Select/Special Topics in Atomic Physics
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Lecture - 18
Relativistic Quantum Mechanics of the Hydrogen Atom

Greetings, we have come a long way in Relativistic Quantum Mechanics, the subject is vast and there is so much one can do, but I would like to introduce you to the solutions to the coulomb problem, that is the hydrogen atom problem. The reason is that when you read any literature in relativistic atomic physics, you are going to deal with atomic wave functions and the probability densities, charge densities, and then various atomic processes such as collusions from atom or photo absorption by an atom and so on.

And the first thing that I am going to hit the i is the atomic wave function, and the atomic wave function will be described by a certain set of what you called is good quantum numbers. And these are not n l, m l, m s these are relativistic quantum numbers, the l n and s, m l m s these are not good quantum numbers, so you need to recognize what the good quantum numbers are. And the subject being vast, I have planned to at least give you a basic introduction to this, so that you recognize what the good quantum numbers are, and how to process information about relativistic wave functions.

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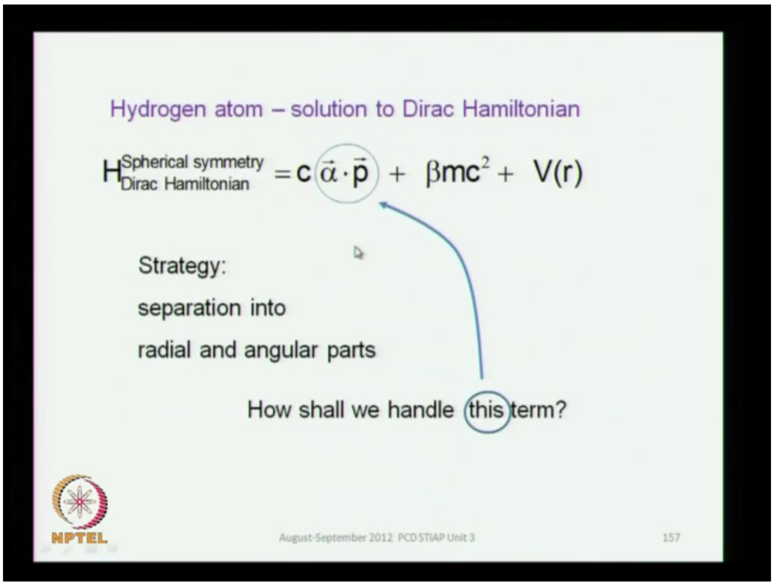

So, let me remind you that our strategy to deal with hydrogen atom is to set up those spherically symmetric Dirac Hamiltonian, so this has got spherical symmetry, this is the Dirac Hamiltonian. And now you need to separate the radial and the angular part, and for relativistic case this is not at all a straightforward task, it is not trivial at all, it is not difficult, but certainly far from trivial. And you will see why it is complicated, because this term alpha dot p has to be handle very carefully, and we are going to do that in today's class.

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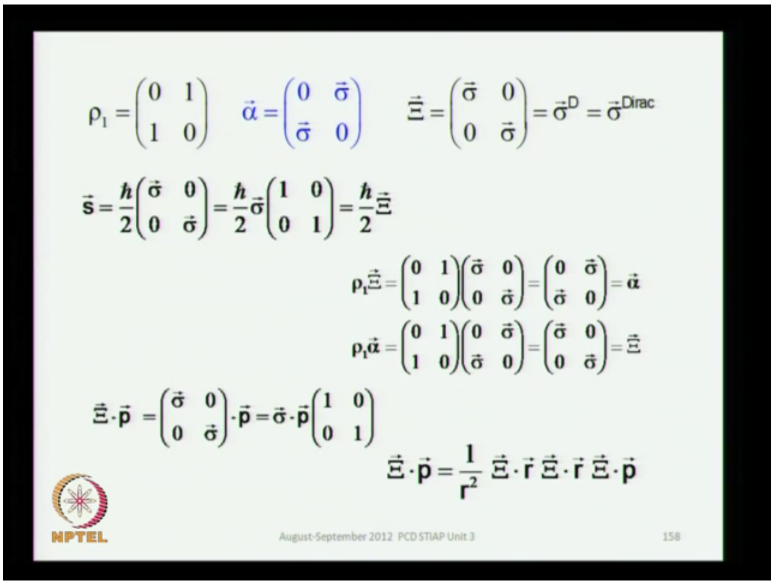

So, in the few minutes, we had before the last class ended, I introduced the sigma matrix with is the Dirac's sigma, so I will quickly remind you about our notation. So, your spin is h cross by 2 sigma and then, depending on how you interpret this sigma or this sigma, this is just a matter of notation. And some books used this, some books used this, but depending on the context whether it is the Pauli sigma or a Dirac sigma. Dirac sigma is the 4 by 4 matrix operator, Pauli sigma is 2 by 2, so that is the essential difference.

And this matrix rho is 0 1 1 0, and rho sigma gives you alpha and rho alpha gives you sigma, alpha is the Dirac matrix, which is the 0 sigma sigma 0 that is the alpha matrix. So, now, let us see how to deal with this term sigma dot p, and sigma dot p I have prefixed this by a unit operator, because omega dot r is projection of sigma along some direction. And this direction, this is only direction because this is a position vector divided by the length of the vector, which is r over r will give you the unit vector; and there are two of these, there is an r position vector here and another over here. So, sigma dot r sigma dot r over r square is the square of the projection of sigma along some arbitrary direction. And that sigma square along any direction is equal to 1 as we know, very well from our experienced with the Pauli sigma.

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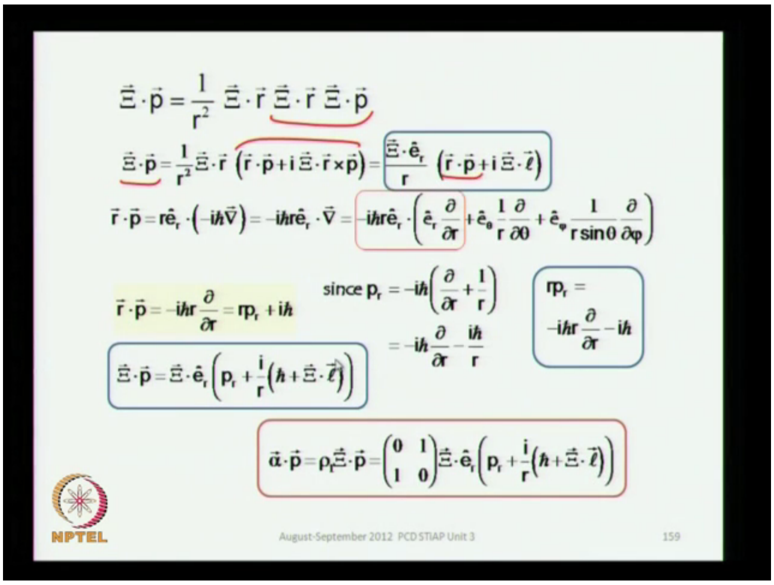

So, now this term can also be treated very easily using the Pauli relation, which we have used very extensively, and this is therefore r dot p plus i sigma dot r cross p. And now from the r cross p you have got the orbital angular momentum vector here and then, from the r dot p you can express the momentum operator, which is the gradient operator explicitly spherical polar coordinates. And then, you are left with only this component, because you will take the partial derivative of some function with respect to r, and then take the dot product with this e r, so only the e r dot e r term will contribute.

So, that is the reason only that one is the significant, you get the radial momentum which is not just minus i h cross del by del r, but this is the same result as in non relativistic quantum mechanics. So, r dot p is this r p plus i h cross. and this is the term that you can use to put in this expression sigma dot p, so sigma dot p becomes sigma dot e r which is coming from here. And then, you have p r plus i over r h cross plus sigma dot l, now pay very special attention to this operator here, h cross plus sigma dot l and this is coming up for very special consideration.

It has got a structure which you have not made before, and it is going to play an extremely important role in relativistic quantum mechanics, as a matter of a fact it will be connected to a quantum number, that we will get for a relativistic quantum wave functions.

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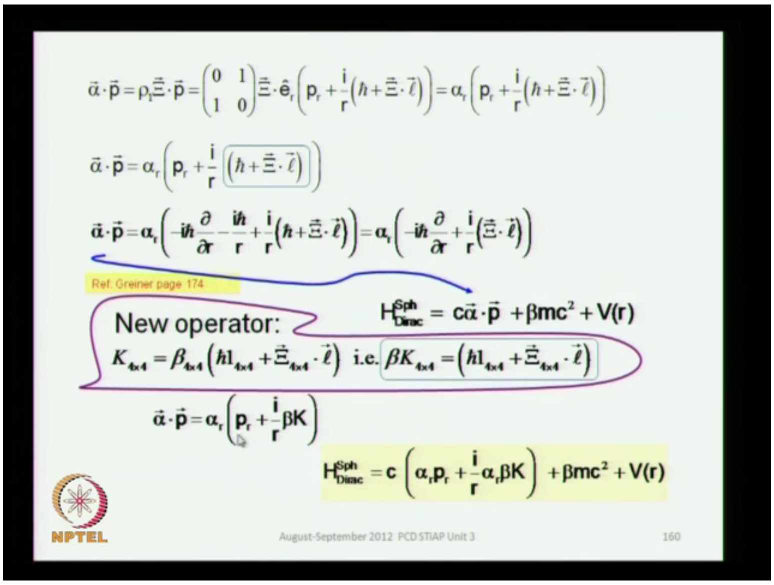

So, you have got this h cross plus sigma dot l coming here, and now you have alpha dot p written in terms of some radial features, but then there are also r some angular features here. So, it is not completely free from angle dependence, but sigma dot p is now, this alpha dot p now has some radial features which will coming handy, when we want to separate the radial part from the angular part, I have alerted you is not a trivial thing. And you will then be able to insert this in the expression for the alpha dot p in your Dirac Hamiltonian. So, this is where we introduce a new operator, and this is h cross plus sigma dot l which is here, but it is pre-multiplied by beta, which is what is gives a new operator K.

