WAVE OPTICS

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Lecture - 41: Fraunhofer diffraction for a circular aperture

Hello, student, welcome to our wave optics course. Today we have lecture number 41 and in this lecture, we will discuss the Fraunhofer diffraction, for a circular aperture. So we have lecture number 41. Today I'm going to discuss what is the form of a diffraction pattern for a circular aperture. So far, we are dealing with a slit whose width is small compared to the length. But now we're going to discuss if instead of having a slit, a narrow window, if I have a circular aperture, what should be the distribution of the intensity due to the Fraunhofer diffraction. So first let me draw the aperture here. We have a circular hole here. Try to draw in 3D. So this is the circular hole we are having. Let us put the x and y axes also, this is x and this is y. So I have a screen here, having x and y coordinates as well. So, I want to find out something about this. So, this is a point P. What is the field distribution at point P, when we have aperture like this? So, I join this P here and I take a small section here over y, I shade this section and then I draw a line from this section to the point p. So this is the point P, where two lines are coming, two rays are coming. And if I do a perpendicular here, so this is from over here, this is y, this is dy, a small section. So here I am, drawing in a magnified version of this. So I have a circular aperture. This is my x coordinate, this is my y coordinate and I have over a distance y, I have a small strip here. So this is my y and this portion is dy and if I draw a line here, that is my bx okay and the radius is r, so I'm just drawing the geometry here, this is the center. So this is the geometry we are having and now we're going to utilize this, so first what we need to do is to find out what is the electric field at point p due to this small area that is drawn over the length d y.

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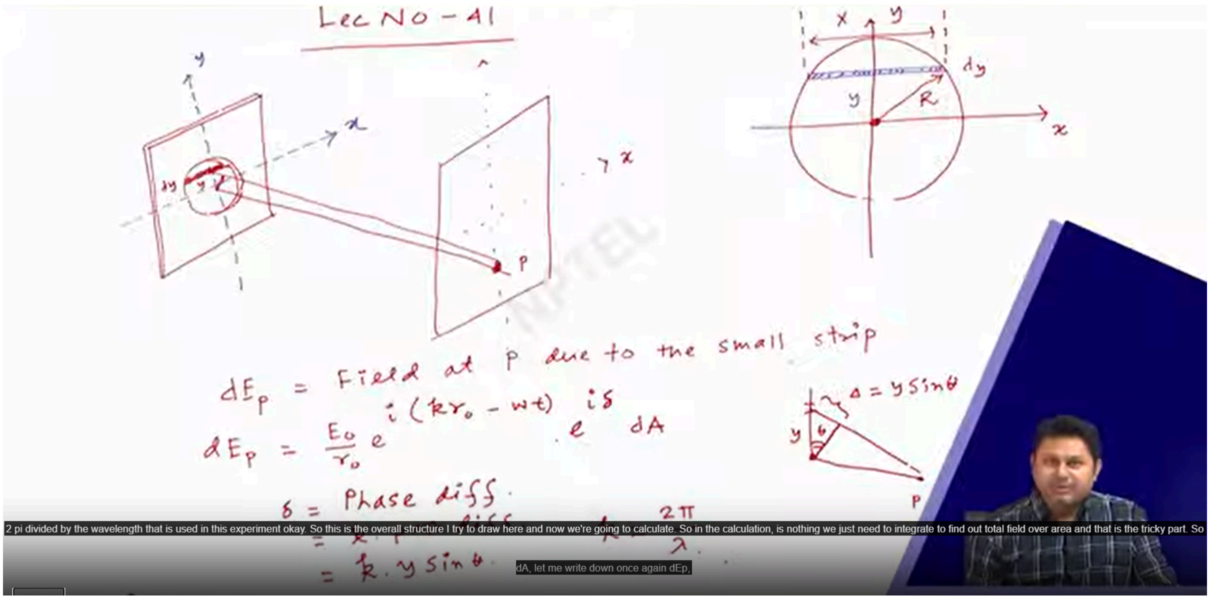

