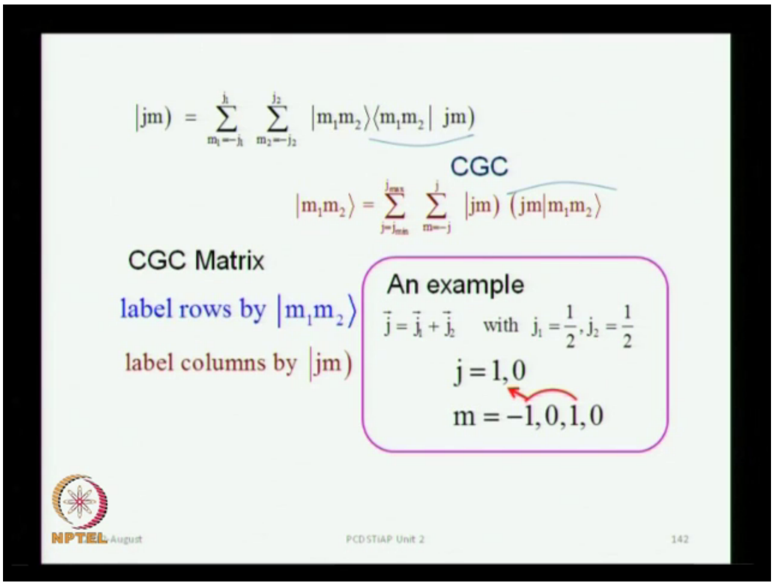
Select/Special Topics in Atomic Physics
Prof. P. C. Deshmukh
Department of Physics
Indian Institute of Technology, Madras

Lecture - 10
Angular Momentum in Quantum Mechanics
CGC matrix, Wigner D Rotation Matrix, Irreducible Tension Operators

Greetings. So we will continue our discussion on Angular Momentum and we will discuss today the matrix of the Clebsch Gordan Coefficients. We will discuss some further properties of the Wigner D Rotation Matrix, and then we will introduce the Irreducible Tension Operators today.

(Refer Slide Time: 00:33)

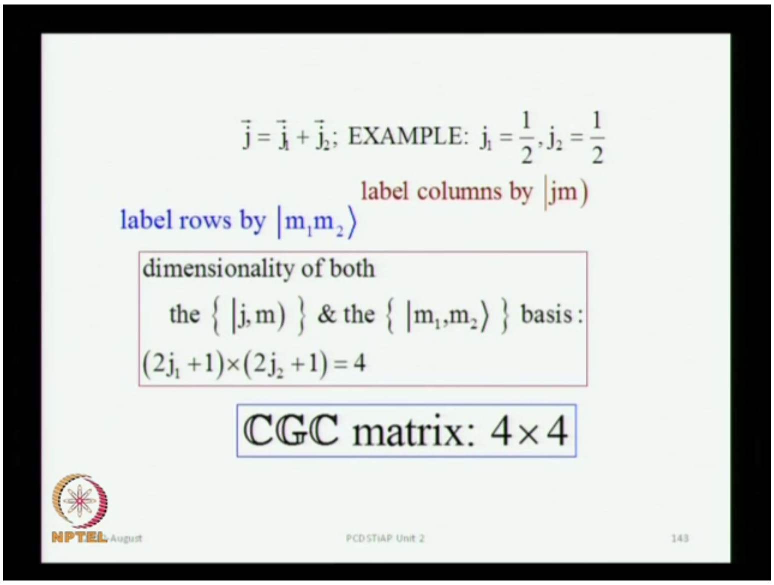


So, we know what the Clebsch Gordan coefficients are these are the vector coupling coefficients, when you couple 2 anglular momenta. And you can express the coupled angular momentum with these circular brackets, in terms of the direct product of the uncoupled vectors. And likewise, you can have the inverse transformation and represent the direct product of uncoupled vectors, in terms of the coupled angular momenta.

So, what we will now compose is the matrix of Clebsch Gordan coefficients, in which we will label the rows by the direct product of the uncoupled vectors, and the columns by the coupled vectors. So, there are two alternate bases sets, both are orthonormal bases sets, and you can always carry out a transformation from 1 to the other, so I will illustrate this procedure by taking a specific example, in which I will couple 2 angular momenta, both of which are equal to half.

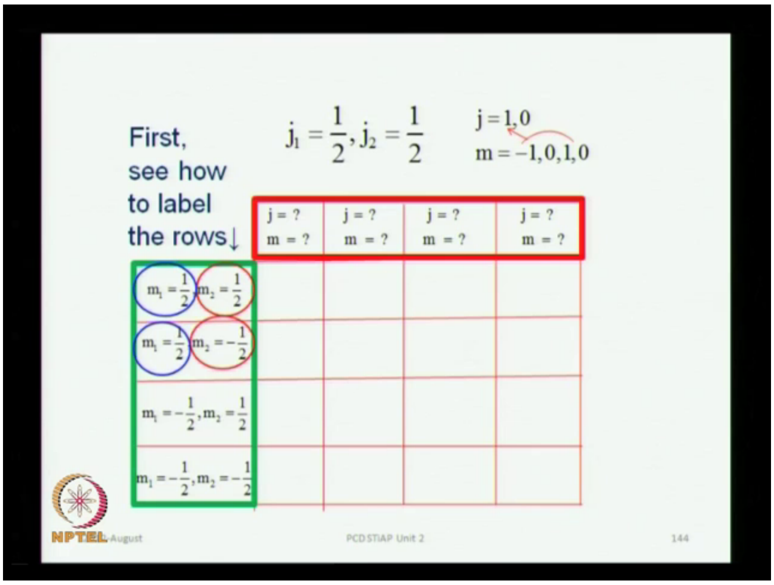
So, j 1 is equal to half j 2 is equal to half and from these we compose a third j, the j 3 or the final j, which is j 1 plus j 2. And the size of j would go anywhere from modulus of j 1 minus j 2 to j 1 plus j 2 in steps of 1, so in this case, it can take only 2 values 0 and 1 right, and for each value of j the corresponding value of m would go from minus j to plus j.

(Refer Slide Time: 02:21)



So, we can already see what will be the dimensionality of this matrix because, we will begin to label the rows by m 1 m 2. So, the dimensionality will be 2 j 1 plus 1 times 2 j 2 plus 1, so both are half, so the dimensionality will be 4 and we already saw in the previous slide the expected this dimensionality to be 4. So, this will be a 4 by 4 matrix which is called as the matrix of clebsch gordan coefficients.

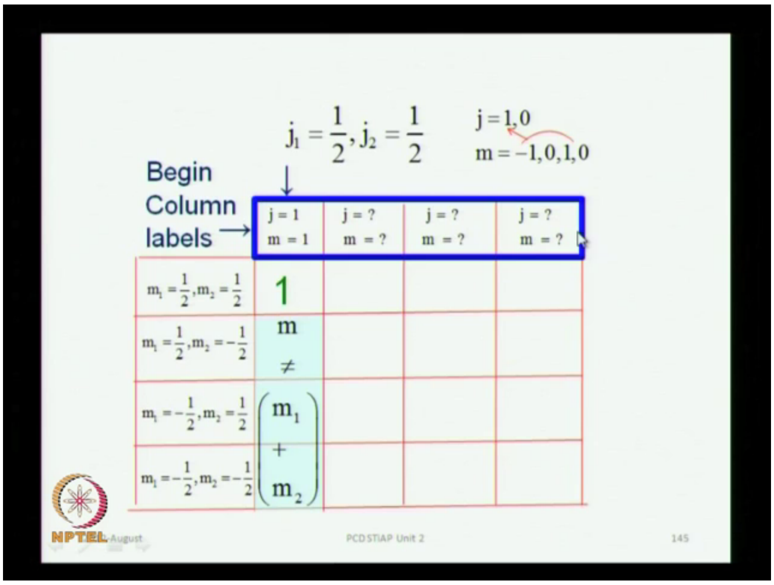
(Refer Slide Time: 02:58)



Now, this is how you compose a matrix, so you first generate a 4 by 4 table, do not even look at the columns first, first see how the labels are row the rows are labeled, these are the labels for the rows. So, the first row is m 1 m 2 both having their highest value, which is half, so for m 1 equal to half what is the maximum value of m 2 that is half, then what is the next value of m 2 for the same value of m 1. So, the next row is labeled by m 1 equal to half, and m 2 equal to minus half.

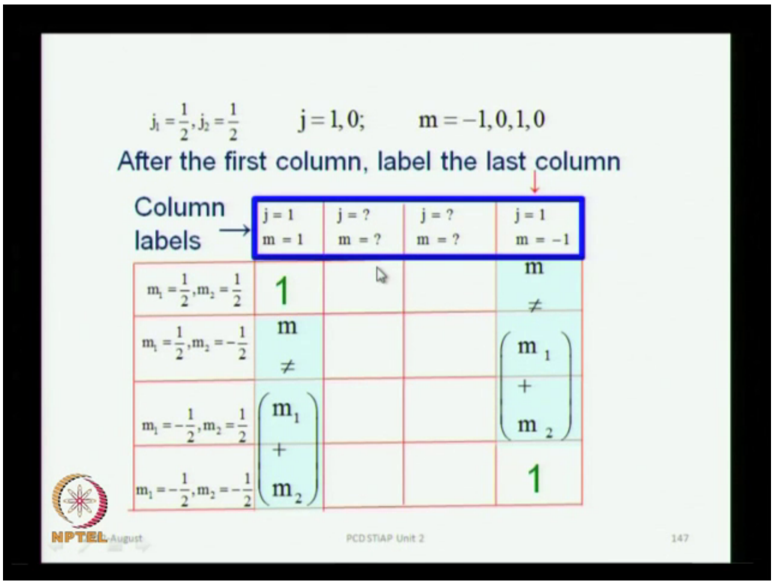
Then the third row is labeled by the next value of m 1, which in this case is minus half and for this minus half again you pick what is the highest value that m 2 can take. And then go to the next value of m 2, whichever it can take and in this case it is minus half, so this is how the rows are labeled and the first thing to do, while preparing your table for the clebsch gordan coefficients is to first decide how the rows are going to be labeled.

