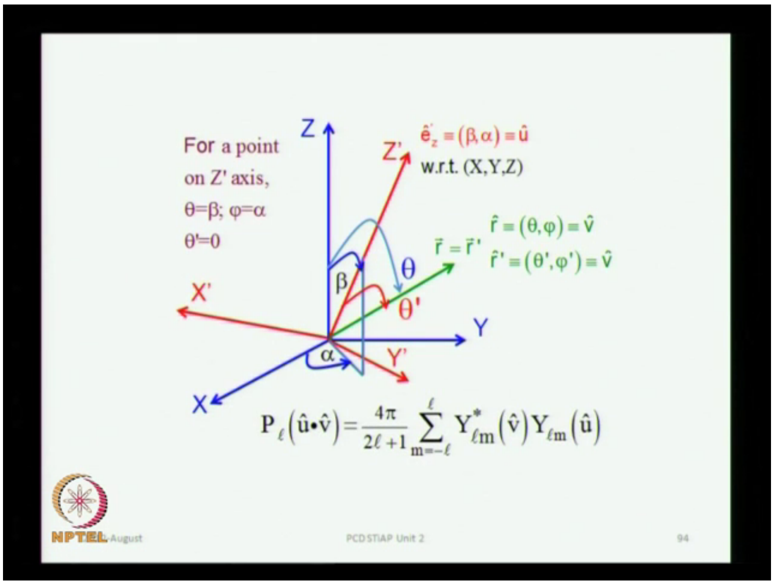

Now, we have these directions, so this is just the remainder of what this geometry is and essentially this angle cosine theta prime, theta prime is the angle between these Z axis and this axis. So, if this is the unit vector u and this is the unit vector V this is the angle between vectors u and V, so you can write this for two arbitrary directions u and V it is exactly the same theorem.

But, now written in the form which makes it look much more general because you can now relate it to any two arbitrary directions no matter what those directions in space are and the direction in space is shown by a unit vector u. And now this result that you now have at the bottom is completely general because it does not make a reference to the red frame or the blue frame and you can tilt these frame, any which way you like for any two arbitrary direction in space.

And that is really what makes it, so powerful for any two arbitrary directions in space, if you want to know what is the general polynomial corresponding to the cosine of the angle between these four directions. Then you can always write it as a sum over products of corresponding spherical harmonics and this is the result.

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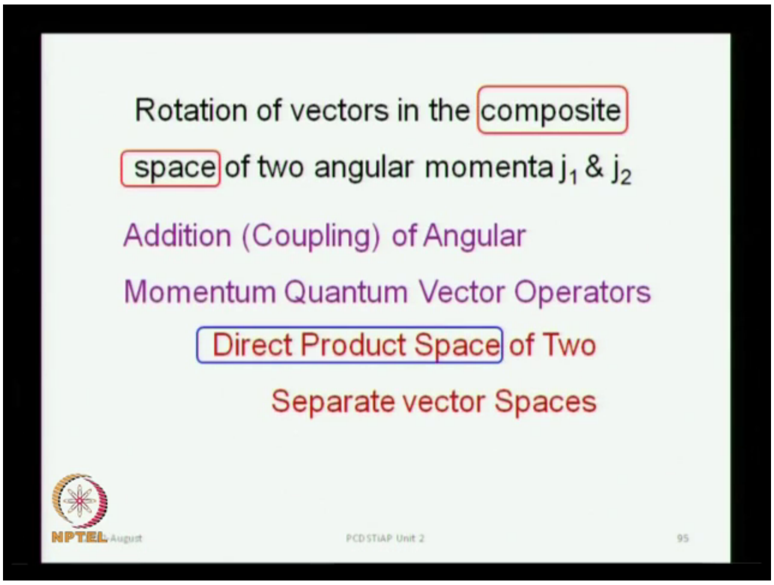

So, this is a very powerful theorem and what we will now do is to consider not one, but two angular momentum why not. Even each electron has got two sources of angular momentum, it has got the orbital angular momentum which has got nothing to do with the orbits. It also has spin angular momentum, which has got nothing to do with this spin over top, but it does have a spin angular momentum which is completely independent right, so it is a different degree of freedom.