So, d e at point p this is a small field, that is the field at p due to the small strip here also i magnify this figure. Now dE, essentially the electric field that is coming from this, is essentially a circular wave that is coming, a circular spherical wavefront that is coming and I write this spherical wavefront in the form of the spherical wave in the form like this, that is the propagating part but also we know that when the waves are moving, then all the waves are moving but they are not in the same phase. So when we have this strip, over this strip, there will be an additional phase compared to the point that is coming from this origin, the electric field that is coming from this origin, so there will be a path difference between these two, I draw that path difference and due to this path difference that leads to an additional phase that we need to encounter and over the area dA. So delta here, so this is the intensity of the electric field per length, and then this equation satisfying this equation such as that this is a total electric field due to this strip. So delta is a phase difference and that is due to the path difference. So, that means we have K multiplied by the path difference. And what is the path difference? Let me draw this part again here. It is like this, so I have origin here, and this is the strip, this is Y, and this is some point P. Suppose this is the point P, so I join this to a line and if I draw a perpendicular here, then that is the path difference we are having and if it is Y and if this angle is theta then this path difference Delta will be Y of sine theta that we already calculated earlier. So that path difference will be y of sine theta. So here I write it is k and y of sine theta. What is k? k is a propagation constant, and it is written as 2 pi divided by the wavelength that is used in this experiment. So this is the overall structure I try to draw here, and now we're going to calculate. So in the calculation, there is nothing we just need to integrate to find out the total field over the area, and that is the tricky part. So dA, let me write down once again dEp, what we find is E naught divided by r naught because it is a spherical wave that should come out from this aperture. So I need to take account of the form and let me draw once again here quickly. Here is the strip and we have y here and that is r, that is x, and that is y, and this is the strip we are talking about. So dA, I first need to calculate what the area is, dA is a strip area, and that is x multiplied by dy.

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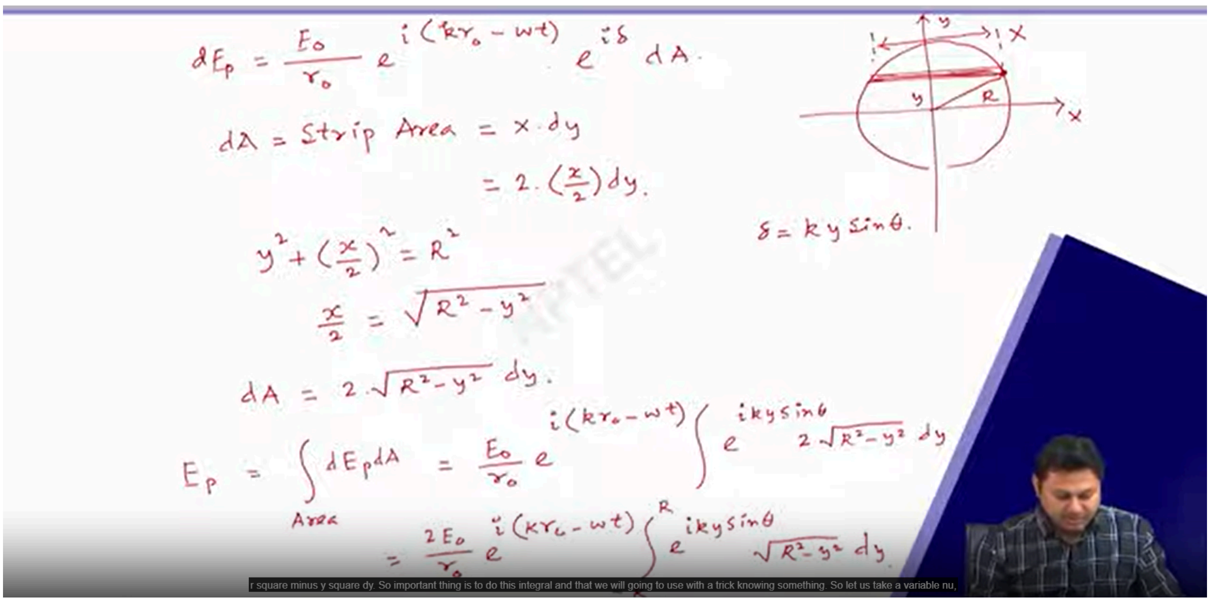