(Refer Slide Time: 04:23)



So, this is how you do that then you begin to label the columns and label the first column first, label the first column such that you have the highest value that j can take, and then the corresponding highest value that m can take. So, this is j equal to 1 and m equal to 1, now what this choice is going to give you is give you the matrix element in the first row at first column, which will be equal to unity, because that is the only transformation which is possible. Now, after you label the first column notice that I have put a question mark in the remaining 3 columns, and now we want to label these columns we have already labeled the rows. Now, we want to label these 3 columns, so after labeling the first column we will label the last 1.

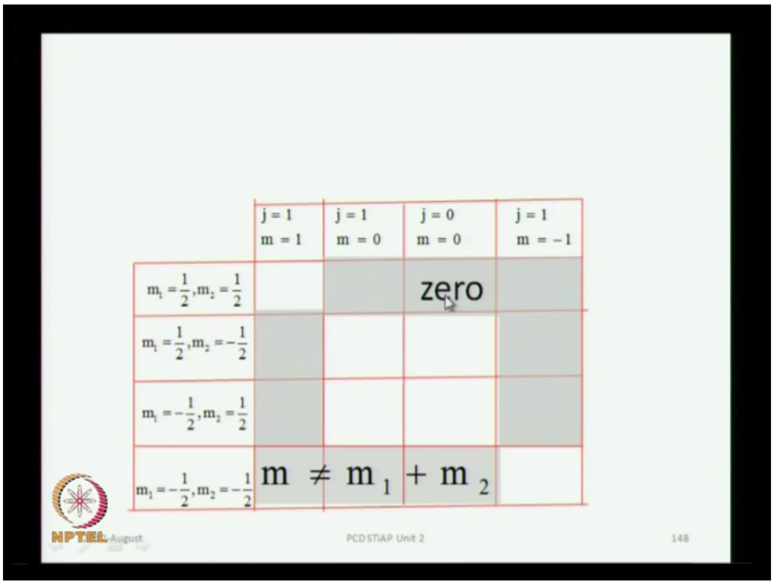
(Refer Slide Time: 05:20)



So, we will now label the last column and last column we will label with j equal to 1 and lowest value, which is the maximum negative value then the corresponding m value can change. Because, what this will do is to give you in this position, which is the element in the last row and the last column, this value will also turn out to be 1 because, that is the only combination which is possible there.

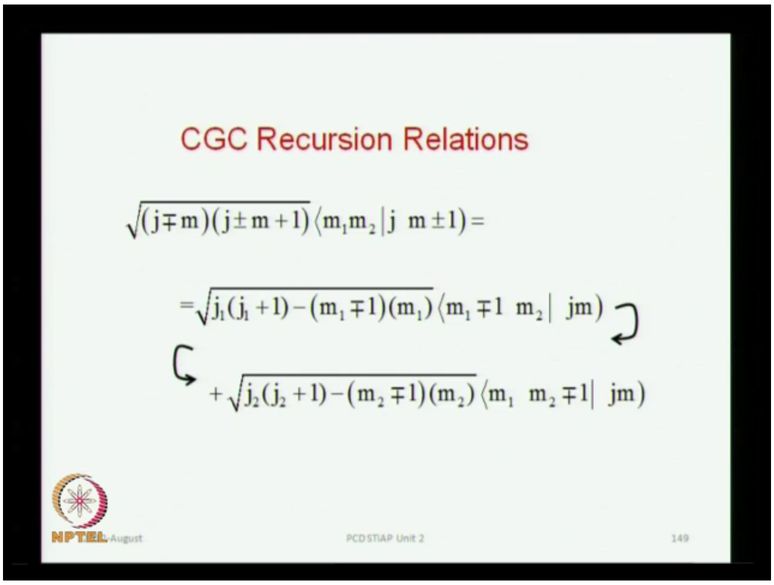
Now, you already know that for the remaining elements in the first row and the last for the first column and the last column m is not equal to m 1 minus m 2, so those clebsch gordan coefficients must vanish, because that is a necessary condition for the clebsch gordan coefficients to be not 0, so put 0 in all those locations.

(Refer Slide Time: 06:32)



Now, you can label the remaining columns and what you do is label them such that you get 0's over here, and 0's over here as well. And you can easily do this by fixing j equal to 1 and j equal to 0 because, let the value of j diminish from here, and then choose the next value of 0 from this set. And now you have got 0's over here and non 0 elements along the block diagonal, so you have. So, it is very easy to compose the matrix of the clebsch gordan coefficient tables, and there is this simple procedure which I just illustrated.

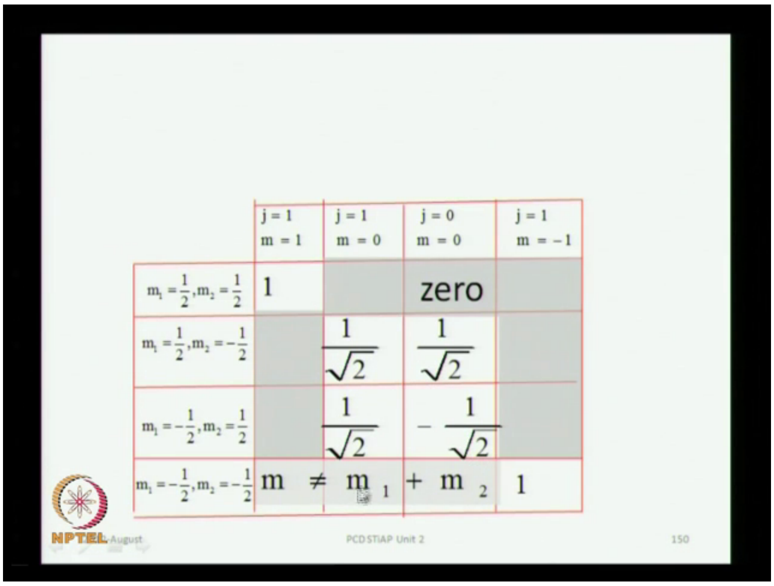
(Refer Slide Time: 07:12)



So, now, we are almost ready because, we can now use the recursion relations, we derived these recursion relations in the previous class. And we have got the matrix elements in the 1 1 location and in the n n location, the last row and the last column and also the first row and the first column. So, the these two have already been pinned down, and now you can go to the neighboring coefficients using the two recursion relations, you have.

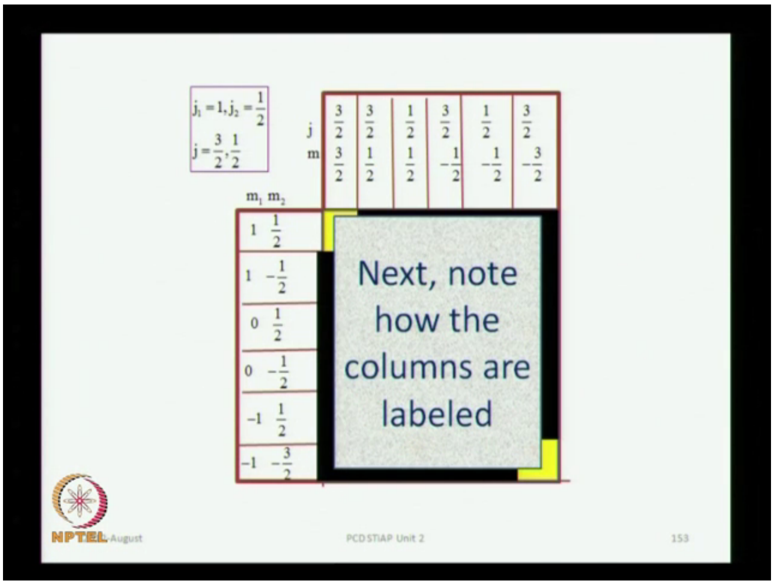
What you find the screen are two relations, one corresponding to the to one sign like, if you take the minus sign over here, you take the upper sign over here and the corresponding upper signs in the rest of the terms. So, this is a set of two equations then with these two recursion relations, you can then generate the entire matrix of the clebsch gordan coefficients. So, you really do not need to consult any book or anything you can do it by hand for coupling of any angular momenta in the exams you will do it by hand, so you know how to do it.

(Refer Slide Time: 08:13)



So, you will need to use the recursion relations to do that, and this table in this particular case of combining j 1 equal to half j 2 equal to half, the table turns out to be 1 then you have got root 1 over root 2 in these two positions, and 1 over root 2 and minus 1 over root 2 over here and then the 0's elsewhere you have got 1 over here as we already illustrated.

(Refer Slide Time: 08:44)



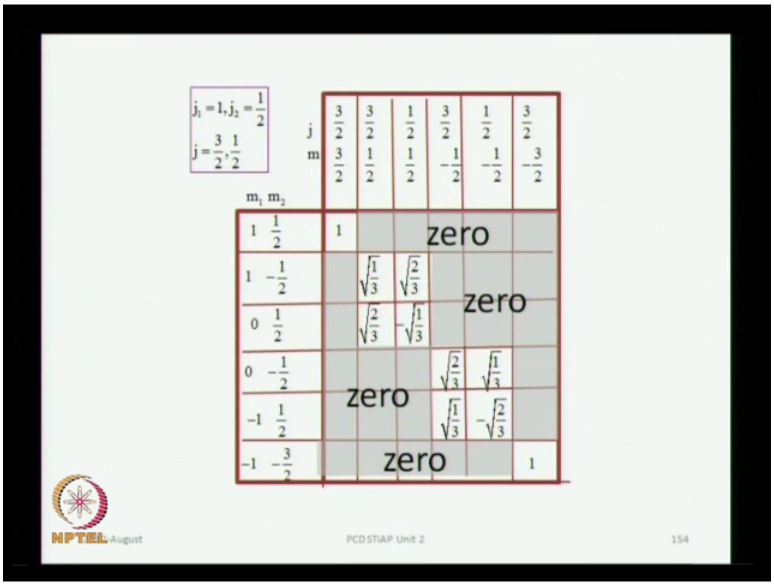
So, let me give you one more example, so that you know the method sinks in this case we will take 1 of the 2 angular momenta to be half j 2 equal to half, and j 1 we will choose to be 1. So, the values of j that we can get out of it would be 1 plus half and 1 minus half, so 3 half and 1 half these are the only 2 possibilities, and what will be the dimensionality of the bases now, it will be 2 j 1 plus 1 times 2 j 2 plus 1. So, that will be 6, so the clebsch gordan coefficient matrix will be a 6 by 6 matrix, how will we compose it first we will label the rows.

