Foundations of Classical Electrodynamics
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Lecture-09
Gradient Operator, Concept of Divergence

So, hello students to the foundation of classical electrodynamics course. So, today we will have lecture number 9, where we are going to continue the gradient operator that we started in the last class and like to understand the concept of divergence.

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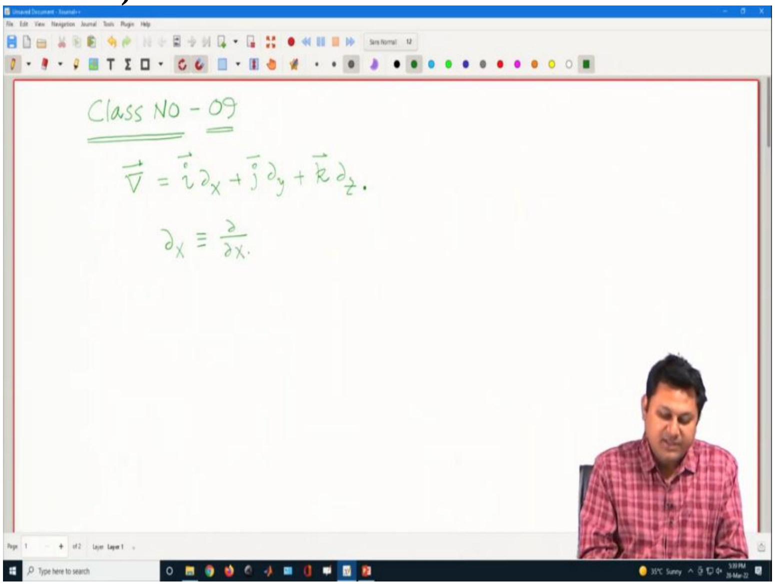

So, today we have, in the last class, if you remember, we define the operator, which we call the del operator, in short if I write it, it was this where this the shorthand notation of the partial derivative with respect to x. So, it is in general a differential operator.

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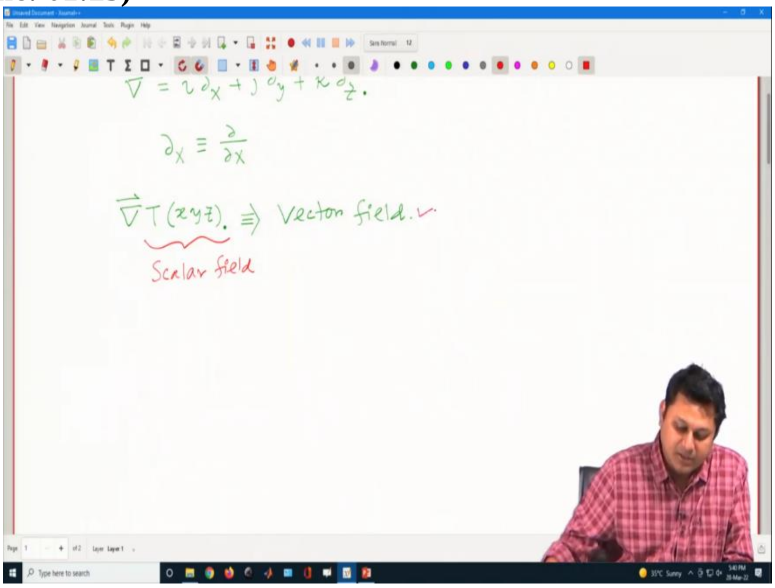

And it will operate over these things we are going to operate over a scalar field say function of x, y, z and if it operates over the scalar field at the outcome, we should get a vector quantity or vector field, so, this operator is going to operate over a scalar field. So, this is a scalar field and we operate this operator over the scalar field and at the result at the end of the day what you are getting is a vector field.