What is x? x is this length; you may remember that I define this as big x. From here to here, this length I write x, and there is a relationship between, so let me write down what this is 2 of x by 2, because from here to here this coordinate is x, so this is small x by 2, multiplied by dy. Also note y square plus this, re this length that is x by 2 whole square is r square. So, x by 2 is simply the root over of r squared minus y square which is the relationship we already have. So, that relationship if I put here in the expression of da then everything will be in terms of one variable, and that variable is y that we want because in the delta, mind it delta was k y sine theta, so the variable here is y because k is constant and we are looking for a particular angle theta, so, da is essentially 2 of x by y is nothing but root over of r squared minus y square and dy that is the area we are having. Now, once we have the area in the form of y and dy, we are in a position to integrate. So, the total field now due to the entire aperture will be the integration of the small field that we are having over this area dA and I am going to integrate over that area. So, that gives me E naught divided by r 0, e to the power of i k r naught minus omega t, where r naught is the distance from the center to the center. Centre to center means I am talking about this distance from here to here, this is r naught, and then I integrate and this, we will have e to the power i delta which is ik y sine theta. And then we have 2 of the roots of r squared minus y square and dy, and then that integration over dy means I need to go; now, the y can go from 0 to r and 0 to minus r, so eventually it goes from minus r to plus r. So this integral, if I write here, then it should be 2 of E naught divided by r naught e to the power of i k r naught minus omega t, and then I have integration, and this integration goes minus r to r and e to the power of i k y sine theta and root over of r squared minus y square dy. So the important thing is to do this integrally, and we will use a trick to know something. So let us take a variable nu, as y by r, so d nu will be 1 by r d y, if y tends to r then plus minus or rather then nu tends to plus minus 1. So, the integral will change. So, let me write down this integral once again. So, E p with this change coordinate will be the initial term which is 2 E naught divided by r 0 e to the power of i k r 0 minus omega t, and then we have integration minus 1 to 1. And then we have e to the power of i, y by r, I put gamma. So I write it as gamma alpha, and then I have r, if I take the common r root over 1 minus nu square and then dr is this quantity, so dy is r into d nu okay.

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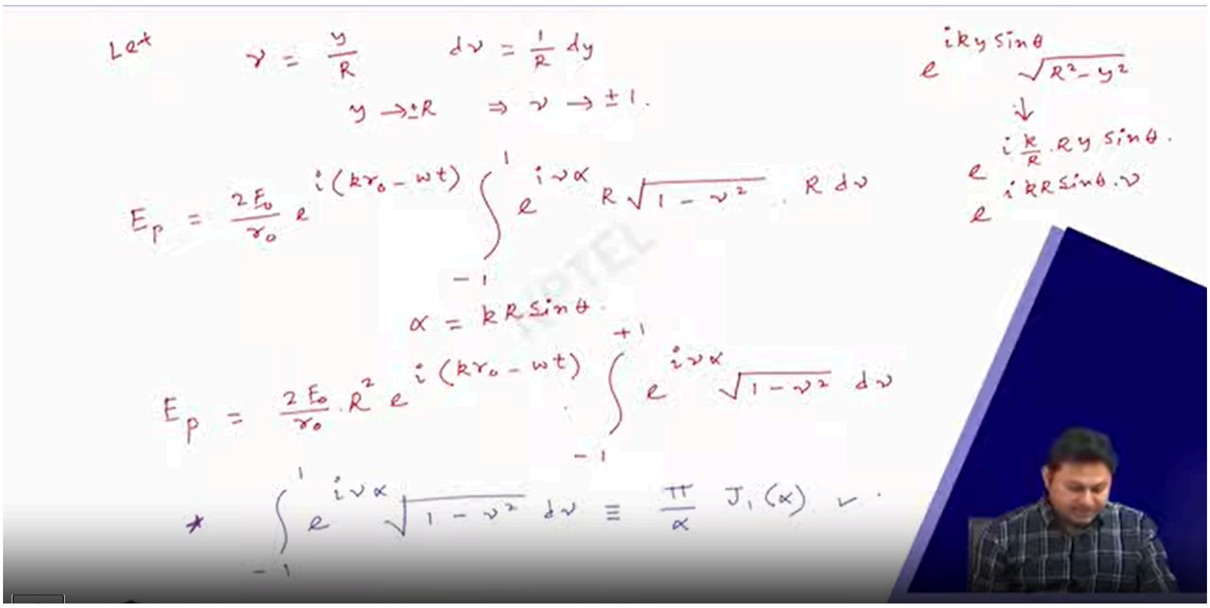