So, we generate an area for a 6 by 6 matrix then we label the rows begin with the m 1 m 2, m 1 taking the maximum value that it can, and m 2 taking the maximum value that it can. So, m 1 can take the maximum value corresponding to j 1 equal to 1 right, the maximum value for m 1 for j 1 equal to 1 is 1, and then m 2 equal to half then the next lower value of m 2, which is minus half. Then the next lower value of m 1 which is 0, and then the two values of m 2, and then you go to the lowest value that m 1 can take and then the corresponding 2 values of m 2.

So, this is how you label the rows, so do not mix them because, if you do that you will not get the clebsch gordan coefficient matrix in a block diagonal form, if you do not label them appropriately. Now, you want to label the columns, so notice that you take the highest value over here, the corresponding highest value over here, the lowest value over here, the lowest value of m which is minus 3 by 2 corresponding to j equal to 3 half.

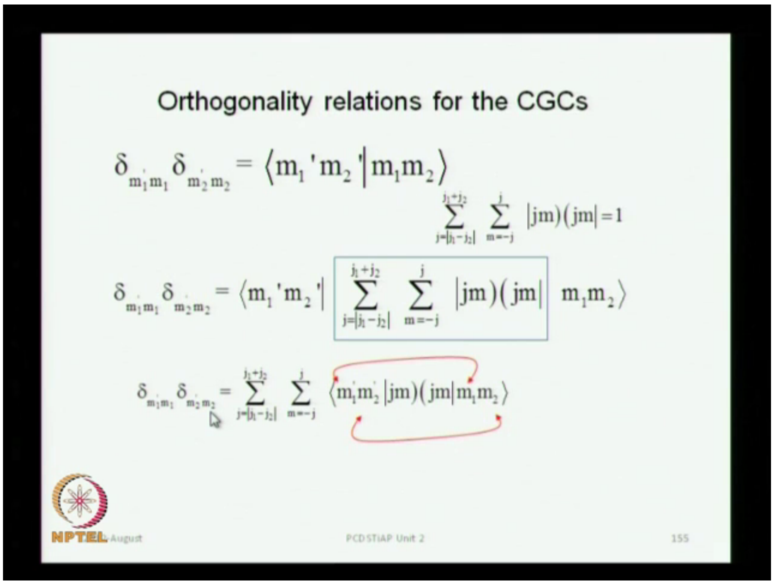
And this is what will give you 1 and 1 in the element position number 1 1 and n n, which is 6 6, 6 over 6 column. And then you use the recursion relations to get the other value, so you can go from one to the other using the recursion relations.

(Refer Slide Time: 11:04)



So, this is what it will turn out to be, so you can generate the clebsch gordan coefficient tables very easily by hand by yourself, these are catalog on the internet if you just Google you will get these tables. So, that you do not always have to calculate this yourself, it is a good idea to may be some of you might want to write a program to generate it, and these are good exercises actually.

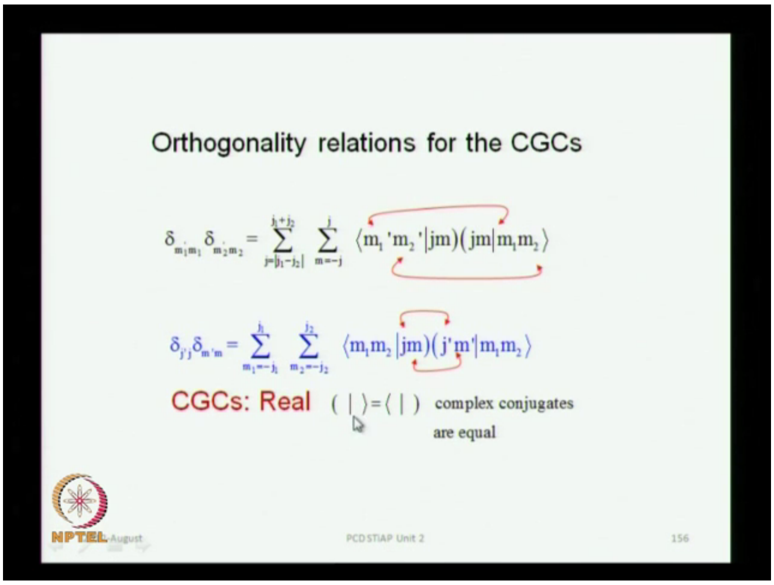
(Refer Slide Time: 11:32)



There are other properties which are very useful, the orthogonality relations of the clebsch gordan coefficients. And you can already see that since you have an orthonormal basis set, so you have an orthogonality between m 1 prime index and m 1 and m 2 prime and m 2. But, you can also plug in a unit operator in between, now when you do that this is a unit operator which is a resolution of unity in the basis, which are the bases of the coupled angular momenta, so this is the j m basis.

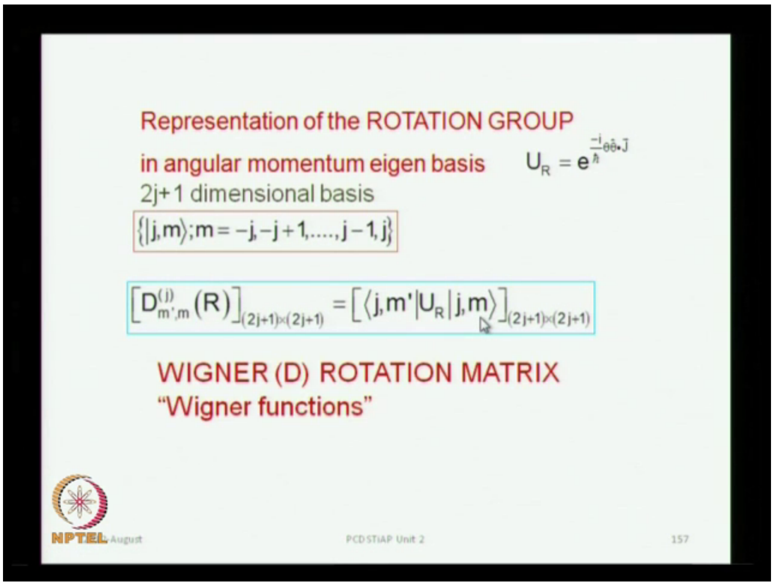
So, which is why I have used the curved brackets over here, and once you plug in this resolution of unity you have this orthogonality relation expressed as the left hand side is the same. And the right hand side is the double sum of a product of these clebsch gordan coefficients, you can also write this element before the other one it does not matter. So, notice that this is an orthogonality between this m 1 prime index and m 1, and then this m 2 prime index and m 2 and what is sandwiched in between is the unit operator. So, it is a very simple relationship all that you have exploited is the resolution of unity.

(Refer Slide Time: 12:54)



So, you have similar relations you can you have to keep track of what the orthogonality is in between, in this expression this is the other orthogonality relation that you get. This is the orthogonality between j prime and j and m prime and m, and you can get it similarly by sandwiching a unit operator. So, it is exactly an identical relation, and these are complex conjugates, so the clebsch gordan coefficients being real you can always write them, with the coupled part on one side and the uncoupled part on the other or vice versa it does not matter.

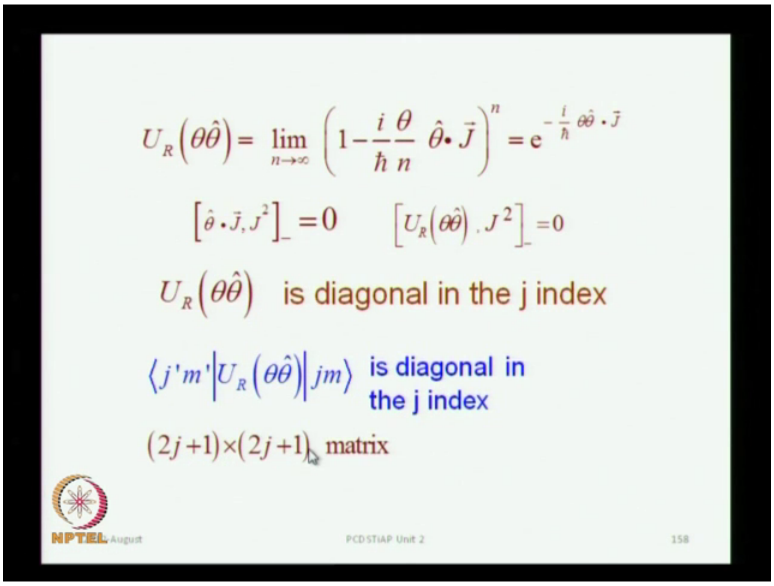
(Refer Slide Time: 13:43)



So, you will see them in different forms, we will now discuss the representation of the rotation group in angular momentum basis, and this is a very simple topic. But, an extremely important one and you will see why it is, so important you will see it being used in atomic physics and also in nuclear physics, very often when you deal with angular momentum properties. So, you already know what the rotation operator is this is the exponential form of the rotation operator.