And if you pre-multiply K by beta, you will get beta square, which is equal to 1 and that is what will give you this h cross plus sigma dot l, which is the operator you have seen here, so this has got K itself, but it is a beta K. So, we now introduce a new operator which is called as K, it is defined as h cross plus sigma dot l and in terms of this K, this alpha dot p now becomes alpha r p r plus i over r and this h cross plus sigma dot l is beta

K. So, this is now your alpha dot p and now you can insert this in your Dirac Hamiltonian, and find the beta K over here.

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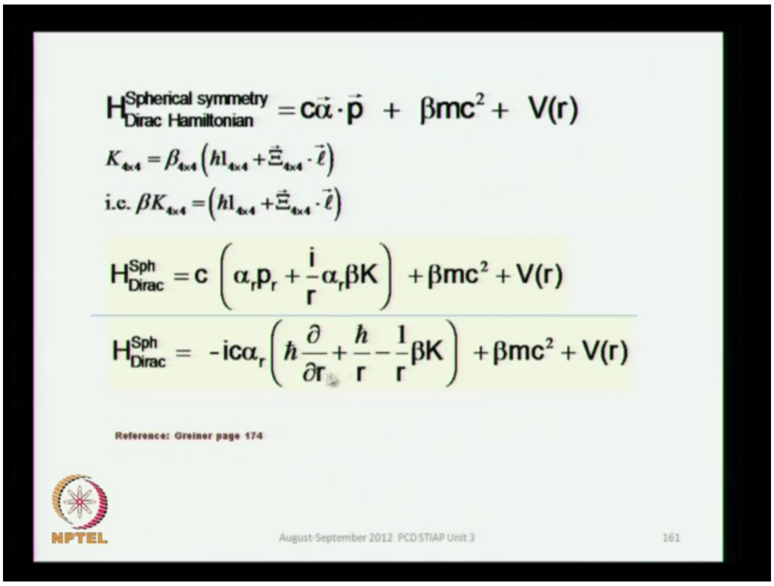

So, this is what we have got, the beta K appears in the Dirac Hamiltonian you can write p r in terms of the derivative operators, and what the radial momentum operator is, so you have two term del over del r and then h cross over r.

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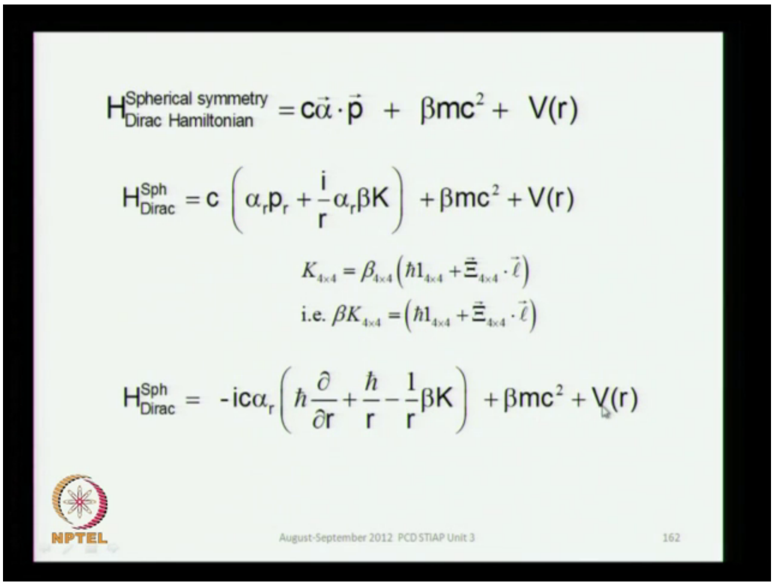

So, you can write it in some equivalent forms, and you have the beta K, so this is your expression for this spherical Dirac operator. It has got the alpha r, it has got a beta K here, and then these two terms beta m c square plus V r of course, we will continue to be represent.

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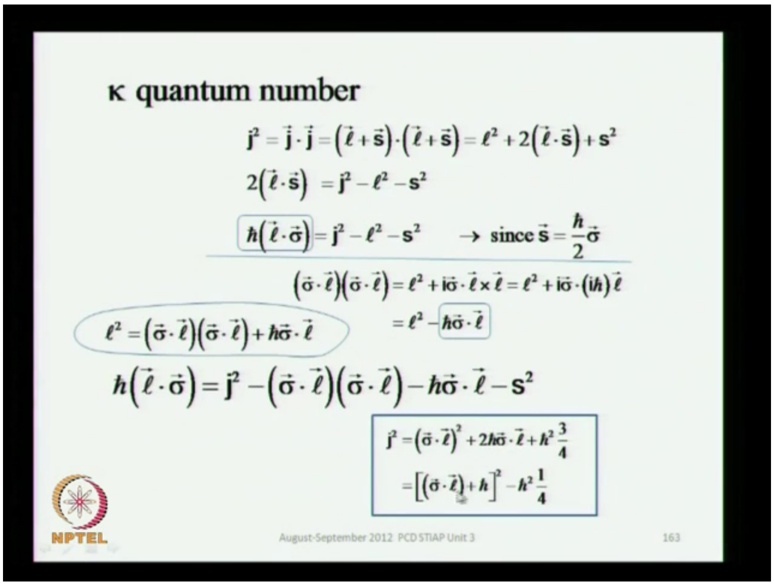

So, the K operator is going to give us a new quantum number, which is called as a kappa quantum number, so this is not the lower case K it is written as kappa that is the notation you will find in most of the literature. And those of you are reading about atomic wave functions, would have met the kappa quantum number. So, this is the kappa quantum number and to understand this, let us look at j square which is l plus S dot l plus S and that gives you twice l dot S equal to j square minus l square minus S square. And since S is a h cross over 2 sigma, you have h cross l dot sigma equal to j square minus l square minus s square, now these relations we have used earlier as well.

So, this h cross l dot sigma, you have another expression for it coming from the Pauli identity which is sigma dot l sigma dot l will give you l dot l, which is l square plus i sigma dot l cross l, but l cross l being angular momentum, it is i h cross l, it is a orbital angular momentum. So, you get h cross sigma dot l coming from the second term, and l square is over here and you can write l square in terms of the sigma dot l. So, l square is now the square of sigma dot l plus this is the minus sign here, so it will come a plus sign here, h cross sigma dot l.

So, you can factor, you can pull out the appropriate factors this l dot sigma is j square minus l square, but l square now has got these two terms and then, you have got this minus h square. So, now you notice that j square, if you write this expression for j square take all these terms to the other side, then j square you will have a squaring dot l, then you have two terms in h cross sigma dot l, which is twice h cross sigma dot l. Then you have h cross square 3 4th, because S square is half into half plus 1 that is 3 4, so you get 3 4 h cross square and you immediately recognize that, this is a whole square minus h cross square by 4.

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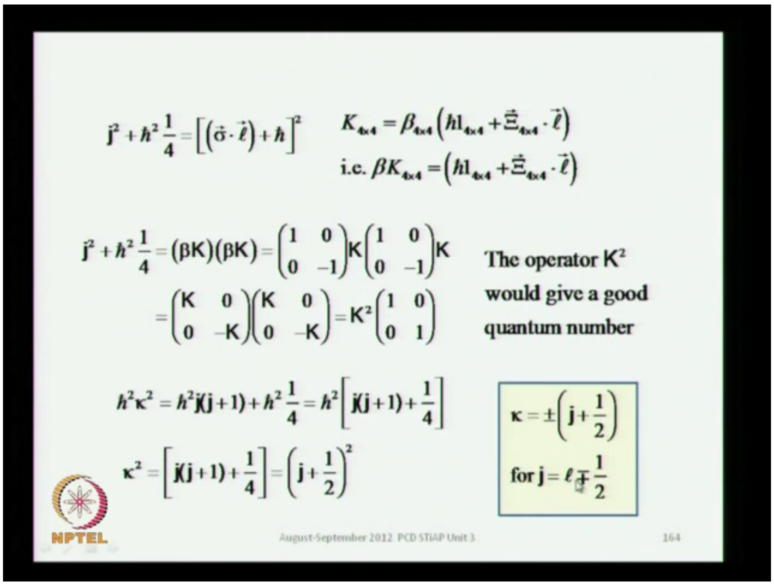

So, this is what you have got and that is suggest you, because you do know that j square is conserved it commutes with the Dirac Hamiltonian, what you have on the right side will also commute and anything that commutes with the Dirac Hamiltonian is a constant of motion. So, it offers itself as a candidate to give you a good quantum number, so you immediately see that possibility and this is what you have got, so j square plus h cross square 1 4th is we defined this operator as beta K if you remember. This is the operator beta K, so this is beta K beta K beta is this 1 minus 1 along the diagonal, so this is beta K and again the beta K, so that will give you k square times the unit matrix.