When my alpha is, so what was there, let me write down here in the previous equation what was there; it is e to the power of i ky, then sine theta. So if I go back and write it was e to the power of i ky sine theta that was the term and then root over r square minus y square term. So, from here if I take r square common then this r is coming out, then in the root we have 1 minus y squared divided by r square which is nu square because y by r is nu. Here in the power of e, if I write e to the power of i k nu then 1 r should be there. So, this term actually is written e to the power of i k divided by r. Then we have r, and then we have y, and then we have sine theta. So, y by r I can write nu. So, then it should be e to the power of i k then r sin theta multiplied by nu okay. So, that means alpha here is k r sin theta. So, alpha is k of r k multiplied by r sine theta that is my alpha. So I can take this r outside and another r is there. So my ep is essentially 2 of e naught divided by r naught multiplied by r square, then e to the power of i k r naught minus omega t, then we have an interesting integral i minus 1 to plus 1 e to the power of i nu alpha root over of 1 minus nu square d nu. So, I put some emphasis here on this integral because it is not possible unless you know what is the value of the integral, I am writing this identity in a different color maybe and this identity is this. So minus 1 to plus 1 e to the power i nu alpha root over of 1 minus nu square, d nu is something equivalent to pi divided by alpha and the Bessel function of first kind j 1 alpha. This is an identity, and this is a very, very important identity. So, I suggest the student remember that in order to solve this problem, we have to utilize this identity otherwise, we cannot execute this integral, and based on this identity the result will come. Okay so, now once we know what is the value of the integral, then we are in a position to find it. But before that, I would like to say something about this. So, J1, this is the Bessel function, with the first order Bessel function, and that is equal to, if I, this is a series function, we know the Bessel function is a series function. And I believe the students that when you derive this, you should have an understanding of the Bessel function; maybe in your mathematical physics course, you already learned about that Bessel function. This is a specific differential equation and Bessel functions are the solution, series solution of this specific integral. And I like to expand this series whatever the series we are talking about.