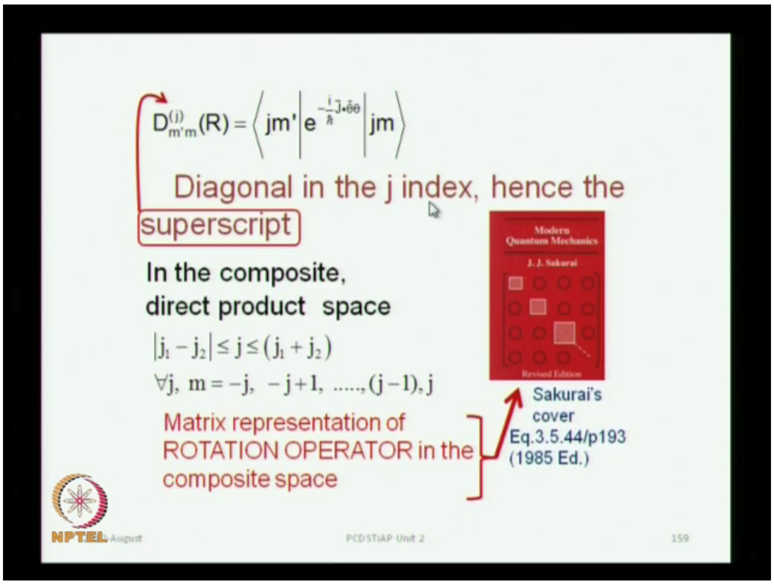
This will be expressed in a basis, which is an Eigen basis of the angular momentum j which will be a 2 j plus 1 dimensional basis. So, the matrix will be a 2 j 1 plus 1 times 2 j 1 plus 1 basis, and each matrix element is the matrix element of the rotation operator in these angular momentum Eigen states right. So, these are known as the Wigner rotation matrices as I mentioned earlier we have met them earlier on sakurais cover.

(Refer Slide Time: 14:53)



So, let us have a look at this in some further detail, now you know that J square commutes with any one component. And since, the rotation matrix for a finite rotation is also made up of a sum of these terms, which generates exponential series the rotation matrix also commutes with J square. And therefore, the rotation matrix will be diagonal in the j index. So, you can always choose J as one of the indices to label it because, it is a diagonal in J index and it will be fixed for every element. So, this property we will exploit that the rotation matrix is diagonal in the j index the matrix size will be 2 j plus 1 times 2 j plus 1.

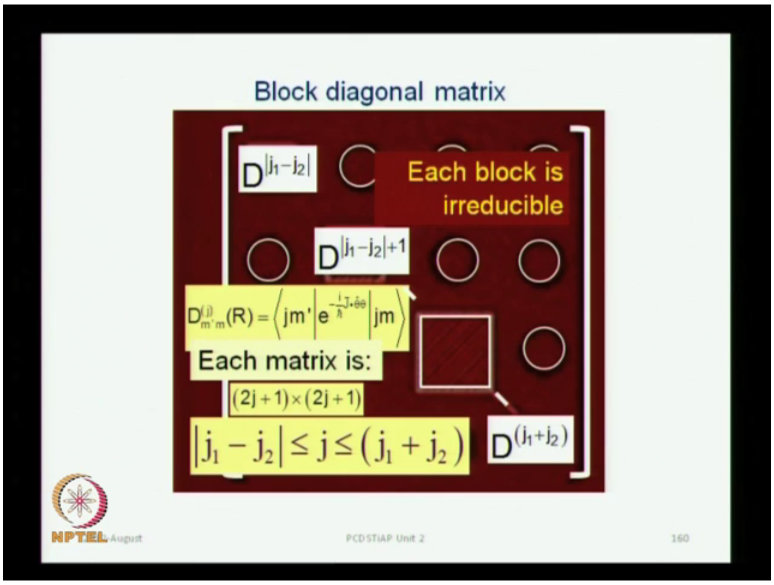
(Refer Slide Time: 15:55)



And you can denote the matrix element by a row index and column index, so this is the row index m prime, this is the column index m. And the super script j will remind you, as to which angular momentum it really belongs to because, so far as this superscript index j is concerned the rotation matrix will definitely be diagonal in this index. So, now let us ask when you have a composition of two angular momenta you take two angular momenta, and combine them generate a composite angular momentum a sum total addition of these two angular momenta.

Then we have seen in our previous class, the j will go from this modulus j 1 minus j 2 to j 1 plus j 2 the triangle inequality that we have seen. And we will now obtain, the matrix elements of the rotation operator when you are dealing with coupled angular momenta. So, these are the ones that really are; obviously, important and have to come on sakurais cover.

(Refer Slide Time: 17:15)

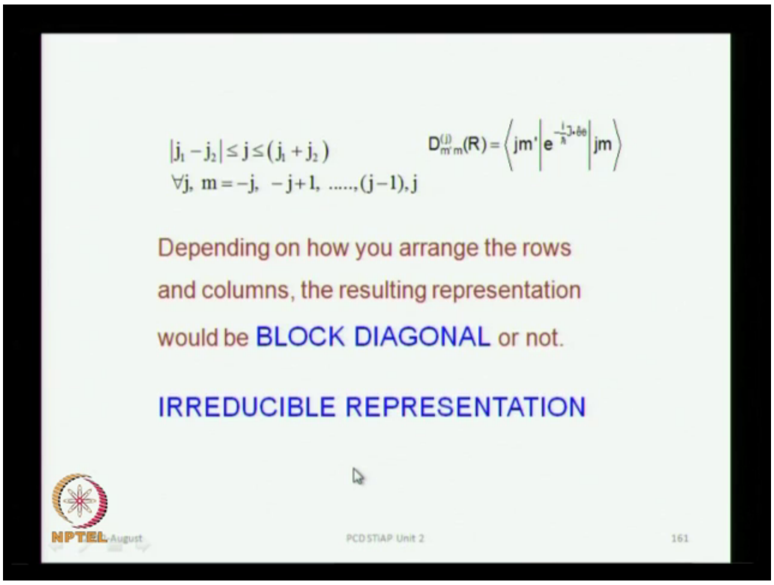


So, the block diagonal form will contain these Wigner D matrices, the smallest block will have a dimensionality which is equal to the modulus of j 1 minus j 2. So, it will be a square matrix, the largest one will have j 1 plus j 2 right, and then you will go one from the other by adding unity. Because, j goes from the lower limit j min to j max in steps of 1, so this is the block diagonal form of the Wigner D matrix, and in each block the value of j will be different, this j is j min for this block it is j min plus 1 for the next block and it is j max for the last block, which is j 1 plus j 2.

So, each block will have a dimensionality 2 j 1 plus 2 j plus 1 times 2 j plus 1, but the value of j will be different it will increase from upward left to the lower right in steps of unity. And there is no way you can reduce any one of these blocks, notice that if you do not label your rows and columns appropriately, you can scramble this or carry out a similarity transformation on this entire matrix. And scramble the elements and get a form which is on in the block diagonal form.

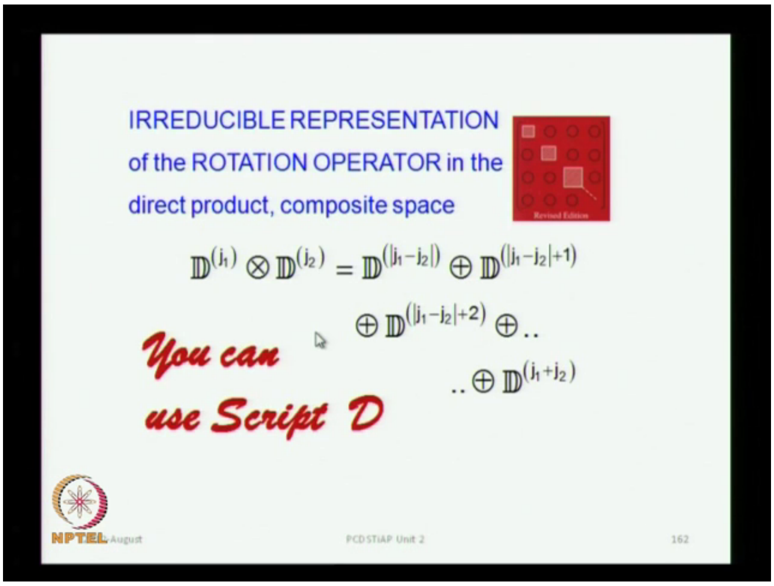
So, typically when you express it in a block diagonal form you have what is known as irreducible representation. A reducible representation is very easy to define it is one which is not irreducible, but you know what an irreducible representation is, so that defines both irreducible as well as reducible.

(Refer Slide Time: 19:16)



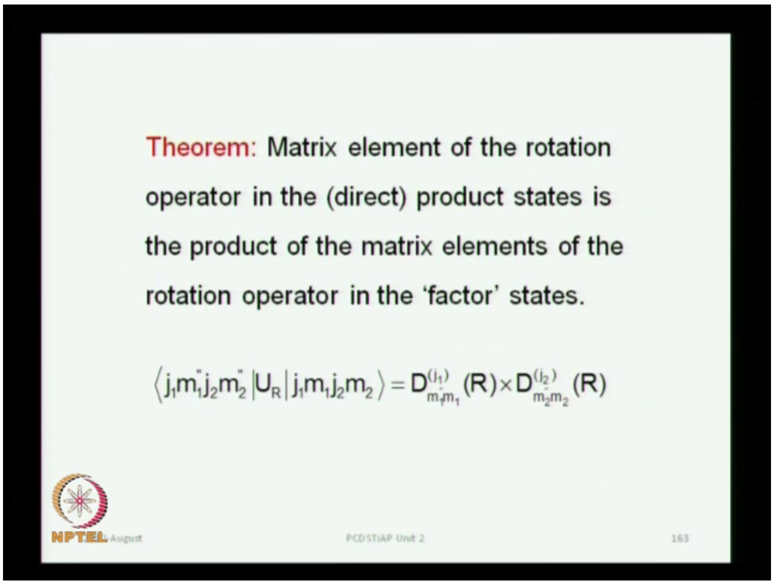
So, essentially it is important to arrange the rows and columns appropriately, so that you will get the matrix in a block diagonal form.