Moreover, you are not going to be doing atomic physics which has the hydrogen atom, and you deal with atoms with more than one electron, you will have 2 electrons, 3 electrons, 10, 20, many right. And each electron will have it is own angular momentum, it will have it is own orbital angular momentum, it will have it is own spin angular momentum. And the atom is going to have a net angular momentum, which is coming from the addition of all of these angular momentum.

So, you need to learn how to add these angular momentum and the angular momentum in classical mechanics is a solo vector, you know how to carry out the addition of these vectors. But, the vectors we are now talking about are not just vectors, they are quantum, vector, operators three attributes right, so you are not going to use the law of addition of vectors, which is the addition law, the triangle law or the parallelogram law of addition of vectors as you call it right, that is not the law that you can use that is a law which you can use for vectors that is not the law for quantum vector operators.

And that is a law that we now have to learn, so this is the addition of angular momenta, and what it will do is this addition corresponds to a rotation of vectors of a composite space. So, you have got two vector spaces now, one which is an Eigen space of one angular momentum j 1, and there is another vector space which is the vector space of the angular momentum j 2, what do we mean by angular vector space of j 1. This is the vector space, which is spanned by the Eigen basis of j square and j z, when j is equal to j 1.

So, for j 1 square and j 1 z you have got a certain vector space, likewise if you have another source of angular momentum, which you call as j 2 it will have it is own Eigen space, which is spanned by the Eigen basis of j 2 square and j 2 z, and you are not going to work with this composite space, made of these two Hilbert spaces. So, you are have to learn how to compose this composite space, composite you have already implied a composition, composition in walls putting things together right.

And there is a certain law that has to be prescribed of how you put it together, so this is called as the product space of the two separate vectors, this is called as the product space. So, these are the keywords the composite space and the direct product space of two spate vectors spaces.

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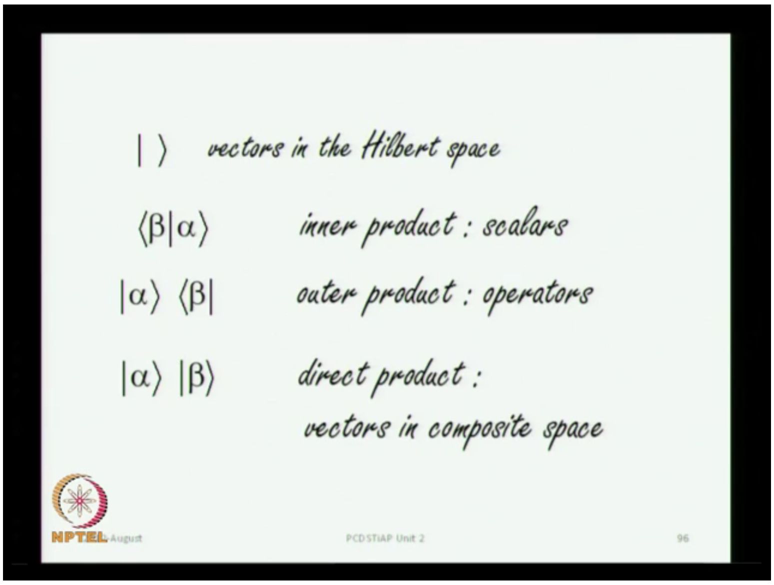

And typically you represent a vector in a Hilbert space by a ket or what can you do with these vectors, you introduce inner products you know what they are. So, corresponding to each vector in the Hilbert space, you define an ad joint vector in the ad joint space right. And between these you introduce products, which you call as the inner products you also introduce outer products.

Now, the inner products are scalars the outer products are operators, so this operator would operate on some of the vector, and give you a new vector, so this is an operator. So, there are different kinds of products that you compose from vectors, one is an inner product which gives you a scalar, you also introduce outer products which give you operators, and now we introduce also direct products. So, one the inner product is a bracket, the outer product kept and the product is a ket, ket.

And this is what is called as a direct product in the composite space, one of which comes from one Hilbert space and the other comes from the other Hilbert space. And then you define this product, so it also called as a product which you do not want to invent your words. So, you use the same words again and again they are all products, but with new meanings, and each meaning is well defined in it is own context.