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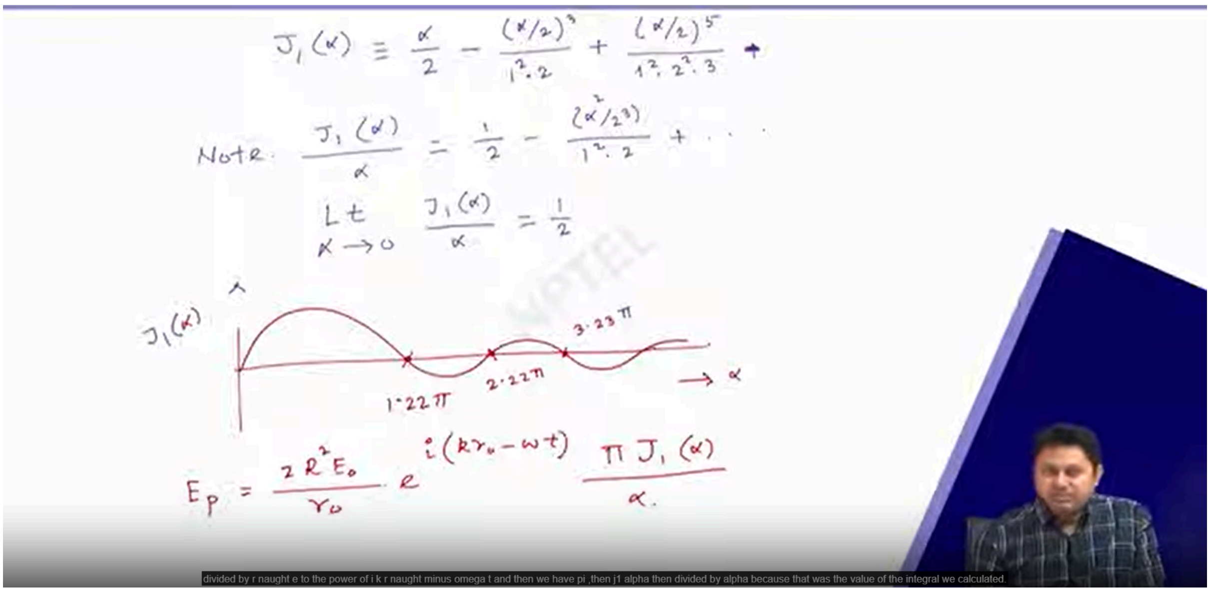

So, Bessel functions j 1 alpha is a series, and it will be defined this alpha by 2 cube, divided by 1 square multiplied by 2, alternative plus alpha by 2 to the power 5 divided by 1 square into 2 square into 3 and so on. This will be an alternative plus-minus sign. So, this will be a minus sign here and so on. So, this is the expansion of the Bessel function, and note that J1 is divided by alpha, this is half minus, this is alpha by alpha square by 2 cubes, and the denominator I have is 1 square 2, and so on. If that is the case then that limit will be there. So when limit alpha tends to 0, then for that limit j1 alpha divided by alpha is equal to half. Okay, so that is the limit we have because that so when we plot j1 that is the first Bessel function of these things, and then it goes from 0 actually. So, there is a, if I plot this, it will be like this. So, this is the form of the Bessel function. When we plot J1 alpha, mind it, I am plotting J1 alpha, not J1 alpha divided by alpha. So, that is 0, this point, the first cutting point is 1.22 pi, this i plot as a function of alpha, this point is 2.22 pi, this point is 3.23 pi, and so on. These are the cutting points. It is important that we note the cutting point here we have a 0 so these are the cutting points ah for this Bessel function of a first kind now let me go back to the field calculation so what was there so ep then essentially 2 of r square e naught divided by r naught e to the power of i k r naught minus omega t and then we have pi , then j1 alpha then divided by alpha because that was the value of the integral we calculated. Now, what is the intensity? The intensity IP will be proportional to the field amplitude mod square. If that is the case, I can write down IP in a very standard form, and that is I naught and J1 alpha divided by alpha whole square, mind it alpha here was r k naught sine theta. It is also written as r is a radius of the circular aperture you may remember. So I can write it as half of D in terms of diameter, then K naught or K whatever, K sine theta I didn't use K naught so, better I remove this, so D is the diameter of this stuff. Now if I want to plot that then how the plot will look, let us quickly understand. So this is the field distribution we are having. So if you plot the field distribution here, it will be like this, like we get for the single slit, but the functional form is different. Mind it, here we are plotting IP, which is proportional to this quantity, J1 alpha divided by the whole square of that.