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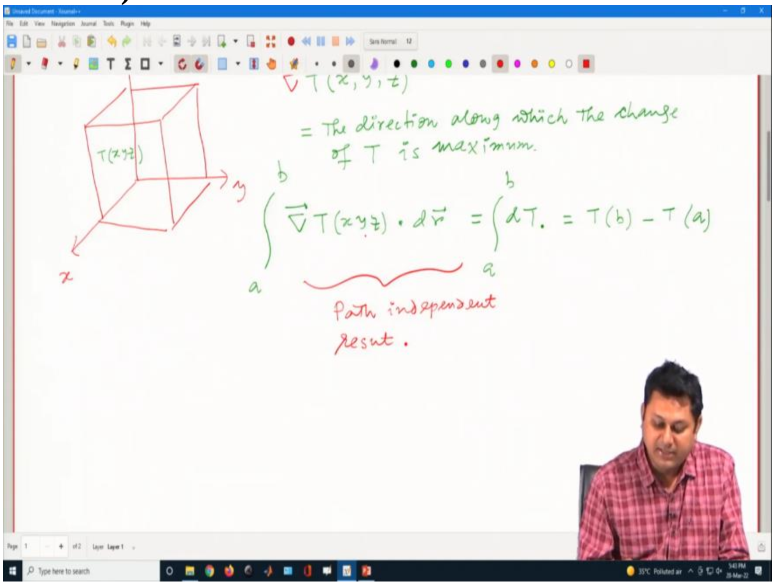

So, the example we already mentioned that the temperature so, I am having suppose, I am having a room here let us consider this is a room and in xyz coordinate system, in this room if I define the temperature, the temperature is varying with different xyz point. Even though when we are talking about the temperature of a room, we mentioned the average value, but in a microscopic level if I go from point to point obviously, there should be a change in temperature.

So, it should make a scalar field. Now, what this divergence will do over this scalar field is this. So, now, I am operating this operator over our scalar field and this quantity gives us the direction along which the change of temperature is maximum, the direction along which the change of T is maximum. We discuss these things in detail. Also, we find one very interesting fact regarding this operator.

And that is if I want to find out this quantity say $\vec{\nabla}$T which is a function say xyz and then dot dr, this quantity is eventually dT and now if I integrate it over a and b so it is a simply line integral, I am doing. So, this side if I integrate a and b, we will simply get the value at this point b and a. So, this is a path independent integral. So, whatever I am getting is path independent.

So, this quantity gives me a path independent result. Even though I am calculating a line integral where path is important, but for this kind of structure if it is associated if it is a vector field if I make a vector field in terms of gradient in this way, then at the end of the day whatever the result I get we find that that should be path independent. So, that is very important information regarding this operator.

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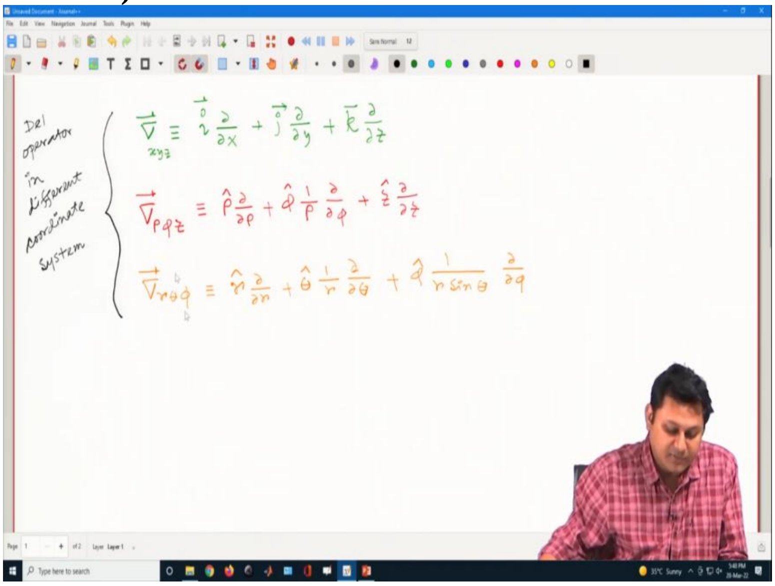

So, now, one thing I need to be very specific and that is when I write this operator, I write it like i $\frac{\partial}{\partial x}$ + j $\frac{\partial}{\partial y}$ + k $\frac{\partial}{\partial z}$. So, you can see that this operator is defined in Cartesian coordinate system, but for other coordinate system we also can define this $\vec{\nabla}$. So, that is why now on let us put a suffix here xyz, which tells me that this operator is defined in Cartesian coordinate system.

Now, the question is if I want to define the same operator in other coordinate system like cylindrical or spherical how this operator will look like. Already I discussed in detail about these 2 coordinate systems, which are very useful in electrodynamics problem. So, the next thing if I want to find that what is the form of say in the cylindrical coordinate system it is ρ φ and z, so, what should be the form.