So, the Eigen values of K square will be the same as the Eigen values of j square plus h cross square 1 4th, and the Eigen value of this sign of the left hand side. So, what are the Eigen values of the left hand side, so the Eigen values of the left hand side are j into j plus 1 times h cross square plus h cross square over 4. And that gives you the Eigen value of K square and we extract the dimensions in h cross square, so that kappa is dimensionless number. And you get kappa square which is defined by this relationship, and now you can strike out h cross from the two sides, and you get kappa square equal to whole square of j plus half.

What it means is that, kappa is either plus or minus of j plus r and then, you see that j will be depending on j begin l minus half or l plus half you pick the appropriate sign over here, that is how it goes together. So, for j equal to l minus half, kappa is plus j plus half and for j equal to l plus half, kappa will be minus of j plus half, but you will see this develop further.

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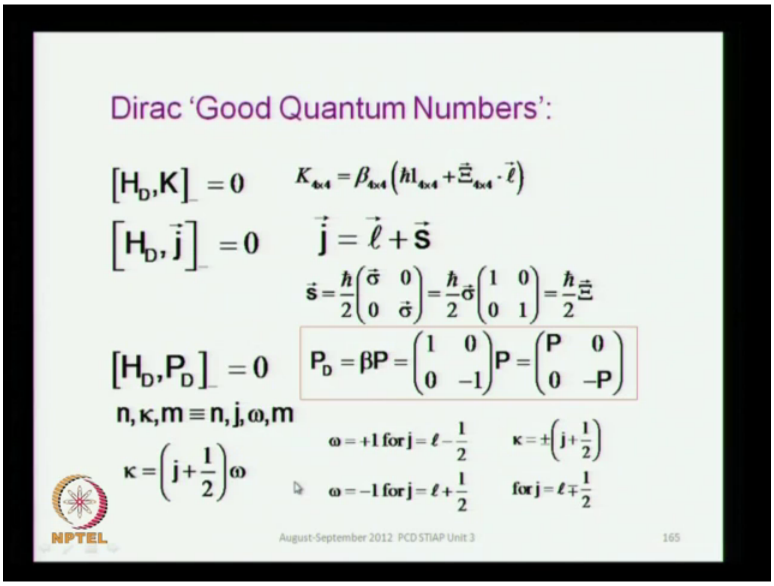

So, now we catalog the good quantum numbers in the Dirac scheme, so the where to get them from, we get them from K, so kappa will emerge as a good quantum number, then we get it from j, because j square commutes, j commutes, so j will be a quantum number. And then, what also commutes with the Dirac Hamiltonian is a parity, but parity has to be defined very carefully in relativistic quantum mechanics, because beta is an operator which is not the diagonal unit matrix. But, it has got the 1 0 0 minus 1 structure, and because of this very special feature of beta, parity has to be defined in a different way.

So, this is called sometimes as a Dirac parity P, written with a subscript d for Dirac parity or sometimes you write it just as P for parity. But, just to rabid in I am using the subscript dot, because the Dirac parity is different from ordinary parity, it has to be pre-multiplied by the beta matrix. So, this will become the 4 by 4 structure will come from P 0 0 minus

P and you will see, how it gives you the appropriate quantum number. So, these are the quantum numbers that you get for the Dirac state, the quantum numbers are n kappa and m, these are the quantum numbers, kappa content information about parity.

So, there is an omega which is introduced, which is defined to be equal to plus 1, when j is equal to l minus half and it is equal to minus 1 when j is equal to l plus half; this is what it gives kappa equal to plus or minus j plus half, when j is equal to l minus or plus half, this is what takes care of it. And omega as you will find it has got information about the parity, so kappa has got information about both about j and omega, because it given by j plus half time omega.

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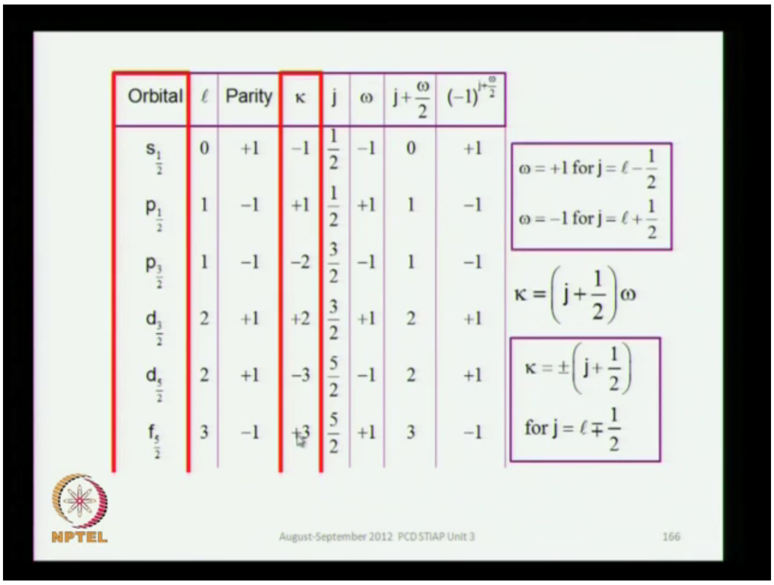

Orbital	ℓ	Parity	κ	j	ω	$j+\frac{\omega}{2}$	$(-1)^{j+\frac{\omega}{2}}$
$s_{\frac{1}{2}}$	0	+1	−1	$\frac{1}{2}$	−1	0	+1
$p_{\frac{1}{2}}$	1	−1	+1	$\frac{1}{2}$	+1	1	−1
$p_{\frac{3}{2}}$	1	−1	−2	$\frac{3}{2}$	−1	1	−1
$d_{\frac{3}{2}}$	2	+1	+2	$\frac{3}{2}$	+1	2	+1
$d_{\frac{5}{2}}$	2	+1	−3	$\frac{5}{2}$	−1	2	+1
$f_{\frac{5}{2}}$	3	−1	+3	$\frac{5}{2}$	+1	3	−1

And if you now, look at the quantum numbers for various atomic orbital's, look at the first column these are the atomic orbital's that you work with S half, p half, p 3 half, d 3 half, d 5 half, f 5 half and so on. They have the orbital angular momentum quantum number, so l is equal to 0 for S, 1 for p, again 1 for p, 2 for these 2 d's, 3 for f and so on. Now, parity is plus 1 minus 1, because parity this is defined by this feature as you will see, because minus 1 to the l is what gives you the non relativistic parity.

So, in this case the omega quantum number gives you that information, the j quantum number is here which is half, it is half for this state, 3 half for this state and so on. And kappa will be integers, because kappa will be plus or minus j plus half depending on j being l minus or plus half, so kappa will go as minus 1 for S, plus 1 for p half, minus 2 for p 3 half, plus 2 for d 3 half and so on. So, these are the quantum numbers that you are going to see in relativistic atomic physics.

You will see the kappa quantum number and they will have the structure, and it is important to keep track of this, that the S half, p half, p 3 half, d 3 half, d 5 half etcetera. Kappa quantum numbers correspondingly will go as minus 1 plus 1, then minus 2 plus 2, then minus 3 plus 3 then minus 4 plus 4 and so on. So, there is a systematic manner in which these quantum numbers will show up, now what we need to do is to separate the radial and the angular part, and that is the non trivial part that I had mentioned. So, let us work with the Dirac Hamiltonian and we have the beta K operator.

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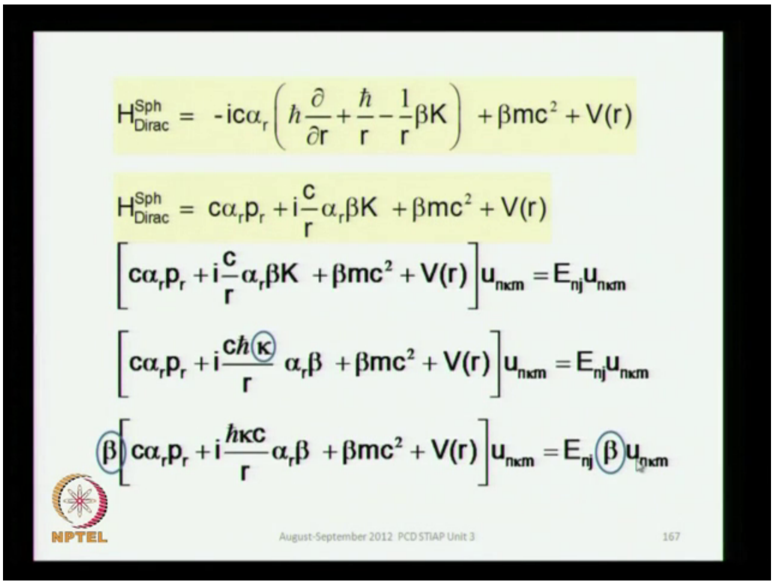

So, this is the Hamiltonian, this operating on an Eigen function which is now designated by the quantum numbers n kappa m, these are the quantum numbers that we have recognized. So, the state is now described by these quantum numbers and n kappa m, so this is an Eigen value equation, this is the Dirac Schrodinger like equation if you like. But, this is of course, the Dirac equation and this is the Dirac equation for the hydrogen atom V r being the coulomb potential.