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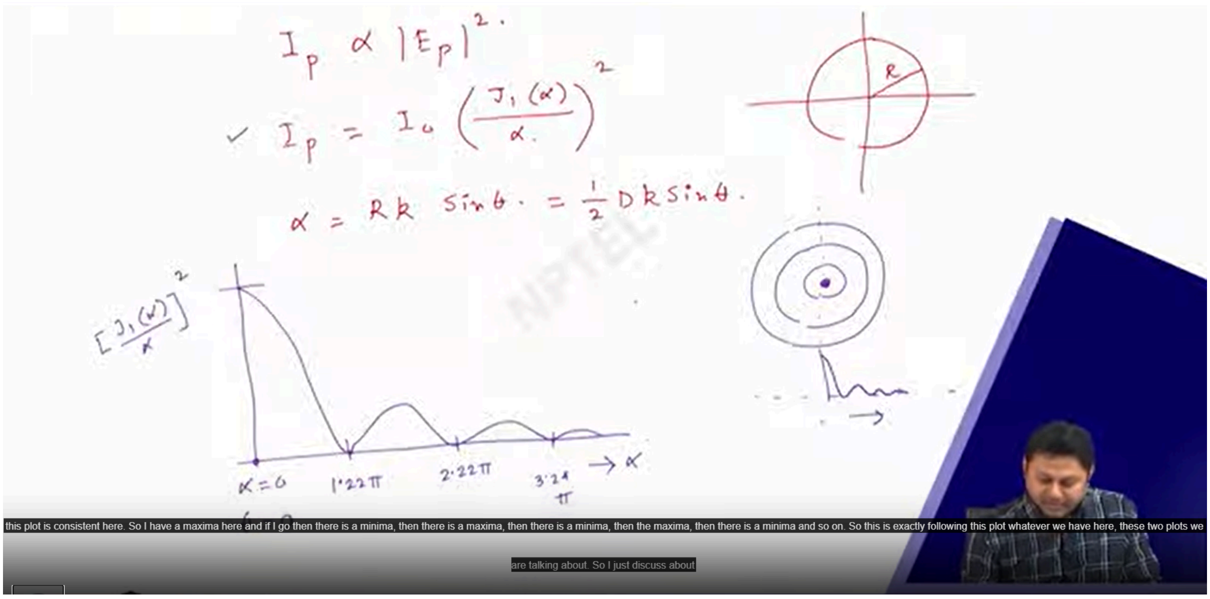

And note, when alpha tends to 0, this is alpha. We show that, when alpha tends to 0, j1 alpha divided by alpha is not equal to 0. It is going to the value of half. That is the limit, so that