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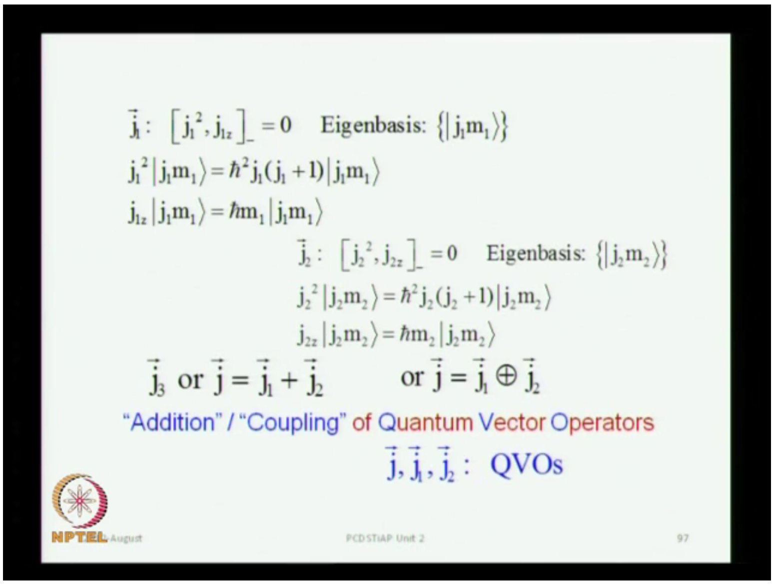

So, you have got an Eigen space of j 1 which comes from Eigen vectors of j 1 square and j 1 Z which can be simultaneously diagonalize. You have got an Eigen space of j 2 which is spanned by the Eigen vectors by the simultaneous Eigen vectors of j 2 square, and j 2 Z. And now we introduce this composition, which is the addition of j 1 with j 2, you can call it as j or as j 3 it is the same that is the third angular momentum, which is the final net angular momentum you get.

And because this is not just the usual addition of two vectors you can represent it with a different symbol, you can put the plus sign in the circle or invent a new symbol to represent these binary operation. It is essentially a new prescription of new binary operation, which is being introduced to define the addition of these two angular momentum. And it is a good idea to use a different symbol at least once in your life the reason is, it tells you that this is not the same kind of addition as addition of two vectors it is not. But, having done it you can use the same symbol, so it does not matter what symbols you use, these are quantum vector operators not ordinary vectors. So, all the three the j 1, j 2 and j 3 or j 3 are quantum vector operators.

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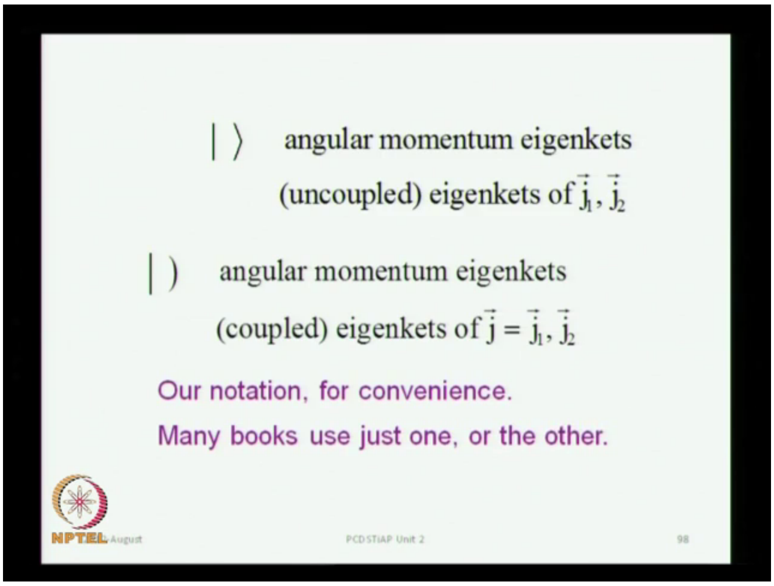

And I will use a notation, which is the angular kets, the angular vectors look at this vector on the screen. And this vector is an angular ket, which I will use to represent the Eigen space of the uncoupled Eigen vectors of j 1 and j 2 I will use a different notation with these circular brackets, to represent Eigen kits of the coupled angular momentum. Now, this notation is not very standard, it is not very specific, it is not essential there are many books, which do not use different notation.