Now, K operating on U n kappa m will give you the kappa quantum number, because that is Eigen state of K, so you get i C h cross kappa over r and then, the rest of the terms have been written just as the r. And then, this whole equation I operate upon by beta, so beta matrix is what multiplies this entire matrix equation, so beta pre-multiplies this left hand side and then, on the right hand side beta pre-multiplies U, which is a 4 by 1 unit wave function vector.

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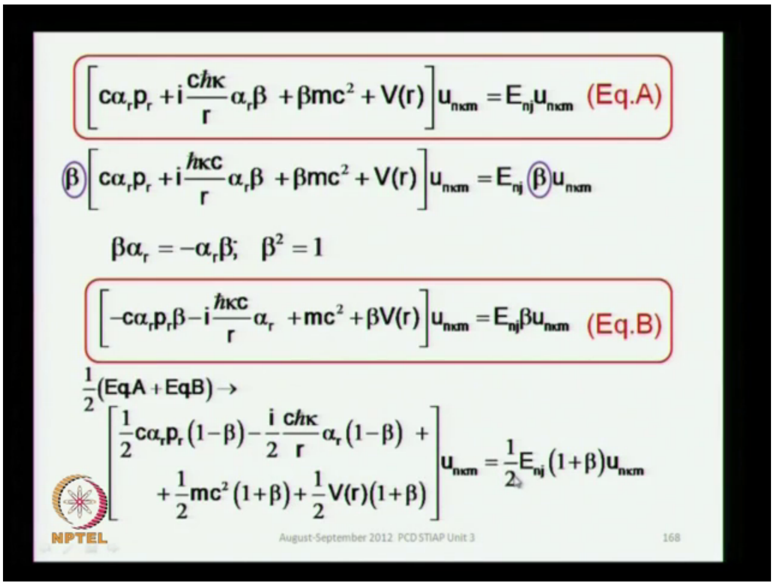

So, this is what you have got, but now that beta alpha r is minus r beta, so if you want to change the order of this beta and alpha r, you have to take care of this sign, it will be the same over here. But, when you interchange the position of beta and alpha r over here, you will get beta square which will be equal to 1. So, you will get some simplification coming from those terms, likewise in the third term you will have beta square m c square and beta square is equal to 1, so you will get good bit of simplification.

So, here I have beta moved to the right of alpha, so I have a minus sign here, the beta square in this term goes to one same thing over here that you have beta V r and beta U. Now, these two equations that what I called as equation A and equation B and we are going to handle them as a pair of equations, what I will do is to take half the sum of equation A and B, so you sum the two equations sum all the terms, and take half of it. What you get you get half of these two terms, so this does not have a beta this is got a beta with a minus sign, so you get C alpha r p r by 2 and 1 minus beta, so very easy to see.

And likewise if you take the next two terms, which has got i C h cross kappa over r which is what you have here as well, but then ((Refer Time: 22:00)) this is with a plus sign and this is with a minus sign and this is got a beta, this one does not. So, if you put the terms together you get minus i over 2 C h cross kappa over r alpha r 1 minus beta and you do the same with the remaining two terms. So, now you see term in 1 plus beta and 1 minus beta coming in and then, we have to see how this function U is going to respond to 1 minus beta and 1 plus beta, that is a first thing that is going to happen.

Because, alpha be lonely operator on the result of that, so alpha as of course, good operate, kappa has already operated and giving you the Eigen value kappa, the K has already done its task. And now we have to see how 1 minus beta and 1 plus beta would operate on the function U.

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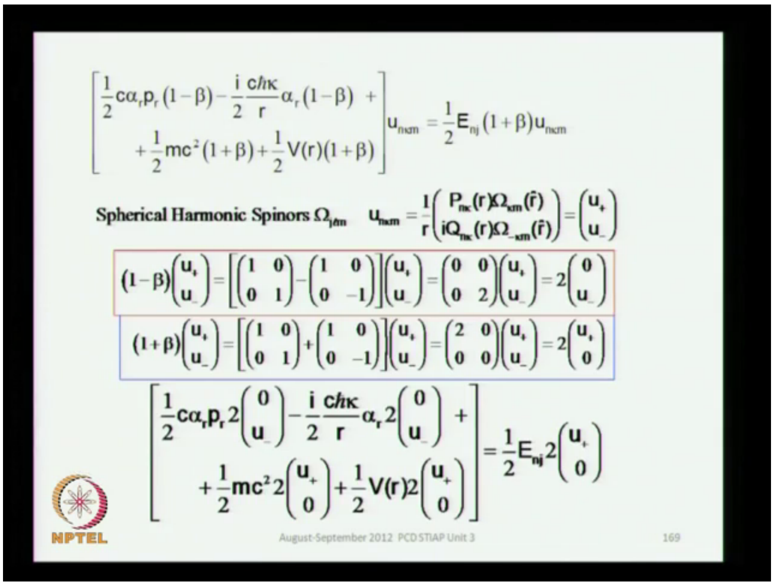

So, the wave function u, we write in terms of these spherical harmonic spinors or spinners and this is 1 over r P omega i Q omega where, these omegas are the spherical harmonic spinners. So, this is written as in a short notation as u plus and u minus, so P and Q are the radial functions which we hope, we will be able to separate, it is not that we have done it yet. But, in anticipation of a prospective separation of the radial part of the angular part, we begin to use that notation and then, we ask ourselves what P should be like and what should be the nature of Q, of the function Q.

That is something which is yet to emerge, it will emerge as we carry out this analysis further, so this is sometimes you used some other notation as well, which I will mention toward end of the class. And then, now you ask what will 1 minus beta do to this wave function, which is written as u plus u minus for probability, and 1 minus beta will give you this one diagonal matrix minus beta, which is 1 minus 1. And ((Refer Time: 24:27)) this matrix when it pre-multiplies u plus and u minus you get twice 0 u minus, that is very easy to see, that is a advantage of having this blog diagonal structure, it is very straight forward.

And then likewise, when 1 plus beta pre-multiplies the wave function, you get twice u plus 0. So, now in place of this 1 minus beta u, 1 minus beta u, 1 plus beta u over here, and 1 plus beta u in the 4th term you get this 2 times 0 u minus, 2 times 0 u minus, 2 times u plus 0. So, those are the terms that get, you also have a 1 half factor here you have got that is coming from the fact that you took half the sum of A and B, but then you get a factor of two. And then there is a similar half factor here and the 2 over here, so the half and the 2 cancel everywhere in all the terms, because they are present in every single terms.

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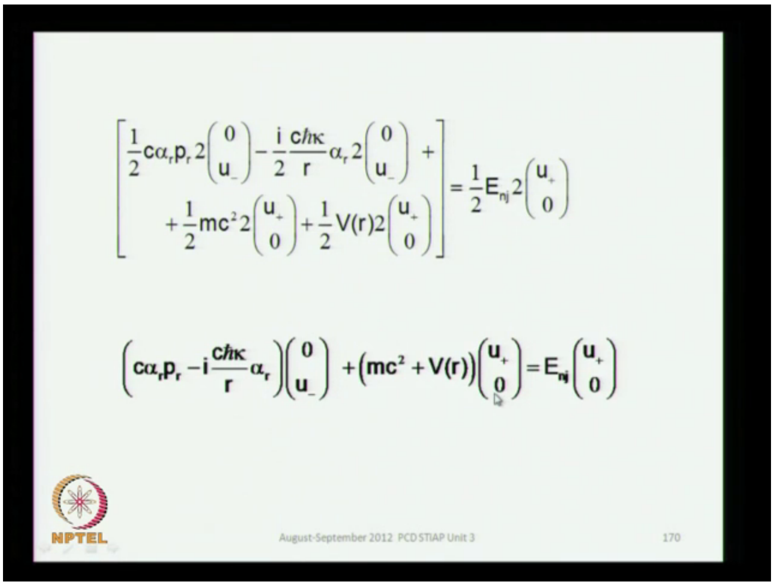

So, they cancel out and you get a relatively simpler expression, but now you have the 2 and half has dropped off, and now you have this C alpha r p r coming from here and then, you combine it with this minus i C h cross the 1 half has gone. So, minus i C h cross kappa over r times this alpha r, and these are the common factors of which pre-multiply the matrix u, 0 u minus which is here. And then you have similar two terms which you can combine for u plus 0 that is good.