means we get maxima here. This is the point where j1 is vanishing, and we know where the j1 is vanishing. We show that it is 1.22 pi, this is 2.22 pi, and it is 3 points around 2, 4, pi something like this. These are the points where we have minima. So, this is the distribution of the intensity where the principal maxima occurs, when alpha tends to 0. So, this is where alpha is 0. And alpha is 0 means from here we can see that when theta is 0. So, theta is 0 tends to 0 where we have a principal maximum. So, in 2D how that figure will look because the spectral will be 2D. So, there will be a pi symmetry here. So, the spectral if I draw that it should be like this, so I have a maxima here, then minima, then I have a maxima here, then I have a maxima here so it will be like a concentric circle, like we have in our other light experiments and how this plot is consistent here. So I have a maxima here, and if I go then there is a minima, then there is a maxima, then there is a minima, then the maxima, then there is a minima, and so on. So this is exactly following this plot whatever we have here, these two plots we are talking about. So I will just discuss how these two plots one can have. So quickly, whatever the time left, let me continue because this is associated with where these secondary maxima, the condition of the secondary maxima etcetera. So the secondary maxima, we have this secondary maxima or the secondary minima, better first you calculate secondary maxima. So, a secondary maximum one can have when d, d alpha of this quantity is 0. So that is a difficult calculation, using the computer you can do that and if you do, you will find that, when the value of alpha is 1.635 pi, then we have a secondary maximum, then when alpha equals 2.679 pi then we have a secondary maximum and so on. And the minima we have already discussed and let me define that. So, secondary minima, so, let me draw the structure here side by side, spiked up, that it is like this. So, this point and this point are the first and secondary maxima. So, this is 1.635 pi and this point is 2.679 pi. These two we have, this and this, we know where j1 is 0. So, that we already calculated. So, j1 alpha, that is 0. Obviously, when alpha is not equal to 0, that is the condition. So, the first minima is around 1.22 pi, that is here, pi that is the value of the alpha for which I have a first minima.
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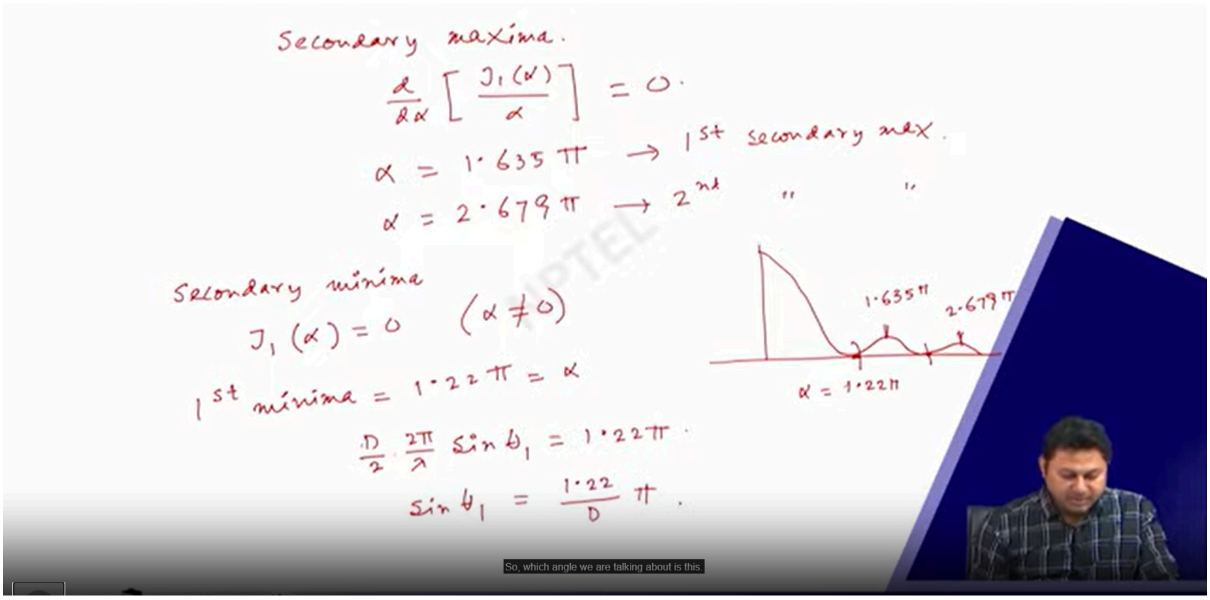

So let us understand what is alpha, that is equal to alpha and how much alpha is d by 2, then k which is 2 pi divided by lambda and the angle for sine, say, sine theta 1 the angular. So, that is the first minima and that is 1.22 pi. If that is the case, we can have sine theta 1 equal to 1.22 divided by d pi, which is the important expression. So, which angle we are talking about is this. So, I have this circular aperture. And for that, I am having these spectra which look very close to the spectra that we have for single slit. And this is the theta, this is theta 1 for which I am getting the minima. So, this calculation such as that sin of theta 1 is equal to 1.22 pi divided by d. If d is small, that is the diameter of this small, this will go to shift to the other limits, to wider, so I have an aperture with the small d, then this spectra will be more spread. It should be like this. So this 1 prime will be larger than this one because the D is smaller here. B So with that note I would like to conclude. Because I don't have the time, so today what we discussed is the diffraction pattern generated due to a circular aperture, very important, normally we have a slit but if somebody does the experiment with the circular aperture, Fraunhofer experiment, he or she can find an intensity pattern that we derived today and it deals with the Bessel function and the first order Bessel function of the first kind. And then we find how it is distributed and what is the location of principal maxima, secondary maxima, and also the location of the minima. And we find that if the D value is less, then the spectra or whatever structure we have gets broadened. So, with that note, I conclude here. Thank you very much for your attention and see you in the next class for more.