WAVE OPTICS

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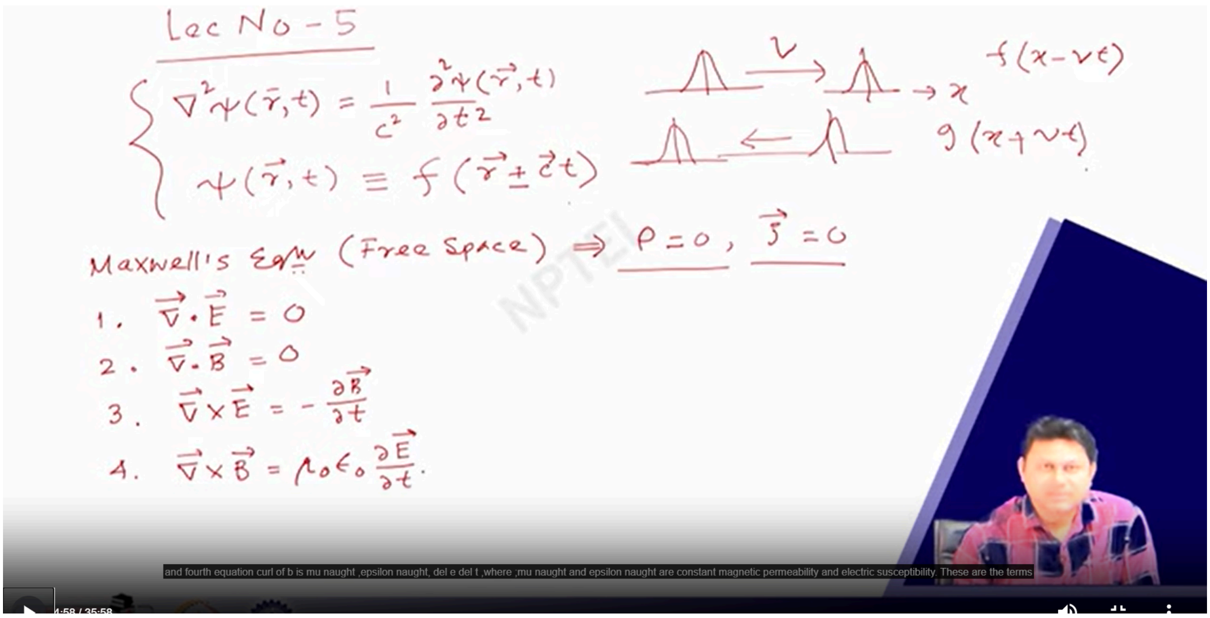
Lecture 05 : Maxwell's wave equation, Poynting Vector

Hello, students in our wave optics course. Today we have lecture number 5 and in this lecture, we will discuss Maxwell's wave equation and the concept of the Poynting vector. So we know that wave, so today we have lecture number 5. And so far what we have done is we derive a wave equation in this form in 3d a wave equation can be represented in this way or 1 by v squared. This is our wave equation and also we know that this wave should have a solution of the form function of in this case R minus CT. In this form, C can also be written as a vector. In general, this solution should be plus minus because if the wave is moving along a forward direction, along this direction with positive x for example; let us consider one dimension then it is f x minus vt with the velocity v and if it is moving in the opposite direction, this direction so this is my initial wave and it is moving along this direction then it should be g of x plus vt something like this. So today we're going to solve Maxwell's equation. First, we derive Maxwell's wave equation and then try to find out the solution of that wave equation. So let us start with the basic Maxwell's equation that we have. So let me write down Maxwell's equations, I'll write and I'll do that for the simplicity of free space knowing that in free space we have the source term that the charge density and the current density are zero. (Refer slide time: 5:55)



This is the condition when we have Maxwell's equation in free space. Now if I write down one by one the four Maxwell's equations it should be something like that first is a divergence of e is equal to 0, since rho is 0 it should be 0, for free space second equation divergence of b will be 0, third equation curl of electric field vector will be minus of del b del t Faraday's law and fourth equation curl of b is mu naught, epsilon naught, del e del t, where; mu naught and epsilon naught are constant magnetic permeability and electric susceptibility. These are the terms we have when we write down Maxwell's equation in free space. Now from these four equations exploiting these four equations it is possible to write down an equation which can be which has a resemblance with the wave equation whatever is written here. Let us do that treatment, maybe you can go to the next page. So, in that case, I need to write down once again quickly the Maxwell's equation. Let me do that. There is no harm in writing this equation several times. So, the divergence of E is 0, the divergence of B is 0, the curl of E, that is minus of del V del T and the curl of B is mu naught, epsilon naught, del E del T. Now from these four equations what we do is like this. We take this third equation and make a curl of both sides. Let us play with this equation in this way. Curl of E, If I do then on the right-hand side also I need to do that and I can write this equation in this way. So curl of e is the vector identity, a very famous vector identity. I hope most of the students attending this course should know and if I use this identity it should be the Laplacian of e vectors with a negative sign plus gradient of the divergence of e. That is a vector identity when you have a curl of some vector quantity E. On the right-hand side on the other hand this curl cross B I can replace from equation 4 which is there. If I do, I am going to get something like a minus of mu naught epsilon naught double derivative of T E.