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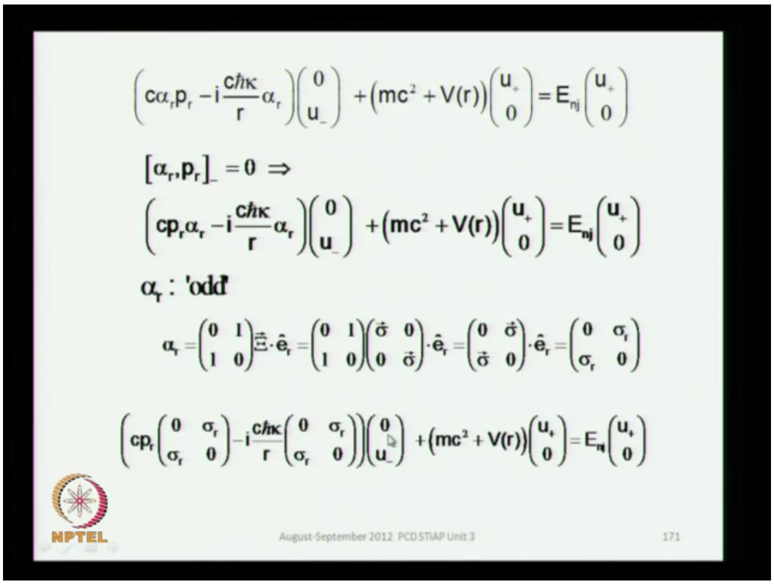

Now, we know that the radial momentum operator the p r commutes with alpha r, and you can therefore interchange these positions. So, you have got alpha r here and when you do this, you take advantage of the fact that you have to find the operation on 0 u minus by alpha r. And what is alpha r, alpha r you remember, this was the matrix rho alpha was rho sigma this is radial component of that, so you get rho which is 0 1 1 0, sigma is sigma 0 0 sigma you take the radial component dot r. So, you get sigma dot r and essentially it is sigma r along these two off diagonal positions.

So, actually you notice that ((Refer Time: 27:39)) this sigma r, and this sigma this uppercase sigma, they play the same role in two dimensional space and 4 dimensional space that is what you find. And using these features now in place of this alpha r, you substitute this 0 sigma r sigma r 0, which is 0 sigma r sigma r 0 in place of this alpha r and you do the same in every term you have got alpha r over here as well. So, you do the same thing over here, and now you are in the position to carry out this matrix multiplication, because you have got a 4 by 4 matrix which looks like a 2 by 2, because each element is 2 by 2.

And then, you have got a 4 by 1 column wave function, which looks like 2 by 1, because each element has got 1 rho's, so that is the structure of these equations. So, let us look at this expression, it is the same one which was at the bottom of the previous slide.

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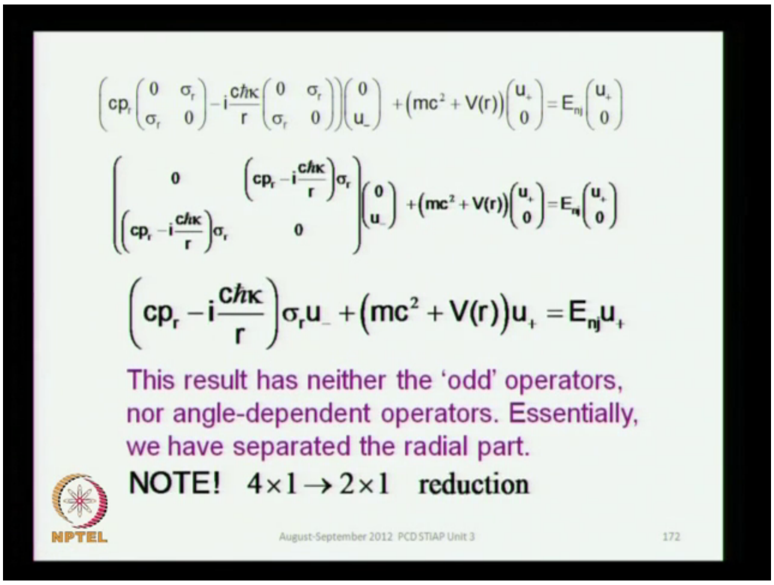

And now you have got off diagonal elements over here, and the diagonal elements are 0 and they must pre-multiply through matrix multiplication, this 0 u minus, so what does it give you, it will give you this term pre-multiplying u minus, just metrically. So, that is what you get from these terms, here you get m c square plus V r that is multiplied by unit 4 by 4 unit matrix and essentially you get a relationship, which is free from all off operators. Now, there is no alpha here, you have gotten rid off odd operators, and there is nothing that is angle dependent.

So, as a matter of fact, we have achieved the separation of the radial and angular part, but this is not the whole story, but this is the essential idea, because we have to do a little more algebra with this to figure out how exactly the complete Dirac equation is handles. So, I will show you how to do that, but now you have a result which is free from odd operators, it is also free from angle dependent operators, and we have also succeeded in having a 4 by 4 structure reduced, 4 by one structure reduced to a 2 by 1 structure. Of course, there is another, it is not that 2 by, the remaining 2 rho's are lost, can you see them what do get from the remaining 2 rho's.

((Refer Time: 30:56)) Here you have 4, so here also two of those rho's are here, what you get from the other 2 rho's you get 0 on the left hand side and 0 on the right hand side. So, you get 0 equal to 0 and there is no physics in, so which is why I am not written that, but essentially you get complete consistency, you are not thrown of anything, from the remaining 2 rho's you get 0 equal to 0, and you are not losing anything in that. As you knew it even before you are born, and you have an essential 2 by 1 structure.

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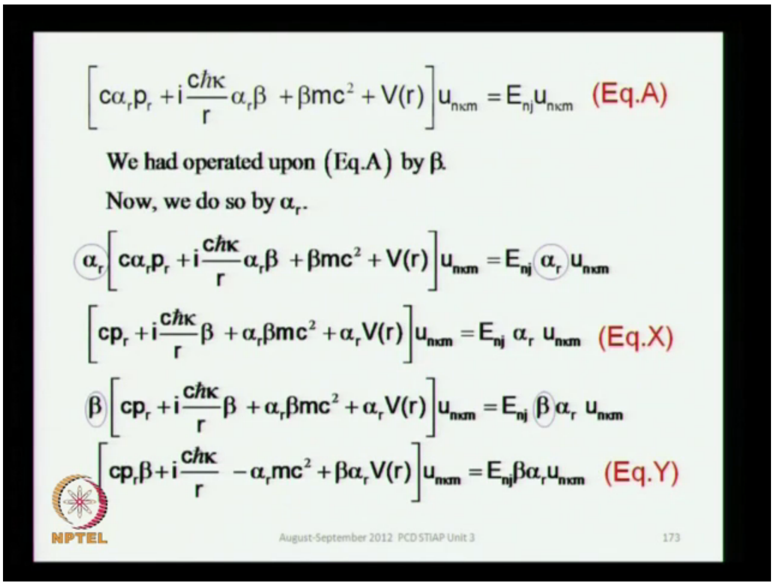

Now, let us back to the equation A, which I had use earlier you remember we take an half the sum of equation A and equation B. So, it is a same equation A which I have written again, but this time I am going to do some different way with it, what we will do is instead of operating upon equation A by beta, we will operate this time by alpha r. The same kind of technique, but now we take the other operator which was there in the equation of motion, so we operate now by alpha r. So, you have alpha r pre-multiplying this whole equation and then, you have alpha r pre-multiplying this wave function which is a 4 component wave function.

So, now you have got alpha r square which will give you unity in the first term, so you get c p r from the first term again from alpha r square you get unity, then you have got alpha r beta and then, alpha r V. Now, on this result you pre-multiply by beta, these are some nice tricks that you play, but it also tells you that ok, when you dealing with this kind of mathematics what is it that you can do, and what you stand to gain by doing so. So, you pre-multiply this by beta and now you get beta c p r, then beta will come here every term here is pre-multiply by beta. And take advantage of beta square being equal to 1, and now what I will do is I will call this result as equation X, and this result as equation Y.