But, I am going to use it because it offers a little bit of convenience which you will find as we go along. Because, once you start putting in numbers over here, once you have angular momentum you do angular momentum addition for j 1 equal to 3 half and j 2 equal to half and so on right you are going to have to put in numbers. And then it becomes easy to keep track which was the uncoupled vector and which is the coupled vector of course, you know it from the context, so it is no not such a big deal.

But, having an additional suggestion as to which is the to uncoupled vector and which is the coupled vector is often useful, at least to begin with till you get used to it. So, we will use angular momentum Eigen kets, with these circular brackets to represent Eigen kets of the coupled angular momentum, and this is being introduced only for notation, and this is not a very standard notation indifferent books you will find different notations and you should not worry if you see different notations.

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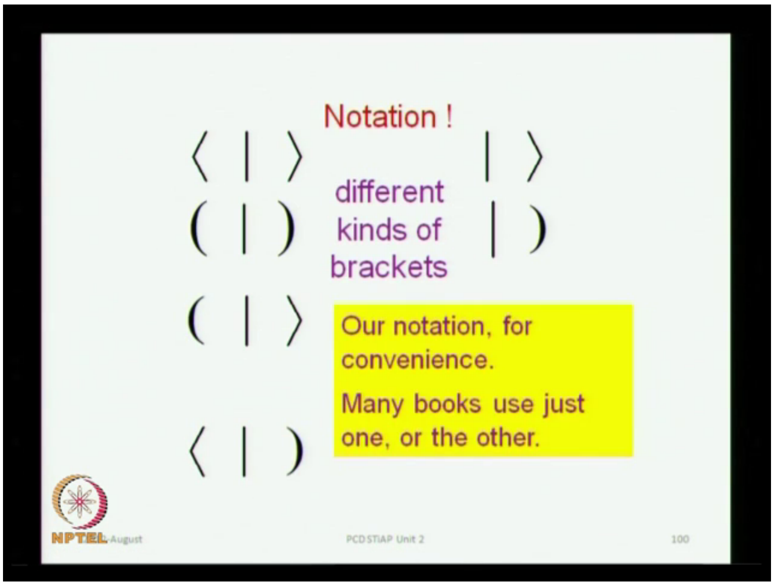

In fact, one could use different kinds of brackets right to represent the inner products as well, you can also use some more notation. And some more and we will use some of them to our advantage, and I will make a distinction between the angular brackets, the circular brackets, and what do you call these beautiful brackets this is what I call as a beautiful bracket. And we will use all of them, it is not mandatory, but it is useful.

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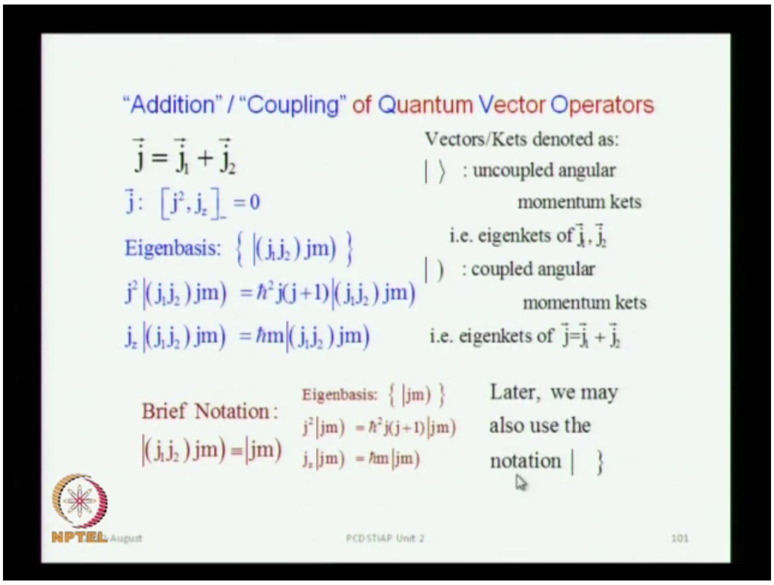

So, you will not necessarily find it in many books because it makes some analysis easy especially when you do problems, you will see how it turns out to be useful. So, now, we have these angular brackets which are the uncoupled angular momenta, and the circular which are the coupled angular momenta. Now, the coupled angular momenta are j m's right and they are coming from coupling of what, they are coupled therefore, there have to be two other angular momenta, which are j 1 and j 2.