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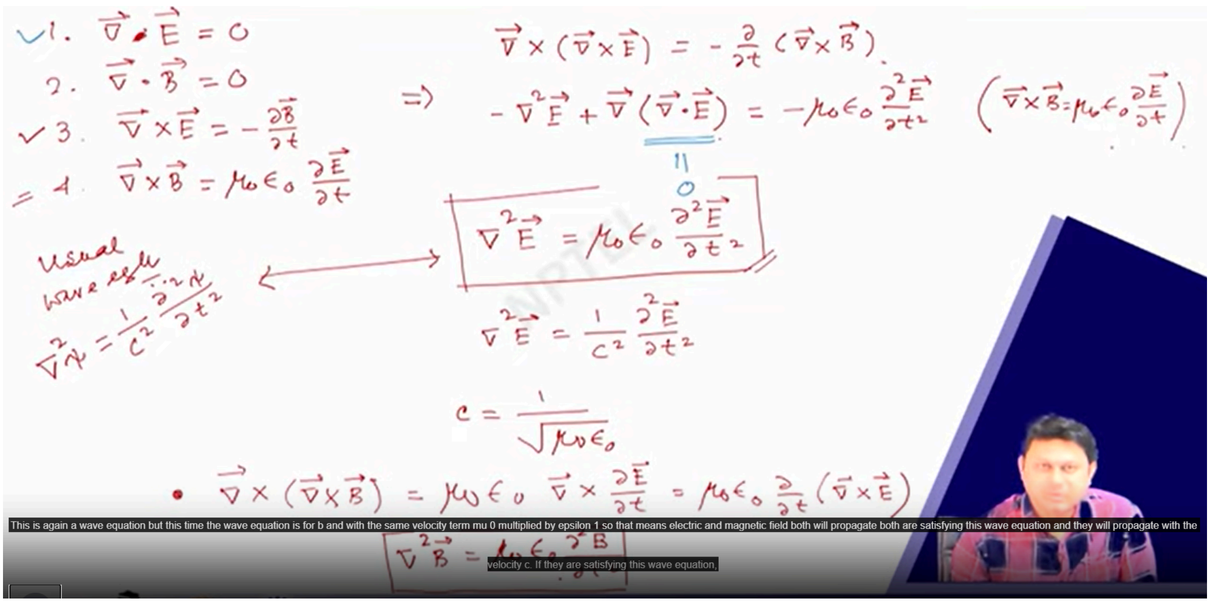

I should write as the curl of B is del E del T with a negative sign and write it clearly. Using this it is equal to minus of this quantity. So if I do that then this quantity should be plus and then what I get is no curl of B is del E del T. So, this is a plus. So, this minus will remain there. There will be no minus sign. This is my mistake. So, it should be simply this. It is okay. I made a mistake here also. This mu 0, epsilon 0 terms I missed.So, let me rectify it first. So there; okay, let me erase this entire thing and write it down clearly so curl cross b is mu 0, epsilon 0, del e del t, and that I replace here. Now this term again, is zero because already we have from equation one, the divergence of e is zero in free space. If that is the case then we have a minus sign here that is already there. So, we will have an equation which is a divergent Laplacian of E is equal to mu naught epsilon naught del 2 E del t square. This is the equation I can get by simply manipulating equations 3, 1, and 4. Exploiting these three equations, we can get this in free space. Now, if I write down the wave equation side by side here, the usual wave equation that is in this form is the divergence of psi that I have written. On the previous page, this is the equation. Now if I tally these two equations, we can see that, these two equations are almost identical; almost identical means -only one quantity is replaced here and that is the velocity term we know that if I write this equation in this way then this equation will be identical to the wave equation but in that case, we need to put the value of c as 1 over root over of mu naught, epsilon naught and indeed if somebody put this value one over mu naught, epsilon naught the actual value of these things which are constant. Then they will go to get the value of the velocity and in this case, this velocity is a velocity of light that's why I define this as c.