(Refer Slide Time: 19:27)



Now, let us consider the irreducible representations in the composite space, the composite space will then have a number of D matrices. So, this is one in the upper left and this is D j 1 plus j 2 in the lower right, and you know sometimes you use the script D, sometimes a fat D to denote these matrices. So, this is just a matter of notations and different books will use different kind of symbols.

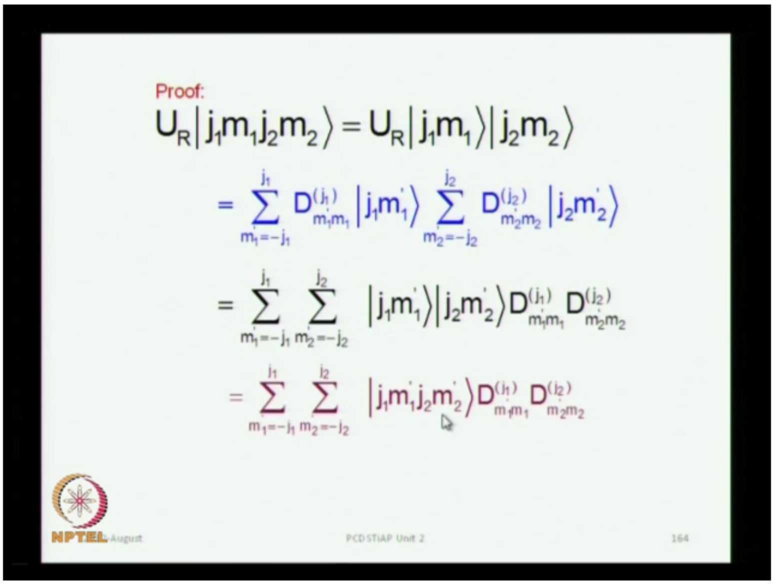
(Refer Slide Time: 20:00)



So, let us do a theorem that the matrix element of the rotation operator in the direct product states, is the product of the matrix elements of the rotation operator in the factor states it is a very simple theorem. But, we will need to prove it this is what we mean, by matrix element of the rotation operator in the direct product state, so this is the direct product right j 1 m 1 j m 2 j 2 m 2 is the direct product of the 2 vectors j 1 m 1 and j 2 m 2, which are independently Eigen vectors of two different completely disjoint angular momenta.

So, there are two angular momenta j 1 and j 2 they have their own Eigen spaces, you combine them to get a net angular momenta. And now we ask, what is the relationship between the matrix element of the rotation operator in the direct product states to what it is in the factor states.

(Refer Slide Time: 21:05)

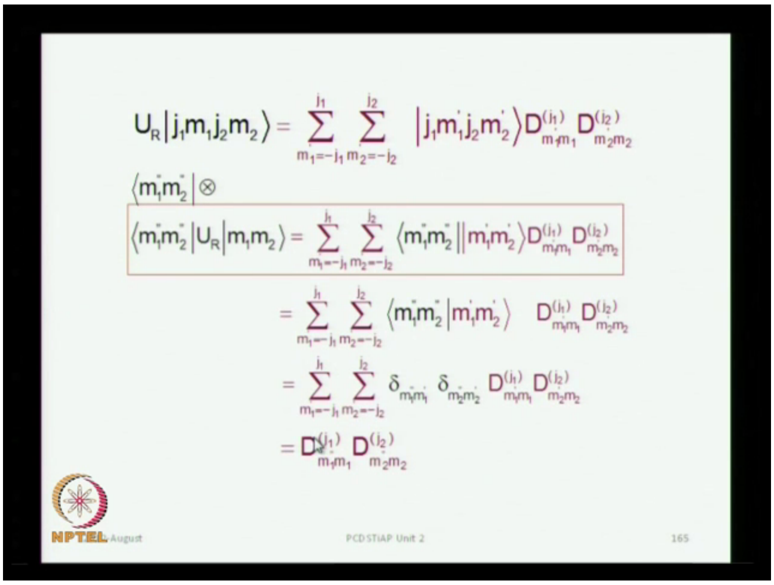


So, we will prove this theorem now and we begin proof by first considering the response of this direct product to the rotation operator, the both being angular momentum states both of these will respond to the rotation. So, j 1 m 1 will respond to the rotation and the result will be expressible as a linear superposition of the component vectors in that corresponding space. So, you will carry out a sample over a dummy index m 1 prime from minus j 1 to plus j 1, and the coefficients will of course, be the Wigner D matrix elements, in the factor states right.

Likewise, this j 2 m 2 will also respond to the rotation and you will get a linear superposition of an mixture of the base vectors. So, m 2 prime will go from minus j 2 to plus j 2 in this, and you combine these base vectors with appropriate coefficients which will be the Wigner D matrix elements belonging to the j 2 space. So, now let us write this in a slightly different form in which all I have done is write the Wigner D matrix elements over here, and then I can once again get these factor states written out explicitly over here.

So, it is the same expression written differently and I compose the direct product states of these two factor states to get j 1 m 1 prime j 2 m 2 prime, and this is the result of the operation by the rotation operator on a direct product state. So, you have to compose this superposition over double, you know this is a double summation over m 1 prime and m 2 prime.

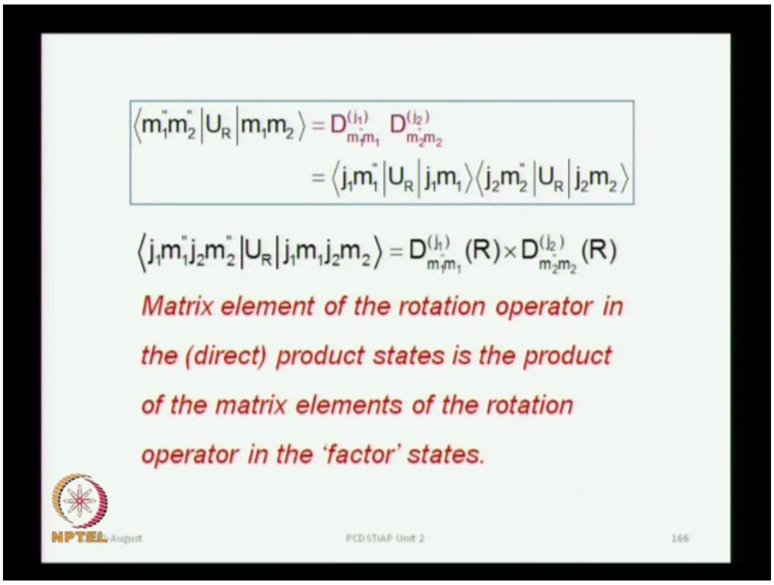
(Refer Slide Time: 23:14)



So, we get this result and we will bring this up to the top of the next slide which is here, and now you what you do is to take it is projection on another direct product state. But, this time we choose this to have different indices, so since we have already used m 1 and m 2 and also m 1 prime and m 2 prime, I have used m 1 double prime and m 2 double prime, so that we do not mix up the indices So, I take the projection of this entire equation on a direct product state.

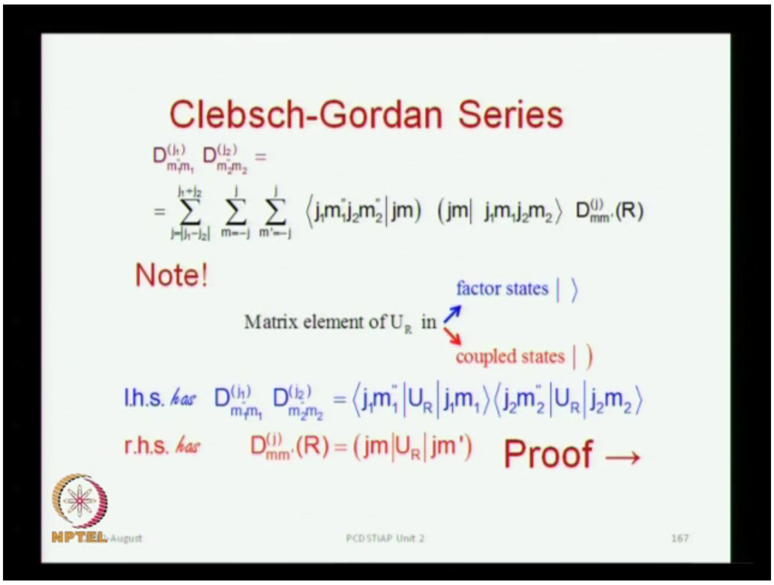
So, now, on the left hand side I will get the matrix element of the rotation operator in the direct product states. And here you have the matrix elements of the rotation operator in the factor states, but now you have this element as well, but you can easily recognize that this is going to give you the chronicle deltas because, this is coming from an orthonormal basis. So, only those terms will survive for which m 1 prime is equal to m 1 double prime, and m 2 prime is equal to m 2 double prime and there is a double summation over here. So, you contract the sums and that really completes the proof, so this is the matrix element of the rotation operator in the direct product states, and you find that it is a product of the matrix element of the rotation operator in a factor states.

(Refer Slide Time: 24:48)



So, that was the theorem that we had set out to prove and it has a very simple proof as you have just seen. So, all you do basically is to make use of the orthonormality conditions of the basis sets, and use the usual transformation properties of how an angular momentum, vector responds to rotation. And that is the basic relation that you already know and you will find that almost all of this algebra is based on very simple properties of the response to the rotation operator.