And that is impassive and you do not necessarily have to write it every time, so you can make this notation compact by suppressing this j 1 j 2. So, this a compact notation for the coupled angular momentum, but if you want you can write all of it as well, and sometimes at least to begin with when you are doing problems if once when you are getting, acquainted with these techniques it is good idea to write all the quantum numbers.

So, this has got Eigen basis which is a Eigen basis of the coupled angular momentum, which is j square and j Z and they will have Eigen states given by two quantum numbers j and m. Because, both of these are simultaneously measurable, and later I am going to use this beautiful bracket as well.

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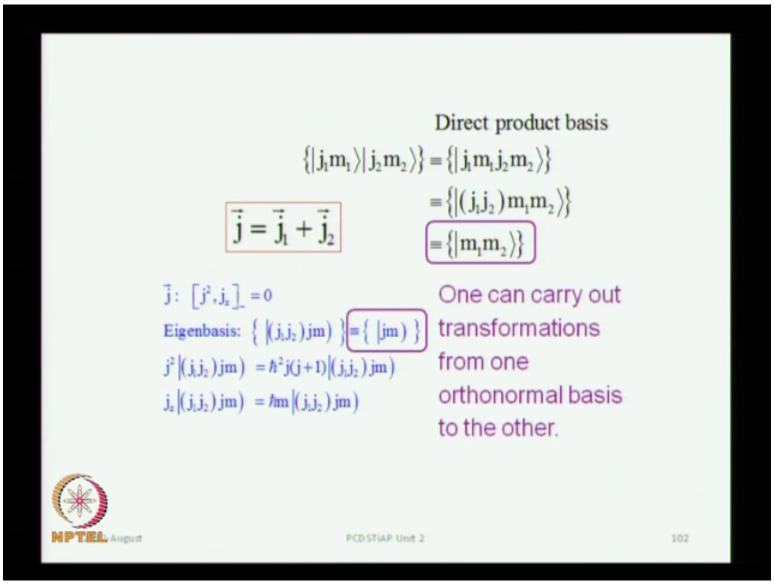

So, here we have the coupling of the two angular momentum j 1 and j 2, so now, you have got two alternate basic sets, one has coming from the direct product of the spaces, which are the Eigen spaces of j 1 and j 2. The other is the Eigen space of j square and j z which is the coupled angular momentum is the same space, you are only referencing it differently. And therefore, you can always carry out ortho normal transformations, unitary transformations from one basis to the other.

So, one basis which is called as the direct product basis, this is a basis if the direct product of the Eigen vectors of j 1, and the Eigen vectors j 2. These are the Eigen vectors of j 2, these are the Eigen vectors of j 1 what you have in this beautiful bracket is the direct product of j 1 m 1 and j 2 m 2. This gives you one basis, you know that the dimensionality this will go from you know each m 1 will go from minus j 1 to plus j 1. So, that dimensionality will come from 2 j 1 plus one times 2 j 2 plus 1 because m 2 will take 2 j 2 plus 1.

So, that will be the dimensionality of this basis and the dimensionality of the basis of the of the coupled vectors better be the same, but that is something that we really have to establish. Because, we know what j and m can take we know that m can go from minus j to plus j, and j will also have a certain range which perhaps you have learned in your first course in quantum mechanics, but we have to establish what that range must be like.