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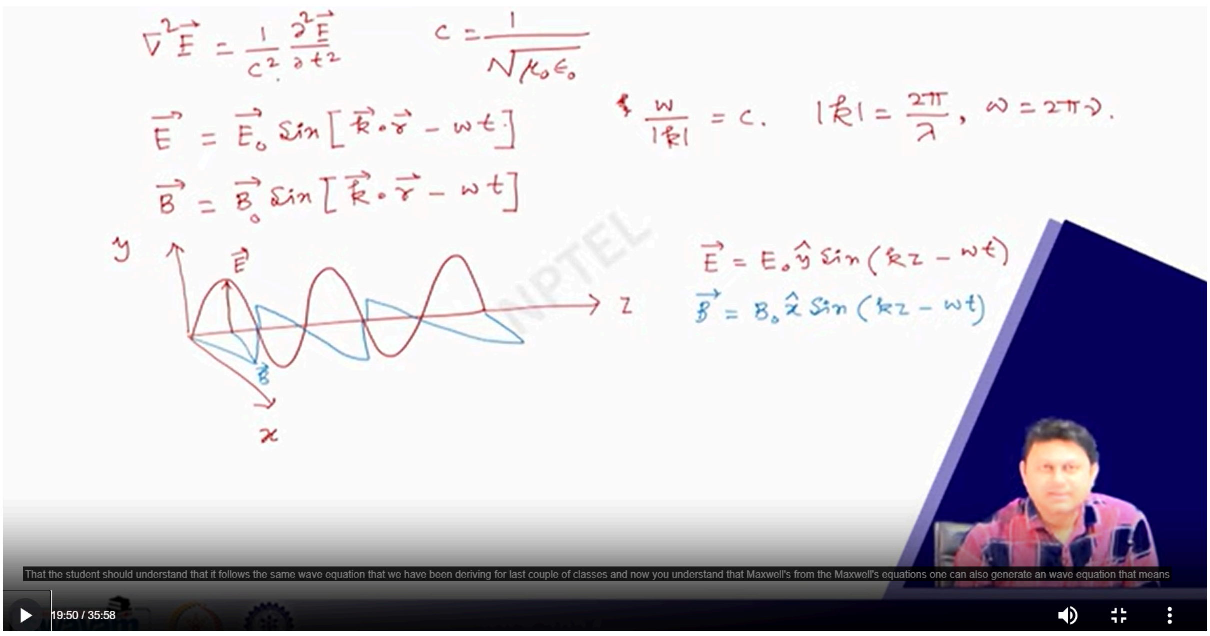
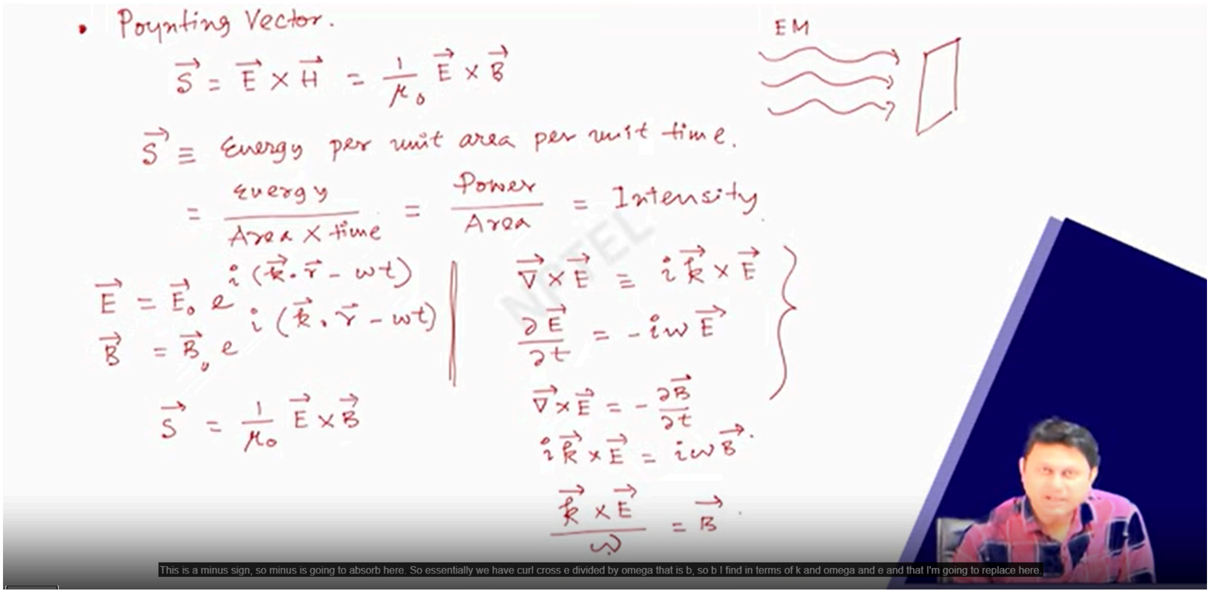
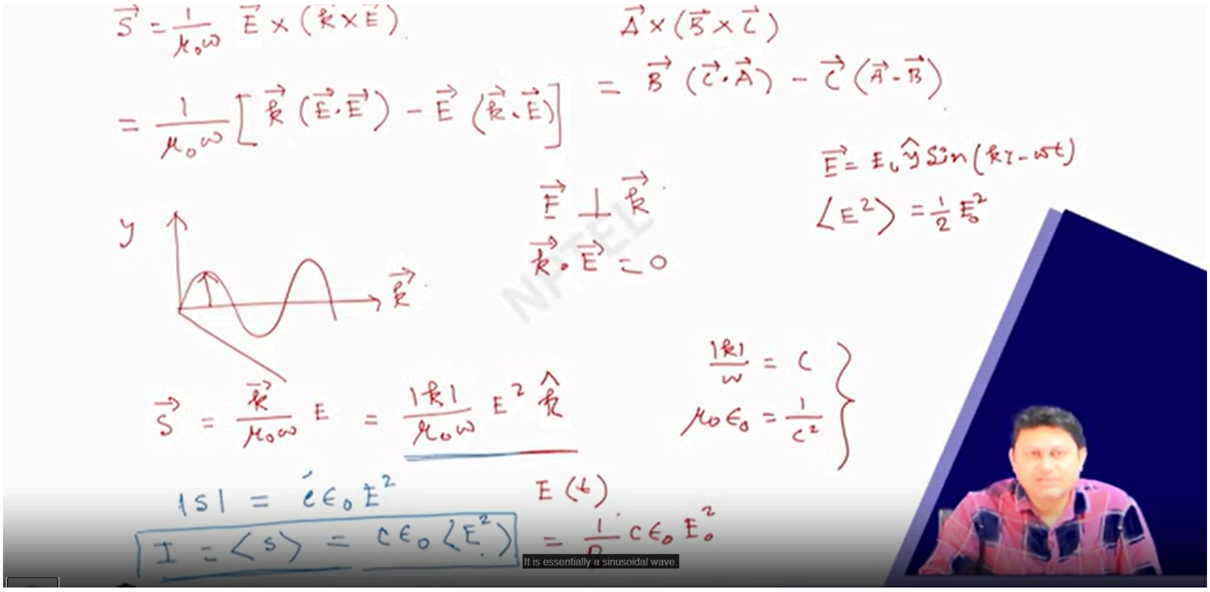


So from this expression, one can understand that the electromagnetic wave or light is nothing but an electromagnetic wave. Which is propagating with a velocity c and this velocity c is light the velocity of the light. So an electromagnetic wave, light, is an electromagnetic wave which is moving with a velocity c and this value is 1 over root over mu 0, epsilon 0. Now I say electromagnetic waves so that along with the electric vector magnetic field can also propagate and that should propagate in the same velocity C. So one can also get a similar equation if they start with this equation for example. So I can, I will not do the entire calculation. Rather I will suggest to the student to do this by himself. If you start the calculation by taking the fourth equation and take the curl out of that, make the curl on both sides like this way. On the right-hand side, you need to do mu 0, epsilon 0, and then the curl of the quantity del E, del T. So that is nothing but mu naught, epsilon naught del del T of the curl of e exactly the same procedure. The only thing is that you start with equation 4 instead of going to equation 3, instead of taking equation 3 and at the end of the day you will get a result which is Laplacian of b by removing e we are going to get an equation like this. This is again a wave equation but this time the wave equation is for b and with the same velocity term mu 0 multiplied by epsilon 1 so that means the electric and magnetic fields both will propagate satisfying this wave equation and they will propagate with the velocity c. If they are satisfying this wave equation, then I write that this is Maxwell's wave equation. So if I write this as 1 by c square d to be the electric field that is satisfying this equation. Which is essentially the wave equation where; c is 1 divided by the root power of mu naught, epsilon naught. Then the solution I can write in this particular form k e is equal to e naught vector and if I write in a harmonic waveform the solution will be k dot r minus omega t where; omega divided by the mod of k is equal to the velocity c and we know that for harmonic waves the mod of k is 2 pi divided by lambda and omega is 2 pi nu. That we described in the last class. In the similar way, we can also write the solution for magnetic field B. The magnetic field solution can be written in this form. If I write in a sinusoidal way, it should be k dot r minus omega t these are called the plane wave solution. Why is it called the plane wave solution, which we have already discussed in the last class? So this is the form of the electromagnetic wave that is propagating over the space in the free space. So if I draw this drawing we had done in our