(Refer Slide Time: 25:21)



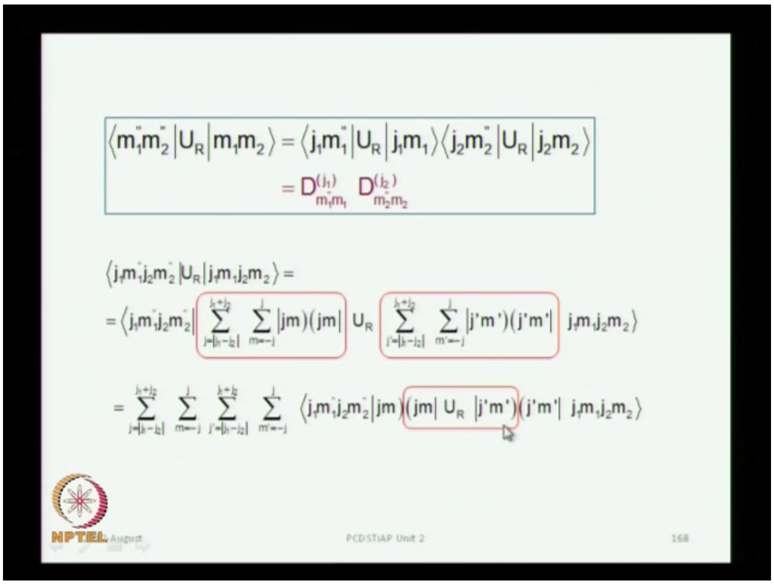
Now, there is another theorem which gives us a result which is known as the clebsch gordan series, this is also a very important relation, so I will establish this as well. And this equation is what is called as the clebsch gordan series, notice that you have on the left hand side product of two rotation matrix elements. And these are in the factor states, on the left hand sides both of these, this is the matrix element of the rotation operator in the j 1 m 1, so this is in the Eigen basis of j 1, this is in the Eigen basis of j 2.

So, the left hand side you have got the matrix elements of the rotation operator in the factor states, on the right hand side you have a triple sum then you have two clebsch gordan coefficients. And you have another rotation matrix element, but this one is not the coupled state, this is in the j m basis, so on the right hand side you have the coupled angular momentum j m. So, this is the matrix element of the rotation operator in the coupled basis.

And now you will begin to see, there is a little bit of advantage of choosing the notation in which you differentiate between the coupled vectors and the uncoupled vectors, both are angular momenta. And it really does not matter because, the bracket is a bracket is a bracket, it really does not matter how you denote it, but I have chosen to denote the brackets of the uncoupled vectors by angular brackets, and the brackets of the coupled vectors by the circular brackets.

Because it helps me keep track of what is what, otherwise they all look the same and then you lose sight of the physics on it sometimes. So, this is just a matter of convenience, and we will prove this theorem this is what is known, what leads us to the clebsch gordan series.

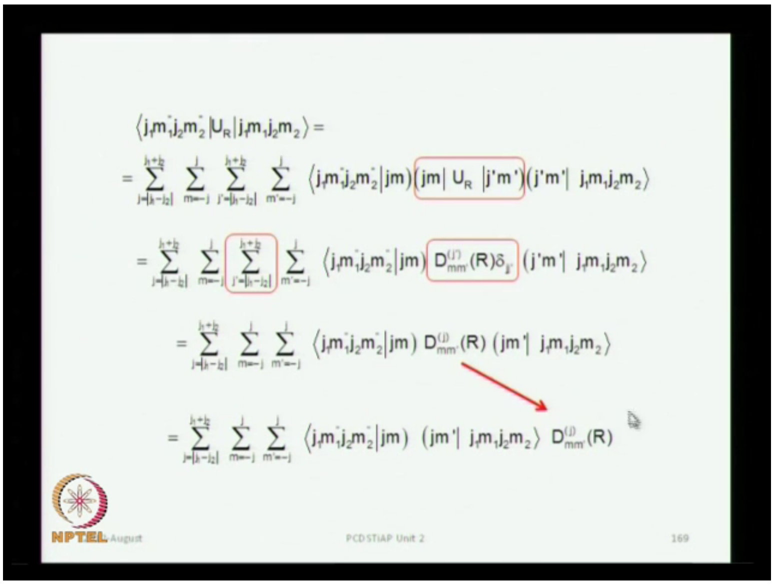
(Refer Slide Time: 27:31)



So, we will use the previous result that the matrix element of the rotation operator in the direct product states, is the product of the matrix element of the rotation operator in the factor states. So, that theorem which we just established we will use that, and using the statement what we do is between this and this rotation operator I sandwich a unit operator over here. And I sandwich plug in a unit operator here as well, and this that is all I have done in time, which is to plug in a unit operator.

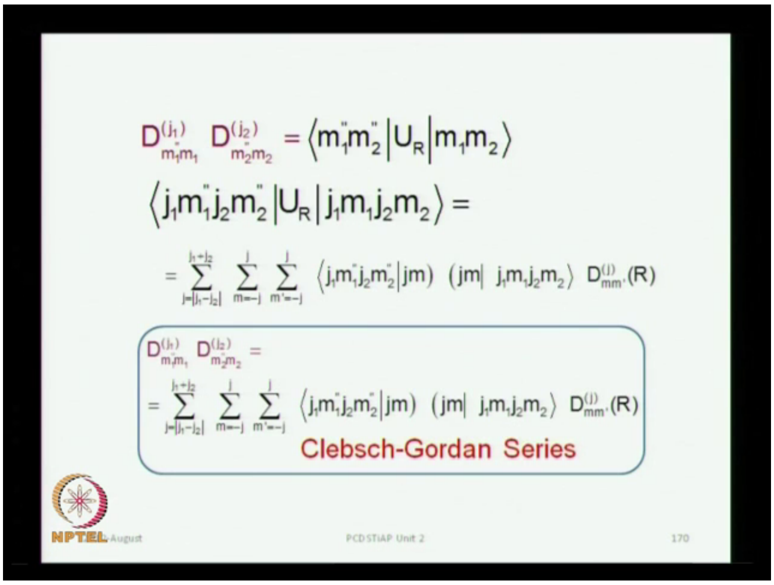
But, this unit operator comes from the resolution of unity in the coupled basis, that is the idea. So, you plug in the resolution of unity, twice one before the operator U and one after it, and now you notice that when you look at this relationship you find in the middle the matrix element of the rotation operator in the coupled basis. Now, this is what you wanted to appear on the right hand side, because on the right hand side in the clebsch gordan series, you wanted to see the matrix element of the rotation operator in the coupled basis. And you can discover it very easily by noticing that right here, you have that element already, so it is a very simple trick.

(Refer Slide Time: 29:05)



And you find this matrix element over here which is nothing, but the matrix element of the rotation operator, but we also know that this is diagonal in the j index, you know that this is diagonal in the j index. So, this will carry the superscript j prime with the 2 j's to be equal, so j must be equal to j prime, but do not you have a summation over j prime somewhere here it is. So, you can exploit that and use the chronicle delta contraction, so the sum over four dummy indices reduces to a sum over three because, you exploit this delta D j prime over here. And, now you have the superscript j prime over here is j prime equal to j, and that pretty much completes the relationship that we are looking for. Because, you can just rewrite this as a product of these two clebsch gordan coefficients, write this matrix element at the last.

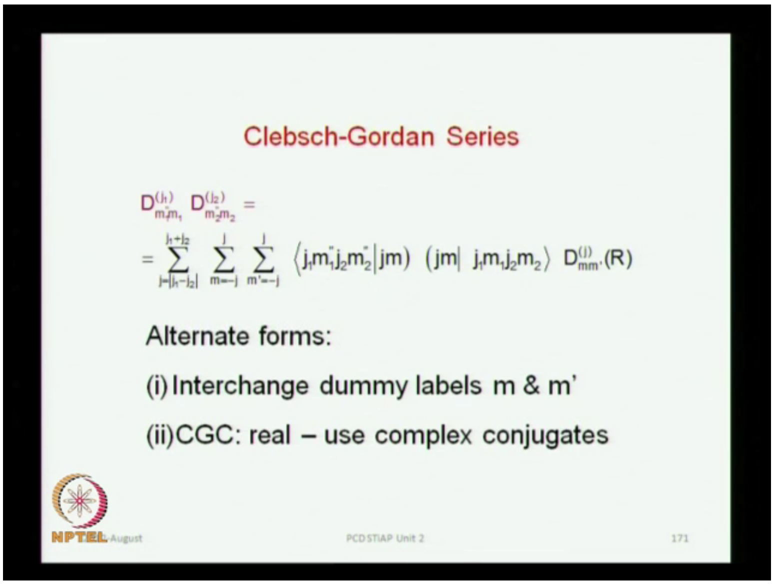
(Refer Slide Time: 30:13)



And you already have this result from the previous theorem that we established right, so we use it and we have the net expression, which is famously called as the clebsch gordan series. It is a very simple proof, and all you have to do is to learn to play with these resolution of unit operators, use the orthonormalities and you will have to do a lot of algebra using these techniques.

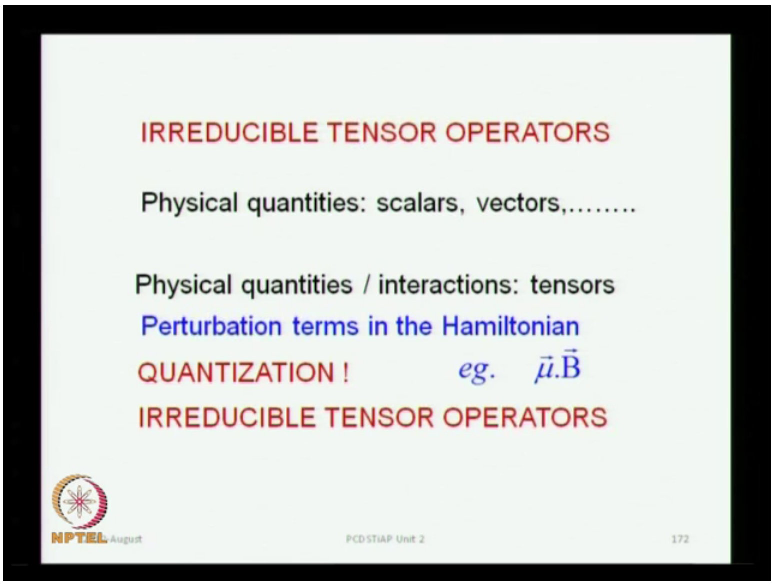
But, you run into a large number of summations sometimes summations over 3 dummy indices or 4 dummy indices 6 and 7 and even 10 and more, when you do complicated you know atomic physics. And it is very easy to make a mistake, so make sure that you keep track of what is what.