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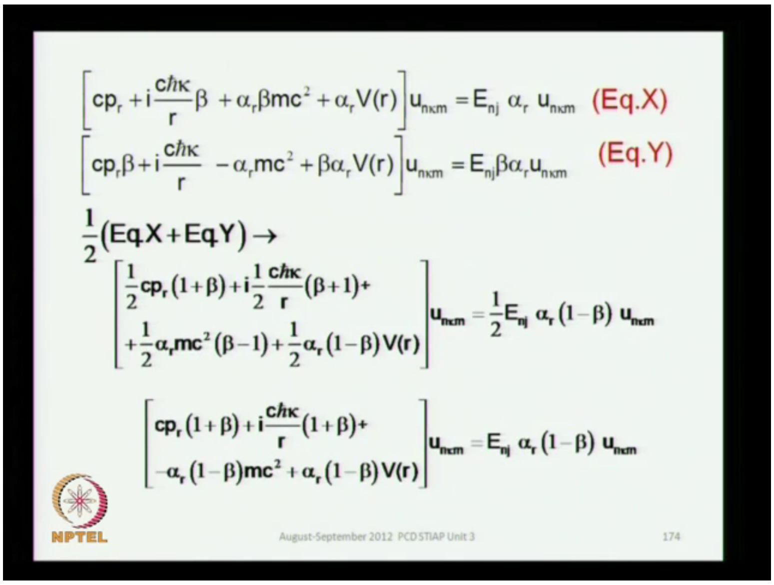

So, named these equations as X and Y and then, take half of X plus Y, so for our convenience, I have rewritten the equation X and Y at the top of this slide, so that we can keep track of each term and now we take half the equation X plus half the equation Y. So, you get the factor half c p r, then you have to take the sum of these two terms, so you get 1 plus beta and likewise, you get 1 plus beta from the second term. Then beta minus 1 from the third term, and 1 minus beta from the 4th term again 1 minus beta on the right hand side.

Now, we have done this earlier, so you can very easily see how these terms fall in place. And once again we can make use of the fact that we already know, what 1 plus beta does to u and what 1 minus beta does to u, so those results we can already use. So, now I have gotten rid of the 2 and 1 half factor, the half factor goes everywhere.

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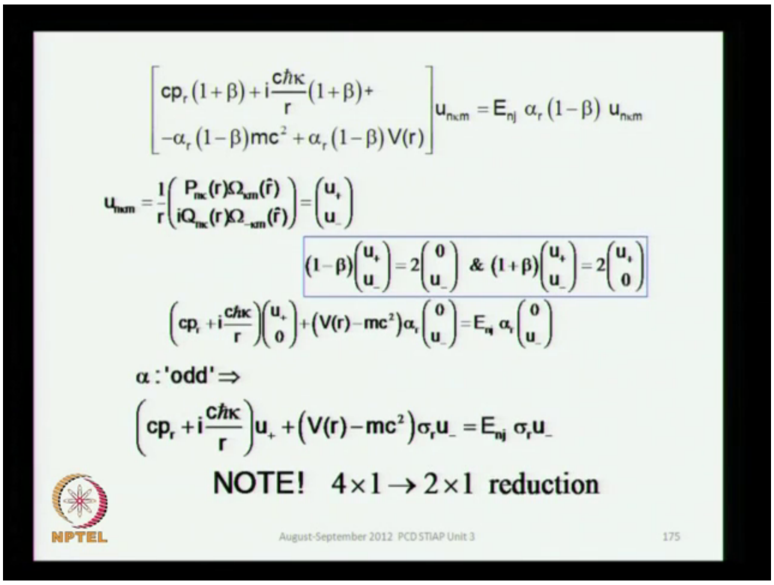

And we have these results 1 minus beta operating on the wave function, pre-multiplying the wave function will give you twice 0 u minus and this 1 plus beta will give you twice u plus 0, where plus and minus are these components of the 4 component wave function. So, these are two component functions, each has got two component. So, you make use of this 1 minus beta u plus u minus, so 1 plus beta u will give you twice u plus 0, so you would get twice u plus 0 over here, but you are going to get two in all the remaining terms, so they will cancel out.

So, in anticipation of that they have already been removed, and now you have over here alpha r which will pre-multiply 0 u minus. So, alpha being odd you can once again do the same kind of matrix multiplication, and you get a simple 2 by 1 reduction. Just a way we did in the previous class and then of course, you will still have two other rho's which will give you 0 equal to 0.

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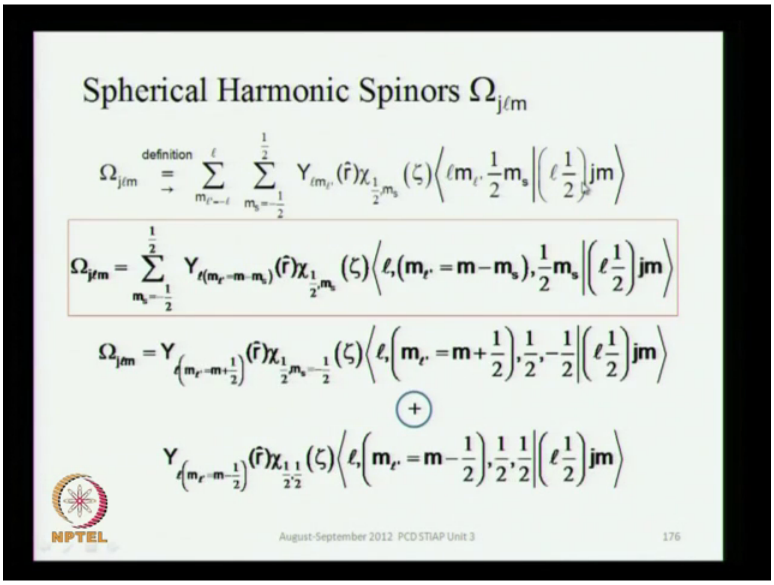

So, now I will spend a few minutes, the scars on the properties of these spherical harmonics spinors omega, so this is the definition of the spherical harmonics spinors, these are made of the regular spherical harmonics, these are the spin functions. So, this is spin functions spin up and spin down, it could be 1 0 for the spin up and 0 one for spin down or up and down however, whatever rotation you want to use, so these are the spin Eigen function.

So, this is essentially a composition of angular momentum of these two angular momentum individual factor states, one which are Eigen functions of the orbital angular momentum, second which are the Eigen functions of the spin angular momentum. And these are coupled using the Clebsch-Gordan coefficients, and you are the worlds expert on Clebsch-Gordan coefficients. So, you have a double sum now, m s going from minus half to plus half and m l prime I have used which will go from minus l to plus l, but you know that the Clebsch-Gordan coefficients would vanish, unless this m is equal to m l prime plus m s..

So, you carry over the sum over m l prime, and you will retain only those terms for which m l prime is equal to m minus m s, all the other terms will vanish, no matter what they are. So, only you the term in m l prime equal to m minus m s, so that pinch down this spherical harmonics, this quantum number m l prime must be m minus m s, otherwise the Clebsch-Gordan coefficient is 0, we know that. And now you have a very simple sum, because this is a sum over m s going from minus half to plus half which means that, you must sum over both of these spin Eigen states which are respectively 1 0 and 0 1. So, you have these two terms, one corresponding to m s equal to minus half here ((Refer Time: 38:56)) and the other corresponding to m s equal to plus half, and then you have the corresponding Clebsch-Gordan coefficients here.

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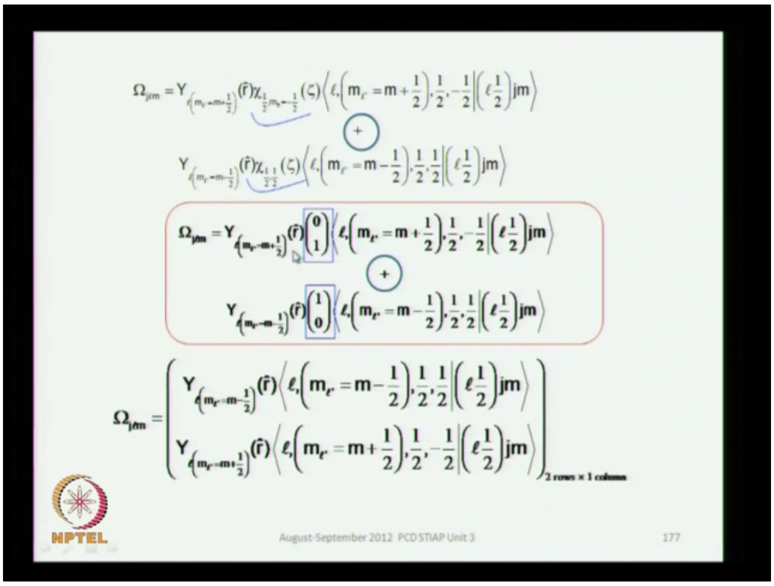

So, let us look at these two terms, so these are the two terms now have summed over m l, you have summed over m s you have got everything has been summed over. Now, these spin functions, this is m s equal to minus half, so this is the 0 1 this is the spin down, this is m s equal to plus half, so this is the spin up which is the matrix 1 0. So, which means that you will multiply this term and this term and it will come in location in the first row, but the element in the second row will be 0.

Over here the element in the first row will be 0, the element in the second row will be non zero and then, you sum the two terms what do you get, you get a matrix of 2 rows and 1 column. The first row comes from the lower term, because this has got this element in the first row to be non zero, so this is what you get and from this term you get the lower row which is this, which is the second row.