first class now I believe you understand in detail. So suppose this is in z direction suppose, these fields are moving in the z direction and this is say x and this is y then along y direction if the electric field is vibrating then I write elliptic field E that is E naught with unit vector y sine of k z . Because it is moving in the z direction minus omega t we write in one dimension and if similarly the B is moving, we know that E and B are perpendicular to each other. Different color I want so, this one so, if b is moving it will move in that plane in this way. So this is the way the b will move, my drawing is not very perfect but hope you understand how e and b will move. So b I right now b naught x is unit vector and then sine of kz minus omega t, that is the way I write e and b, e vector, b vector will be along this direction and e will be along this direction. This is the e vector that is vibrating and this is my v vector that is vibrating along this direction and they will propagate along the z direction. This is the simplest way one can describe the electromagnetic wave but the point is why I should write the electromagnetic wave or electromagnetic field in this particular form kz k dot r minus omega t or kz minus omega t. The student should understand that it follows the same wave equation that we have been driving for the last couple of classes and now you understand that Maxwell's from Maxwell's equations one can also generate a wave equation that means an electromagnetic field that is propagating as a wave and electromagnetic optics is nothing but an electromagnetic wave. So if I define that the light is moving in a particular direction, that should be the representation of the propagation of a wave if it is a light. Okay, so next the important thing that I like to discuss is associated with this concept as well and that is called the Poynting vector. Later we are going to use that. So the next concept is the Poynting vector. So, what is a Poynting vector? By definition, a Poynting vector is a quantity S which is E cross H or 1 divided by mu naught E cross B. By definition, that is the form of a Poynting vector. Now, what is a Poynting vector physically? So, this quantity is nothing but energy flow per unit area, per unit time. So electromagnetic waves that are coming from certain sources will come like this and they also carry some sort of energy. So I have a unit cross section like this and calculate how much energy the electromagnetic wave is carrying that quantity per unit area. If this is a unit area per unit time that means per unit second how much of energy is coming through a radiation that is the quantity defined by the Poynting vector. (Refer slide time: 26:55)