I am directly writing this form. This form will be this $\hat{\rho}\,\frac{\partial}{\partial \rho} + \hat{\varphi}\,\frac{1}{\rho}\,\frac{\partial}{\partial \varphi} + \hat{z}\,\frac{\partial}{\partial z}$, this is in cylindrical coordinate system. So, you can see that it is not that straightforward that we had in the previous case. In Cartesian coordinate system, I had simply $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ but here I find that we are having $\frac{\partial}{\partial \rho}$, $\frac{\partial}{\partial \varphi}$ but $\frac{1}{\rho}$ term is present here.

And $\frac{\partial}{\partial z}$ it is fine, but it is not like the way we define it in Cartesian coordinate system. Now, things become even complicated it looks even complicated in if I want to present it in spherical coordinate system. For example, so, this is the same operator I am writing this in r θ φ coordinate system, which is spherical coordinate system. So, how these things will look like. We should have a $\hat{r}$ and then $\frac{\partial}{\partial r}$, then I have $\hat{\theta}$, then $\frac{1}{r}\frac{\partial}{\partial \theta}$.

And then I have $\hat{\varphi}$ $\frac{1}{r\, sin\theta}$ then $\frac{\partial}{\partial \varphi}$, this is the form of $\vec{\nabla}$ in different coordinate system. So, I am writing the $\vec{\nabla}$. So, this is the $\vec{\nabla}$ in different coordinate system. So, this is in different coordinate system I am having this. Now the question is how I am getting from this to this, I mean, then you need to use the transformation and how the vector is transformed.

So, I am doing only for one case how to change this coordinate system xyz to ρ φ z, rest of the thing that how xyz it can be converted to r θ φ you can do by yourself as a homework.

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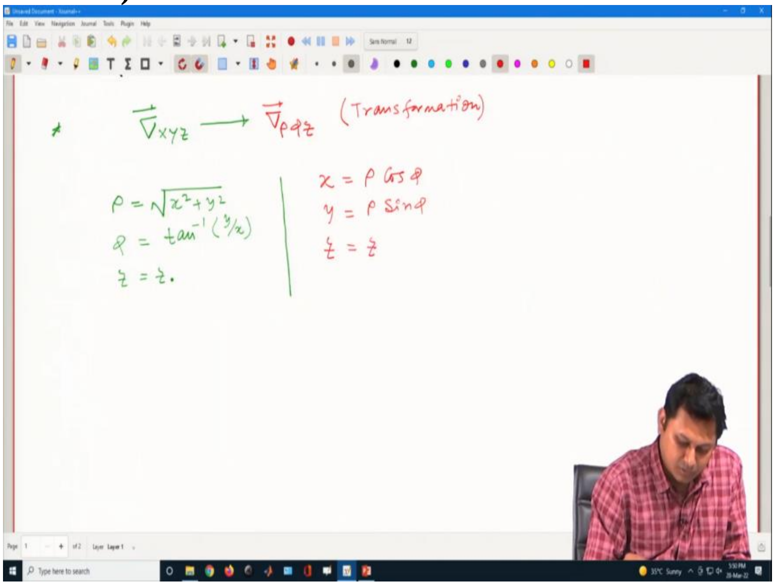

So, let me quickly try to find, at least, like to show that how it is happening. So, what I am doing? I am transferring this operator to this operator, the form, I mean I am going from 1 coordinate system to other coordinate system, how we will do that. So, this is a transformation, a rigorous way is there so I am just trying to show you quickly how to do that. So, I know the transformation rule that how the ρ φ z it is related to xyz.

So that I am going to use, so ρ we know is $\sqrt{x^2 + y^2}$, φ I know this is $\tan^{-1}\frac{y}{x}$ and z is simple z. In a similar way, if I want to find out how the xyz is changing, it should be x = ρ cos φ, y = ρ sin φ and z = z. With these transformations, with these relations actually, we can use the partial derivative because at the end of the day, these partial derivatives are there in the operator. So, how do I do that?