(Refer Slide Time: 31:02)



So, you will find alternate forms in different books because, you can interchange the dummy labels. Since the clebsch gordan coefficients are real, they can be transposed and you will find them written differently, but they will all be completely equivalent of this, and you have very simple ways of going from one form to another.

(Refer Slide Time: 31:22)

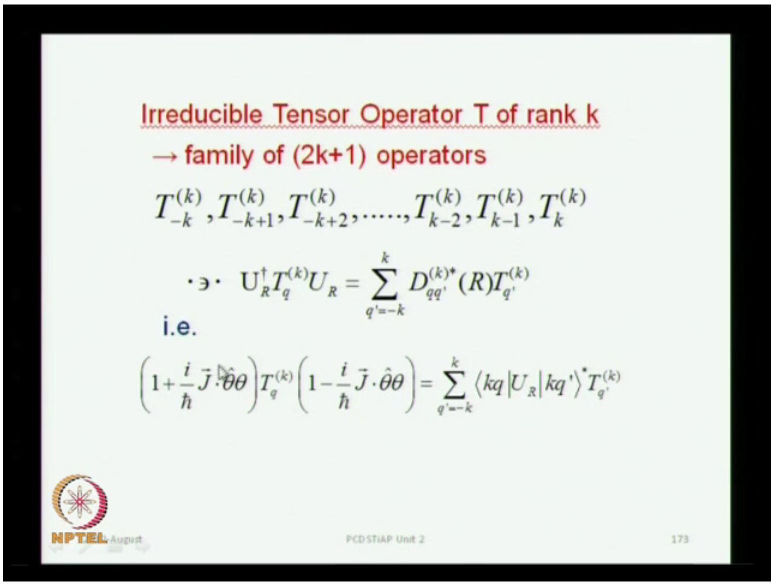


So, what I am going to do now is to define what is known as irreducible tensor operators, and these are really important. These are, you know in classical mechanics also we deal with physical quantities, and these are scalars, vectors and so on right whatever, you work with physical quantities like temperature, energy, velocity, force, momentum you know these are physical quantities that you deal with. And these are all tensors of various ranks.

So, 0 tensor of rank 0 as a scalar tensor of rank 1 is the vector, tensor of rank 2 sometimes you call it as a die deck, and you know there are various tensors of various ranks that you work with. And these physical quantities will, then correspond to certain dynamical variables in classical mechanics, for which you will have corresponding quantum operators because, that is what quantization is about. So, interactions will also be represented by some operators, which will be some sort of tensor operators.

So, perturbations for example, in the Hamiltonian, so here is an energy perturbation if you have a magnetic dipole, which you put in a magnetic field. So, this will be some sort of a perturbation, and you will express it as some tensor and then you will quantize it have the corresponding quantum form. So, all of these are made up of you know fundamental tensor operators, and it is for this reason that this is an extremely important topic.

(Refer Slide Time: 33:04)



So, I will first define what is called as a Irreducible Tensor Operator ITO, an irreducible tensor operator is defined to be 1 of rank k. And it is actually not just a single operator, but it is a family of operators it has got number of you know members, this family has got a certain number of members. The number of members is 2 k plus 1, where k is a rank of this operator.

So, the irreducible tensor operator rank k is a family of 2 k plus 1 operators, and you designate these by writing the rank k as a superscript. And the members, as subscripts and 2 k plus 1 you designate by an index which goes from minus k to plus k in steps of 1. So, it has got a terminology which is very similar to what you have found in angular momentum designations right, so it is the same kind of notation that you carry.

And the defining relationship, and this is what is called as a defining criteria, this equation is what defines this irreducible tensor operator. So, it is a family of 2 k plus 1 operators such that a certain relationship holds, now what is that relationship that is the defining relationship, it is this equation it tells you how a member operator transforms under rotations. The responsor rotation, and this is again something this is not new to us because, we know how tensors are defined even in classical mechanics right.

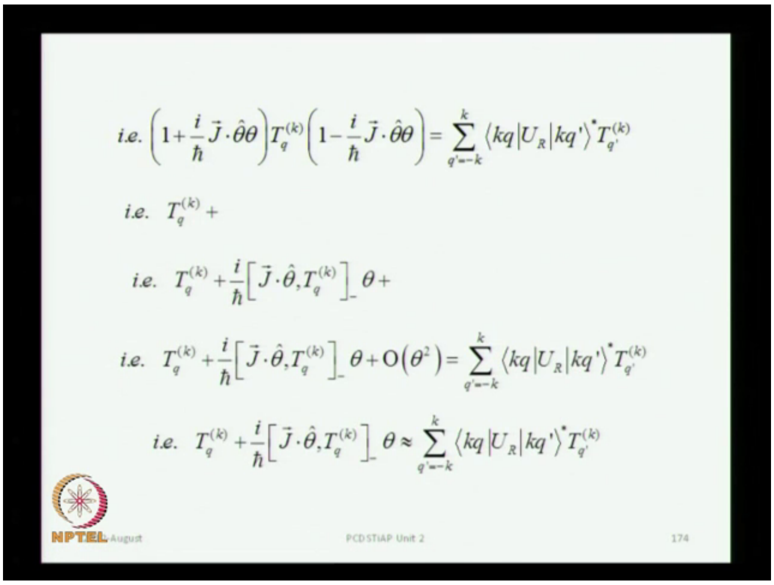
That basically a scalar is a it is characteristic feature is that it remains invariant, under the rotation of a coordinate system right. Vectors are those whose components, transform according to the cosine law, so the law of transformation how the system response to rotations is what defines a tensor. So, it is the same idea it is exactly the same idea, what you must examine is how these quantities respond to rotations, and we know that the response of any operator to another operator is shown by U dagger T U, right.

So, if U is a certain transformation right U dagger T U will tell you how it response, so this is what defines the response of the operator T k q to rotations. And the law of this transformation is defined by the right hand side, the right hand side tells you that the response of any member to rotation is that the result is expressible, as a linear superposition of the entire family of the tensor operators, with appropriate coefficients. And these coefficients are the Wigner D matrix elements.

Now, you see that these Wigner D matrix elements are you know showing up, at many new places which is why they are really, so important. And the Wigner D matrix elements appear as coefficients in this expansion, so this is the law of transformation, this is the definition of what an irreducible tensor operators. Now, we know what the rotation operator is, the rotation operator is 1 minus i over h cross j dot theta, where this is any direction in space, this is the rotation operator, this is the ad joint. So, the left hand side is a product of these three operators.

So, we have written the rotation operator explicitly now, so it is just that definition that we have rewritten explicitly in terms of the rotation operator. And we can find what we get from the product of these three operators, and you do it term by term because, you will multiply this is a unit operator. So, you will get a term in 1 into 2 into 1 then you will get a term in 1 into T into this operator, which is the J z or J U which is the component of I in one direction. Then you will get this term J z or J U into T into 1, so term by term you can expand the left hand side.

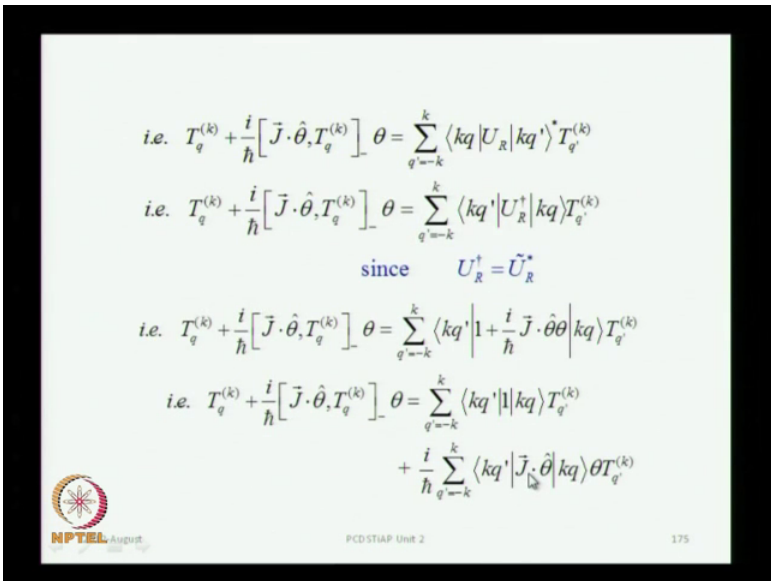
(Refer Slide Time: 38:08)



So, let us do that the first term is the operator T which you get from the product of 1 into T into 1 right. Then you get these 2 terms J dot theta into T and then you get another term with a minus sign here, which is T times J dot theta right, so you will get the committer because, the first one comes with the plus sign, and the second one comes with the minus sign. And I have extracted the i over h cross as common, and then since you get the first term with the plus sign and the second minus, you get a committer of these two. And then you get a third term, which will be quadratic in the small infinitesimal angle, and the square of a small angle is ignorable, so you can throw that term you get a quadratic term which is coming from the last term. So, essentially all we have done is to rewritten the definition of the irreducible tensor operator, explicitly in terms of the definition of the rotation operator, and expanded the algebra.

So, now, after ignoring this quadratic term, you have the remaining two terms which is the first term and the second term, which is nearly equal to the right hand side. And this nearly equal to sign only reminds me that I have ignored this quadratic term, it is of no consequence in linear superposition's.