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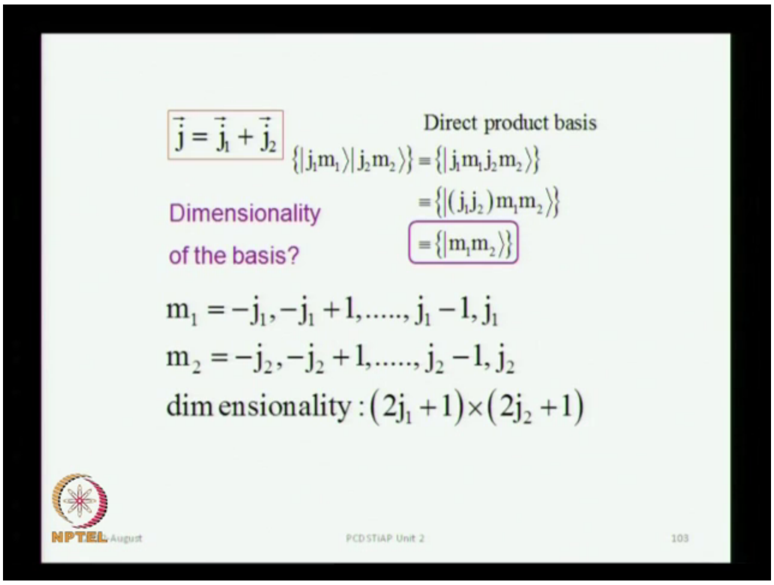

So, that is going to be part of our exercise, so we have to find this dimensionality it better turn out to be what we expected to be, it will turn out to be what it is, but it is not a result that we will take for granted, we will actually prove it. So, this dimensionality of the direct product basis, this is not very difficult to see it is; obviously, 2 j 1 plus 1 times 2 j 2 plus 1.

The dimensionality of the basis of the coupled basis is not obvious, we expect it to be equal to this. But, it is not obvious at least for the beginner for the experts amongst you know the answer, you know the answer and I hope that you know the proof for them, so we will discuss the proof for that.

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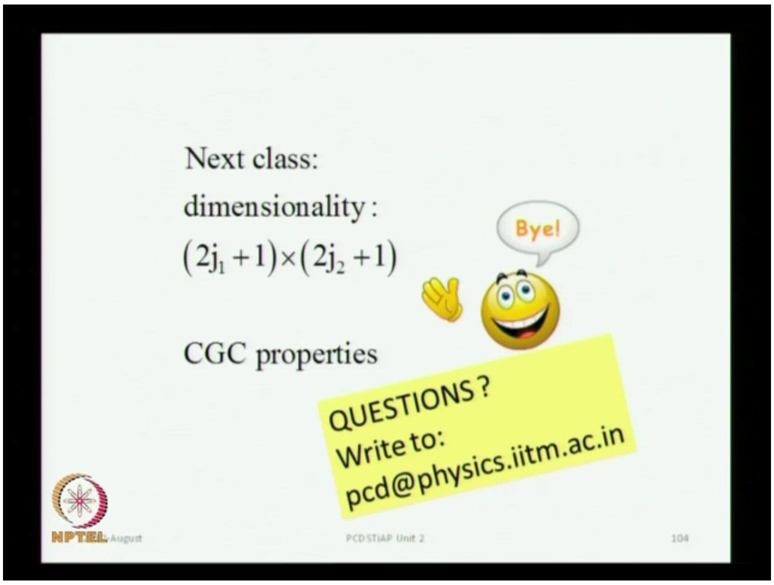

So, I guess I am going to take some questions at this point because essentially what I am going to introduce now are what are called as the clebschgorden coefficients. These are the scalar coefficients, when you go from one expression of any vector and you can express it, in terms of super position of Eigen vectors of the uncoupled vectors. All as a super position of Eigen vectors the coupled vector, and you can carry out transformations from one to the other.

So, that is a complete comprehensive topic by itself which I thought I will in a separate class which will be the next class I will take, in the meantime if there are any questions over here I will happy to discuss. Yesterday's question I assumed that you already found what the answer is, so why need an inhomogeneous magnetic field for the ((Refer Time: 40:45)) because u dot v is just the energy, but you need a force right to separate the spin up component from the spin down component, and that force can come only from the gradient. So, that is where you need the inhomogeneous magnetic field, so that is no big deal about it right, any other question.

So, thank you all very much.