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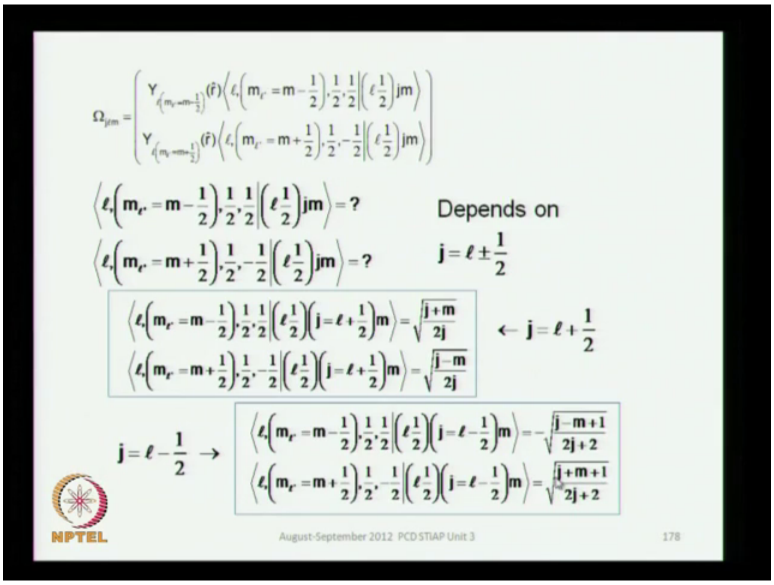

Now, this is your simplified expression for the spherical harmonics spinors, and now you need these Clebsch-Gordan coefficients, you have to determined that. And you can determined them quite easily from standard, tables Clebsch-Gordan coefficients and so on, and their values actually depend on whether j is l plus half or l minus half. So, you have to handle these cases separately, because the value of the Clebsch-Gordan coefficients will certainly depend on this.

And then if j is l plus half, then this Clebsch-Gordan coefficients the upper one is root j plus m over 2 j, the lower one is root j minus j over 2 j on the other hand, if the j is equal to l minus half, then the Clebsch-Gordan coefficients are given by this. So, this is something that you can extract from standard tables of Clebsch-Gordan coefficients, now you have got everything, you plug in the corresponding Clebsch-Gordan coefficients.

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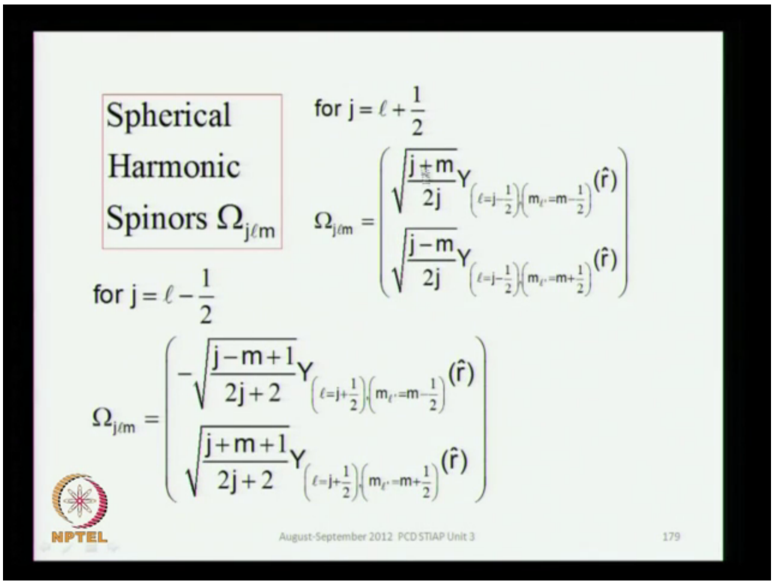

And you will have two different sides, one for j equal to del plus half and the other for j equal to del minus half, because a corresponding Clebsch-Gordan coefficients are different, I have just compiled those relations over here for simplicity.

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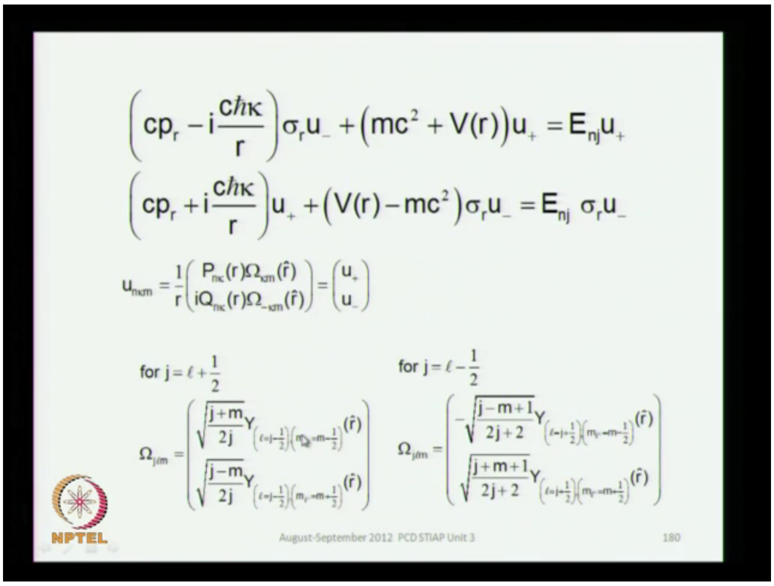

And now you have got all the elements that you need, you have got the radial equation you have got the wave function, which is written in terms of this u plus and u minus. And you have got the spherical harmonics spinors which are now written in terms of the spherical harmonics and the Clebsch-Gordan coefficients, so all the elements are there.

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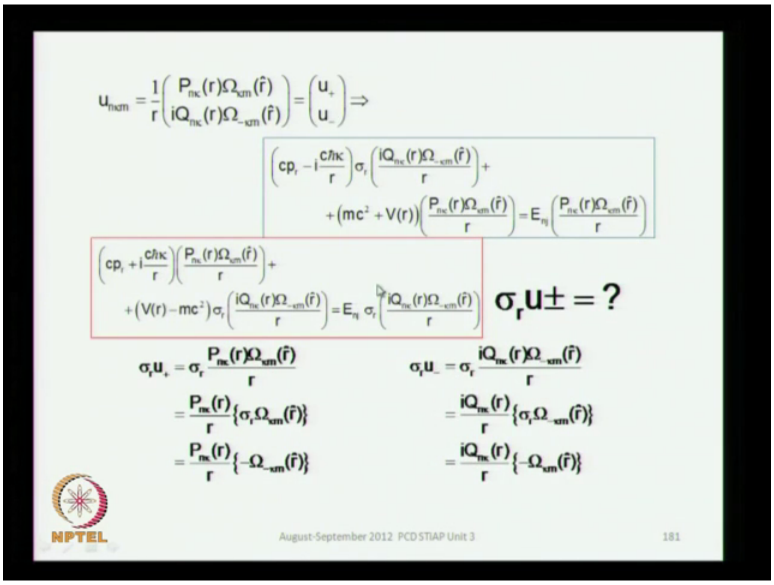

Now, all you have to do is to see what these equations simplify to, so now this u plus u minus which we have used, I use explicitly the functions P and Q, so this is i Q this spherical harmonic and here I have got P omega. So, I have got two sets of equations if you remember, so I use both of them and what we need is sigma r U plus U minus, because that is still to be determined. So, you have got a sigma r over here do not forget that, so this sigma r what it does to the spherical harmonic spinors, and the wave function u plus u minus is something that is yet to be explode.

So, this is something that you can work out that when sigma r pre-multiplies u plus, you get minus omega minus kappa m, and this comes just from the definition of the spherical harmonic spinors, because sigma r has got this matrix structure. So, you just use the matrix structure and the definition of the spherical harmonic, and you will find these quantum numbers. This omega kappa m when pre-multiplied by sigma r will give you minus of omega times omega, but the quantum number kappa will replace by minus gamma and you have got a similar relationship over here. And you can now use this result back in these two relations, in both of these you need these two relations.

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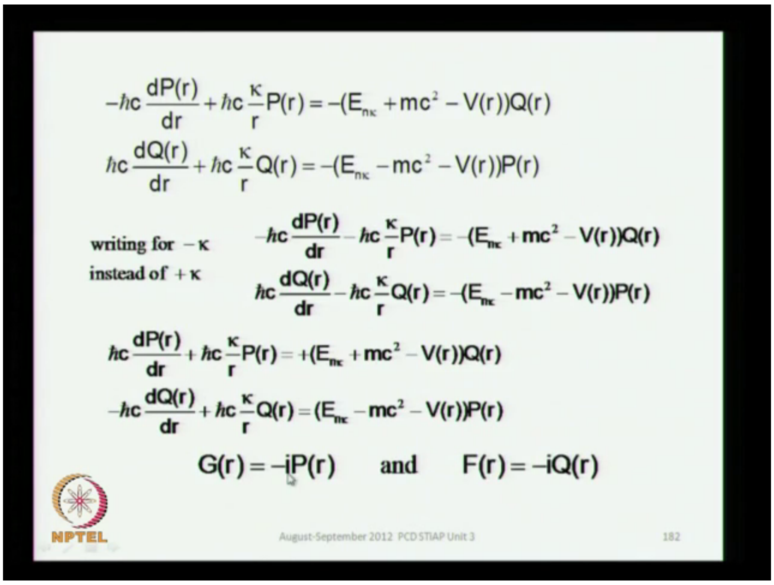

So, plug it back and you get now the angular part is completely separated out, so the partial derivatives with respect to r del by del r can be written as d by d r, and you get two coupled equations P and Q, and this is the completely radial equation. And this really comes as a surprise, because it was not at all obvious from the beginning that the Dirac equation can actually be separated into the radial part and the angular part, because it had a very complicated structure, in terms of the operators that we were working with.