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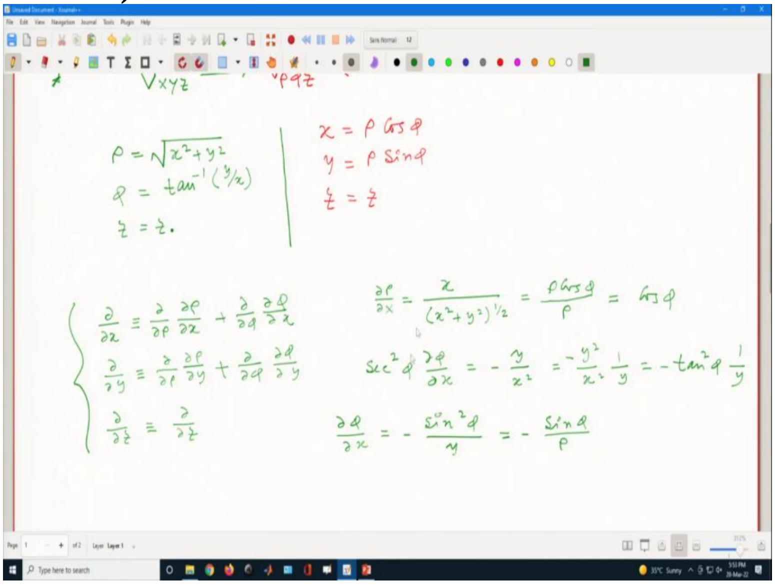

So, $\frac{\partial}{\partial x}$ if I want to find this is equivalent to because x is a function of ρ and φ, so I can write by using chain rule this quantity. Similarly, $\frac{\partial}{\partial y}$ is $\frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial y}$, and $\frac{\partial}{\partial z}$ is simply $\frac{\partial}{\partial z}$, this is the chain rule I am having. Now, what is $\frac{\partial \rho}{\partial x}$? Because I know what the relationship here how the ρ is related to this.

So, I can do that and this value is simply $\frac{x}{(x^2 + y^2)^{1/2}}$, which is eventually ρ cos φ because x is $\frac{\rho \cos\varphi}{\rho}$ or simply cos φ. So, that we know, now, similarly, so, I can use these things so, $\frac{\partial \varphi}{\partial x}$ because it is a tan function. So, simply I can have this if I execute this, it should be I know $\sec^2\varphi \frac{\partial \varphi}{\partial x}$, will be because I am executing this tan function.

So, it should be -y x^2, which is equal to if I write $-\frac{y}{x}$ and make a square of that and make $\frac{1}{y}$, and then it should be simply written as $-\tan^2\varphi \frac{1}{y}$ that is useful relation, we are doing, then $\frac{\partial \varphi}{\partial x}$ is simply $-\frac{\sin^2\varphi}{y}$ from here and which is simply $-\frac{\sin\varphi}{\rho}$ because of I am now using x so I am during this calculation you just need to understand that I am using all the relationships, all the relations that I am having here, because at the end of the day, I need to find out this $\frac{\partial \rho}{\partial x}$, $\frac{\partial \varphi}{\partial x}$ in terms of φ, ρ, because I am exchanging $\frac{\partial}{\partial x}$ to the other coordinate, that is ρ φ z coordinate.

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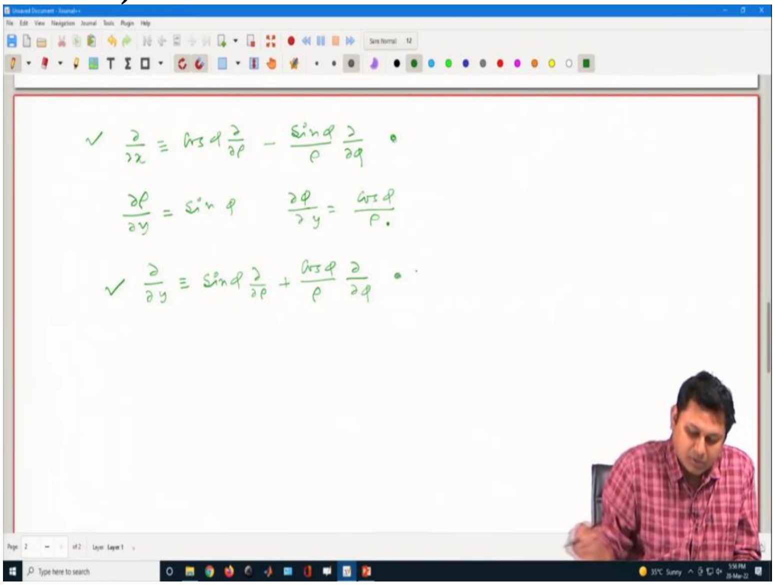