So essentially this is energy per area into time or other way energy per time means it is power divided by area, now the power divided barrier is nothing but Intensity. So the Poynting vector is nothing but measures the amount of intensity of radiation that is carried through. So now I know what is my e if e, we know how the electric field and magnetic field vary so e is e 0. If I write in the complex notation that was also discussed in the last class the harmonic wave can also be represented in terms of this complex notation, very useful notation, and several places we are going to use in the upcoming lectures. So it should be like this. Okay now, If we have a solution like this, the interesting fact is if I make a curl of E of this specific plane wave solution, then these come out to be these things come out to be I k vector cross E. That is an interesting fact also from simple just by looking at the expression one can also find that if I make a time derivative of this quantity. If this is the form then one can get minus of i omega and multiplied by the elliptic field. So these two identities are one can use to figure out what is the value of the S. So if I write S here then this is 1 divided by mu naught E cross B. If I write the S in terms of e only then I replace this b in terms of e and that we can do by using this equation. So we have one equation curl cross e is equal to minus of del b del t. If I have that, then curl cross E, I replace, if this is my E and B, I can replace that I K cross E and minus of del B del T, I can write as I of omega of B. This is a minus sign, so minus is going to be absorbed here. So essentially we have curl cross e divided by omega that is b, so b I find in terms of k and omega and e and that I'm going to replace here. And if I replace it, then I am going to get something. So, S will be 1 divided by mu naught. Then we have E. Then I will write B here as 1 divided by omega. And then K cross E. Omega is a scalar quantity, so I can take this quantity out and I am going to get like this. 1 divided by mu naught omega and then I have E cross K cross E. So it is like A cross B cross C the situation that we have is like A cross B cross C. (Refer slide time: 35:15)



So let me go to the next page. So my S is now let me write down once again. S is 1 divided by mu naught omega and then I have E cross, k cross p. So we know from the vector identity that a cross b cross c is b, c dot a minus c a dot b If I use this identity here I'm going to get 1 divided by mu naught omega and then k e dot e minus e k dot e well this k dot e that means so, if I have an electric field that is vibrating along this direction say this is my y direction and this is my electric field that is vibrating and y direction it is propagating along z direction. So k and e will always be perpendicular to each other. So e is a homogeneous medium in free space that is the case a e is perpendicular to k. So the vibration of the electric field, the polarization of the electric field and the propagation direction