Now, you have these two operators, two sets of coupled differential equations for P and Q, you will find them in different forms, in different books and papers, because you can replace kappa by minus kappa, and vice versa. And you see them in a slightly different form, but it is completely equivalent, it is just minus kappa instead of kappa that is it. And then you can also have these relations, you multiply everything by minus 1 and again you see them in a slightly different form. So, you will see these equations in somewhat different forms in literature, and sometimes instead of P and Q, you find these equations written in terms of G and F, rather than P and Q, where G is usually defined as minus i P r and F as minus i Q r, so you will find coupled equations for G and F rather than P and Q. So, different author used different notations, but that is just a matter of very minor detail and it is not going to worry you.

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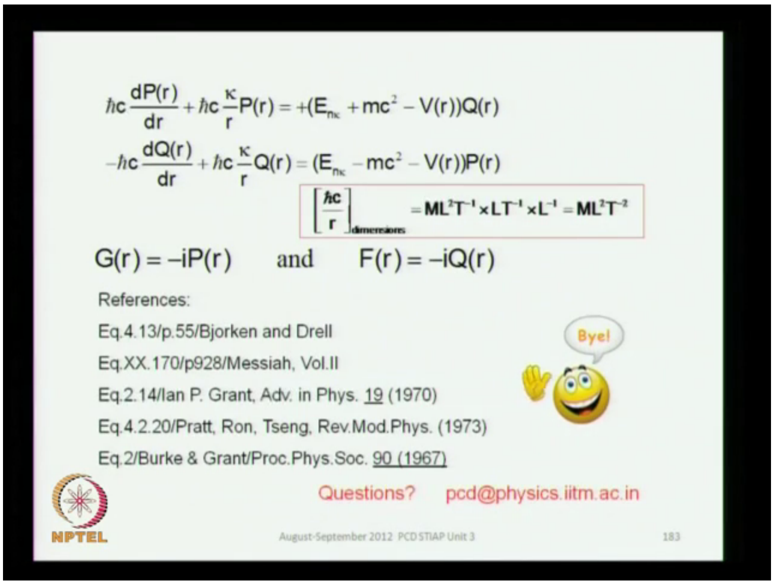

So, these are the coupled equations for P and Q, these are the radial equations you can solve them, and depending on what source you are using, whether you may be reading Bjorken and Drell or Messiah has got a very good chapter on relativistic quantum mechanics in volume 2. The article which I strongly recommend is an article by grant in advances in physics 1970, and you will find these relations in grant paper as well Pratt, Ron, Tseng have a review in modern physics, Burke and Grant have got a nice paper.

And you will find equations radial equations of this kind or some equivalent forms, either in terms of G and F or P and Q or the same equation multiplied by minus 1 or kappa replace by minus kappa. So, some small transformation you may have to do, but essentially what I was hoping to do in this class, is to introduce you to the radial differential equations. And then, you can go ahead and solve at like any other system of coupled differential equations, that is technique by itself, but that is not new to you, if you have come this part how to do that.

So, I am going to stop here for this class and with this I can conclude this unit 3, a whole lot of physics that you do with relativistic atomic wave function. In fact, anything that you deal with atomic structure and atomic processes require you to use relativistic wave functions. For the simple reason that nature is relativistic, there is no escape from it, there is no escape from fact that the speed of light is finite and everything is begins there.

So, the speed of light is finite, laws of nature are quantum mechanical and therefore, there is no escape from relativistic quantum mechanics.

And we have bare introduction all we did was the hydrogen atom, which is the smallest thing that we can think about, once you take any atom which has got more than a single electron, then it becomes that much complicated. And then you want to do relativistic quantum mechanics with it, it becomes even more complicated, so we first of all we need to learn how to deal with, atoms with more than one electron, and that is more than one is many. So, we will get into what is formally called as a many electron formalism of atomic structure.

So, the next unit will be on the Hartree-Fock self consistent fields method of dealing with many electron atoms, but if there is any questions on the relativistic hydrogen atom, at least I hope that you have got basic introduction to the topic. You have your handle on the techniques, and these are the methods that you have to use means, I am sure that you will be confronted with various problems, which will require some effort. But, you will not require new techniques, you those techniques with you, if there is any question I will be happy to take otherwise, thank you for now, and then we will meet for unit 4.

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But, that has been separated out it does not come here, in this equation there is no spherical harmonic.

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So, in the previous result where we used that, the separation was not explicit, we used that to arrive at this, the original expression of course, has got the radial part and the angular part, just like the Schrodinger equation has the radial part and the angular part. So, the angular parts are there, they are sitting in the spherical harmonics, which sit in the spherical harmonic spinors, but then using this algebra of the Dirac matrix operators, the sigma's j particular.

We dealt with their radial components of sigma's and then, when we had the radial component sigma are operating on the 4 component wave function, we used the result in the previous relation here.

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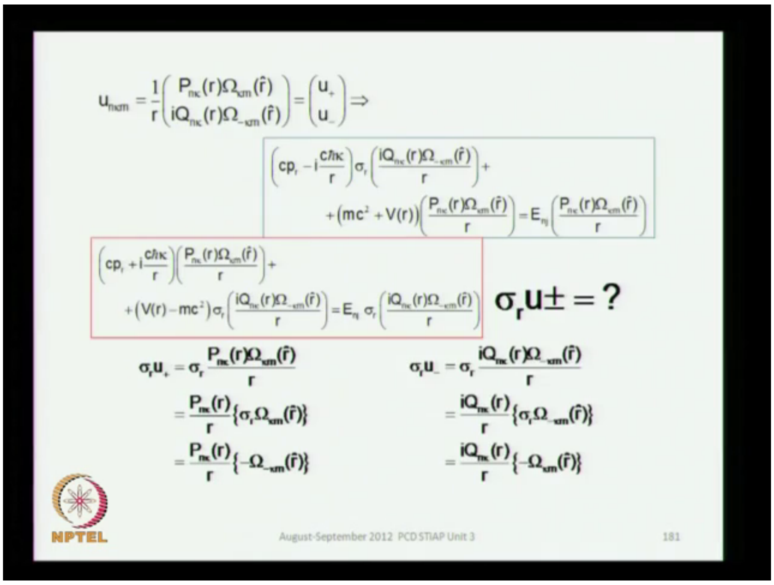

And then all the common terms cancel out and what you are left with is only the radial equations, but this will give you a set of terms, which will cancel out completely, so all the angular parts go away and you are left with, only the radial part.

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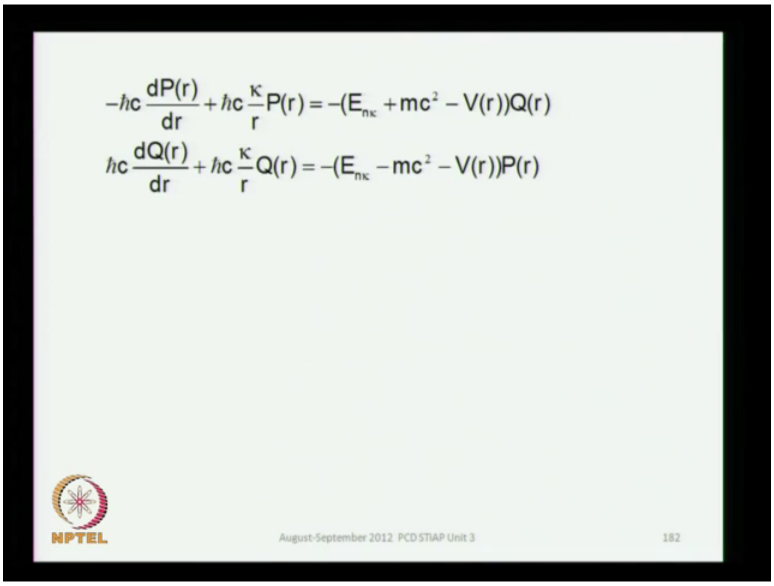

And this is great wonder which is why it is often said that the Dirac equation can be solve for a few problems, for which you have exact analytical solutions, the hydrogen atom is one of them. It is not for many situations that you can really do, but then that is not much of a concern, because we have already reconciled with a fact that in most situations the challenge for a physicist is not really to get the exact solutions.

But, to get the best approximations, exact solutions do not exist there are exist in theorems, which tell that the exact solution does not even exist. So, your challenge is not really, so much in looking for an exact solution, because that could be a search for the impossible, but to get the best approximation. And for the hydrogen atom you do get exact solution, but when you go beyond the hydrogen atom you do not, then your challenge is what is a best is that you can do. And the many electron theory the Hartree-Fock formalism is a wonderful exercise in getting approximate solution to many electron problem. But, of course, there are ways of improving on the Hartree-Fock as well, so we will talk about it in the next unit, any other question, so thank you very much and goodbye for now.