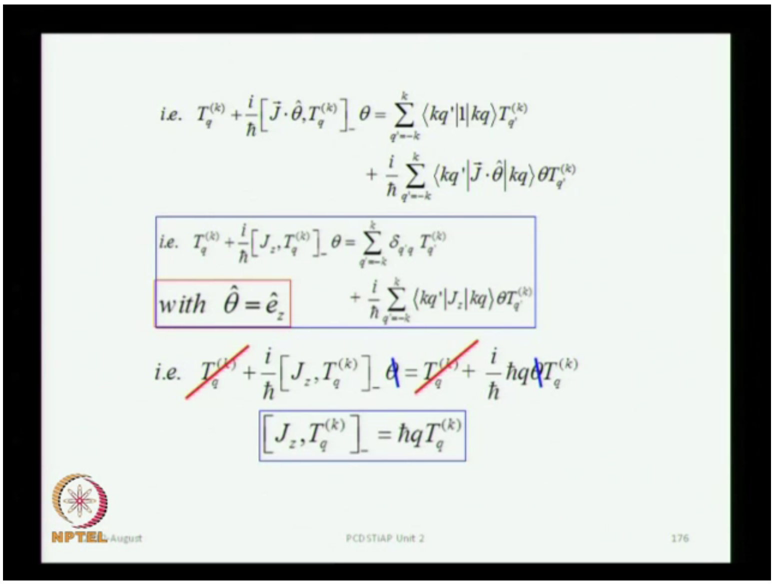
(Refer Slide Time: 39:51)



So, we will throw that term completely and we will write an equivalence here after, so now we have this relationship, and here these coefficients are exploit the fact that this is a unitary operator. So, I can do the transposition and complex conjugation, so I have written this in terms of matrix element of the joint of the operator, and the joint operator you know is 1 plus i over h cross J dot theta. So, now, you have a very simple relationship, and right hand side also now you can simplify because, here you have got the unit operator.

So, when you take the projection of this k q and k q prime you will get a delta q q prime, and summing over q prime you can contract the chronicle delta. So, you should always be looking for these orthonormalities wherever you can find, them as that will simplify the algebra. So, now you have the left hand side written just as it is, the right hand side has this unit operator, the matrix element of the unit operator in k q prime and k q, which is the first summation over q prime. The second term is again a summation over q prime, but a summation of matrix elements of J dot theta, theta is just a multiplier which I have written outside this bracket. Basically you are getting the matrix element of this J dot theta which is a projection of the angular momentum in some direction does not matter what.

(Refer Slide Time: 41:40)

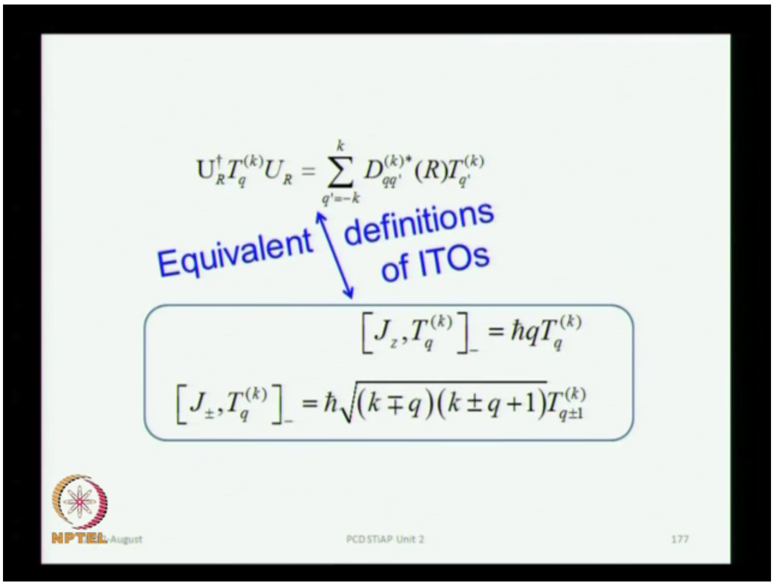


So, now you have a contraction showing up from this first term because, this is a matrix element of the unit operator. So, you get the delta q prime q I choose the axis of quantization to be e z just for simplicity does not matter whatever it is, and then you have the matrix element of j dot theta, which is a matrix element of J z over here. So, this is the matrix element of J z and J z operating on k q will give you q times h cross times the same vector right.

And that q time h cross comes out as a multiplying factor you get the orthonormality between k q prime and k q, and then you sum over q prime. So, that only the term in q prime equal to q survives, and there is only one term surviving over here, and now you see that you have got this operator, which is equal to this operator. So, these two terms cancel, in the remaining term you have got theta which is just as tiny angle which is this common angle in both sides of the equation.

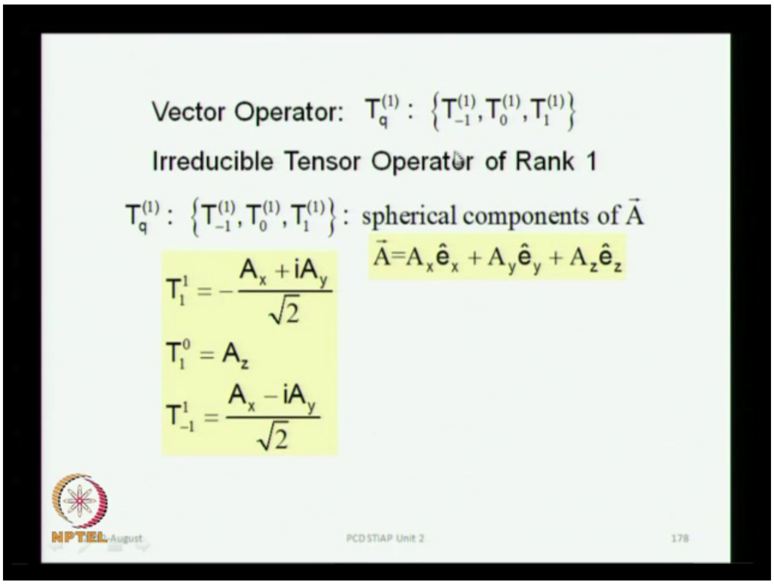
So, that cancels and you get a commutation relation between the irreducible tensor operator and angular momentum theta. We have got this essentially from the fundamental criterion of how we define the irreducible tensor operator, so there is nothing new that we added, all we did was to manipulate the terms. In other words, this relationship can actually be used to define an irreducible tensor operator.

(Refer Slide Time: 43:30)



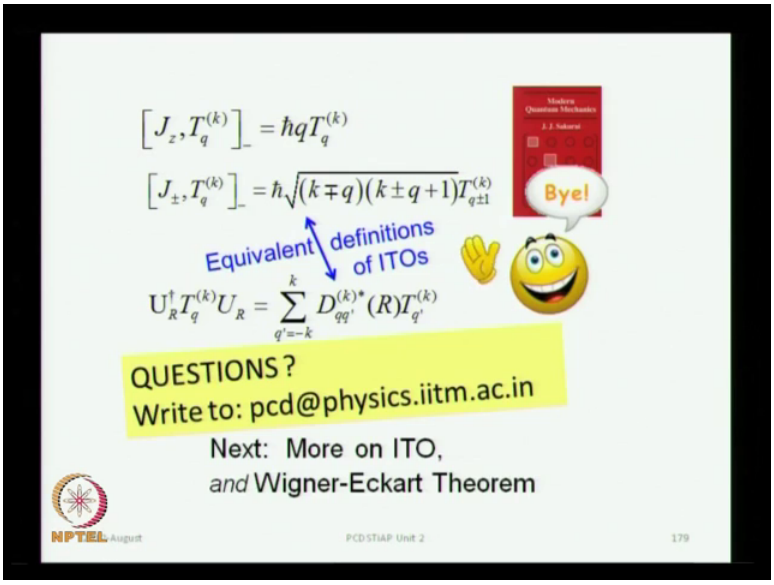
And you can certainly define the irreducible tensor operator using this, so these are equivalent definitions, you can define it also with respect to the J plus and J minus by working out similar algebra I will not work out all those details. Now, it is a very similar exercise as we just did, and you will find in various books an irreducible tensor operator defined, either by the relation at the top or by the relationship that you see at the bottom, which is in terms of how the components. The member of the irreducible tensor operator, how the family, each member of the family commutes with the angular momentum operators J x J y and J z, so J plus and J minus you know are made up of J x and J y, so the commutation relation of the irreducible tensor operators can also be used to define irreducible tensor operators.

(Refer slide Time: 44:30)



And that with you can take an example over here of an irreducible tensor operator of rank 1, you expect it to be a vector operator. So, you write the spherical components of this the Cartesian components are A x A y A z, and the irreducible tensor operators are minus 1 0 and 1. This is an irreducible tensor operator of rank 1, so it is members will go from minus one to plus one in steps of 1 and these are the transformations between Cartesian and you know the spherical components.

(Refer Slide Time: 45:02)



And we will be using this, the next we will be introducing a very important theorem in spectroscopy, which is known as a Wigner eckart theorem. So, now, we have the tools to discuss the Wigner eckart theorem, it needs the angular momentum algebra which we have, it needs a good handle on how angular momenta are coupled. We have that we know how to use the rotation matrix elements, we have the clebsch gordan series with us, we have got the addition theorem of the spherical harmonex with us.

So, now, we have all the tools to establish what is an extremely important theorem in quantum mechanics, and specially in spectroscopy in atomic spectroscopy, molecular spectroscopy, nuclear spectroscopy any branch of spectroscopy. Any branch of spectroscopy, you it could be nuclear magnetic resonance spectroscopy if you like in any branch of spectroscopy. Because, what you deal do in spectroscopy is to look at transitions, and these transitions are from a certain initial state to a final state, induced by a certain physical interaction.

And this physical interaction is represented by an irreducible tensor operator, so what you are looking at is the matrix element of a tensor operator. And the states will have some angular momentum elements, so you will have to extract the matrix element of these tensor operators, in angular momentum states or what the Wigner eckart theorem does is to help you, look at this in a very simple fashion.

So, this is at the very heart of any branch of spectroscopy, atomic spectroscopy, molecular and Zeeman spectroscopy starc affect, Zeeman affect, NMR spectroscopy, double resonance spectroscopy, Mossbauer spectroscopy whatever you are referring to and that is what a physicist does. Because, ultimately you are going to look at interactions and transitions between you know a certain initial state to a certain final state, those are the observables of a quantum system. So, we will introduce the Wigner eckart theorem in the next class.