are perpendicular to each other. If that is the case then this k dot e term should vanish. This is 0 and essentially we get s equal to k divided by mu naught, omega, k unit vector, and e dot e I can write it as e square. Now I can write it as k mod of k mu naught omega, e square, and k unit vectors. So that means the Poynting vector is moving along the direction of the propagation here in free space which is the k direction now mod of k divided by omega is c and also mu naught, epsilon naught is 1 divided by c square. Exploiting these two expressions here, it is easy to show that the mod of s value is equal to s equal to c epsilon naught and then e square. Now, e is a function of time so the intensity that we have should be the average value of s and that means it is c epsilon naught, the average value of e square. Now e is a sinusoidal wave we already write the expression of b. Let me write it here in this space e is e naught and if it is in one direction, say y direction let me write it in y direction, and sine kz minus omega t then the e square average value will be simply average of sine square value and that is half. I'm going to get half e naught square so that I can write as half of c epsilon naught and then I'm going to get E zero square. So intensity one can write in this particular form and that suggests that intensity is proportional to the amplitude of the electric field multiplied by C into epsilon naught. That is an important expression especially, this is an important expression that intensity is the average value of these things. Later in a few cases, we're going to use that. So this expression one should remember that how the intensity varies with respect to the amplitude of the electric field or the average value of the electric field. With this note I would like to conclude here; so, today what we understand is electromagnetic waves are generated and it is generated from at least the electromagnetic wave equation from Maxwell's 4 basic equations we can derive first the wave equation. And that wave suggests how the electromagnetic wave propagates in space. It is essentially a sinusoidal wave. And if we consider the sinusoidal wave as a solution, we find out the expression of the intensity. The intensity is found to be proportional to the electric field. In the next class, we will try to understand more about the wave, and the nature of the wave. And this nature of the wave, we will go to understand in terms of the superposition principle. And two waves, if they interact, two electromagnetic waves or two electromagnetic waves, two light with a different frequency or same frequency when they are going to interact with themselves, how the superposition will take place. We will discuss it in the next class. So with that note, I will have to conclude here, thank you very much for your attention and see you in the next class.