So, now still I am having, so, $\frac{\partial}{\partial x}$, what is $\frac{\partial}{\partial x}$, I am getting so let me write it $\frac{\partial}{\partial x}$ is equivalent eventually cos $\varphi \frac{\partial}{\partial \rho}$ because cos φ that I figured out here, because this quantity become cos φ and $\frac{\partial \varphi}{\partial x}$, which is sitting here which I am getting as -sin φ. So, I just simply write $-\frac{\sin\varphi}{\rho}$ and then $\frac{\partial}{\partial \varphi}$. During the calculation, I must mention here that during the calculation if you somewhere become confused that what should be the term, but you should care you should look carefully the dimensions here. It is $\frac{\partial}{\partial x}$, x is a dimension of length. So, right hand side whatever the combination you have it should be 1 by length and if you look carefully, the combination is such that here also you have 1 by length because ρ is $\sqrt{x^2+y^2}$, so, it is also the unit of length and $\frac{1}{\rho}$ is here and φ sitting here. So, it seems that at least dimension wise it is fine. Next $\frac{\partial \rho}{\partial x}$ I evaluate, $\frac{\partial}{\partial y}$ is sin φ because ρ and y is related to here.

So, I can find it from ρ is related to y in this way, the way I calculate cos φ in the similar way if you calculate you will find it simply comes up to be sin φ and then $\frac{\partial \varphi}{\partial y}$ we will get as $\frac{\cos\varphi}{\rho}$. So, $\frac{\partial}{\partial y}$ again I will get simply sin φ $\frac{\partial}{\partial \rho}$ + $\frac{\cos\varphi}{\rho}$ and $\frac{\partial}{\partial \varphi}$. Again, I am getting this quantity so, I am getting this one, I am getting this one what is next.

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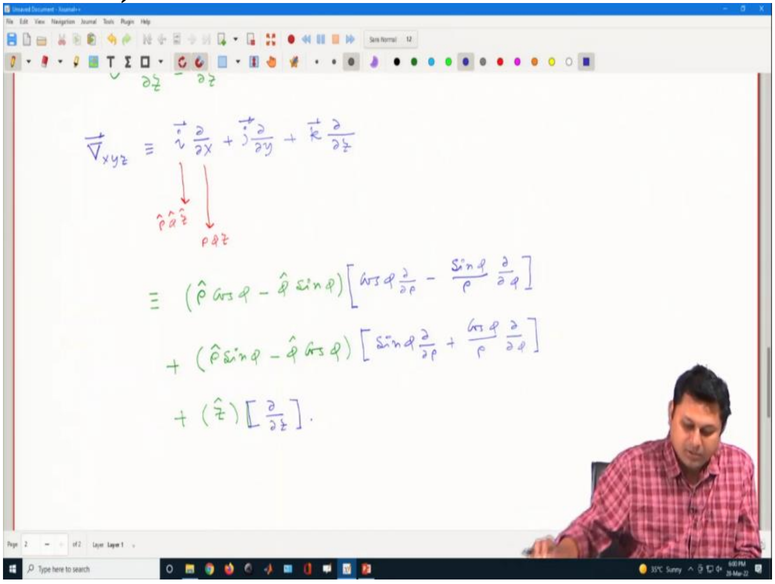

The next is $\frac{\partial}{\partial z}$, which is simple because it is simply $\frac{\partial}{\partial z}$. So, this also I get so, this having these 3 relations now, I am in a position to write this operator, x y z is equivalent to $\hat{\imath}\frac{\partial}{\partial x}$ let me write it once again + $\hat{\jmath}\frac{\partial}{\partial y}$ + $\hat{k}\frac{\partial}{\partial z}$. After that what we do we just replace this, I know how i is changing in terms of the unit vector ρ φ and z, we did it earlier.

And now I know how earlier $\frac{\partial}{\partial \rho}$ is changing in terms of so, i is changing in terms of $\hat{\rho}$, $\hat{\varphi}$, $\hat{z}$ that I know and how this operator $\frac{\partial}{\partial x}$ is changing in terms of ρ φ and z that I just derive here. So, these 2 we put together then the entire operator will be in terms of ρ φ z, that is all. So, these things are equivalent to, so, what is i in terms of ρ φ z it is $\hat{\rho}$ cos φ - $\hat{\varphi}$ sin φ we derived this. What is $\frac{\partial}{\partial x}$ that I just derive now.

So, it is cos φ and then $\frac{\partial}{\partial \rho}$ - $\frac{\sin\varphi}{\rho}\frac{\partial}{\partial \varphi}$ this is the first term I am having. What is second term, + + $\hat{\jmath}$ I can write it in terms of ρ because I know the matrix ρ sin φ - φ cos φ, $\hat{\varphi}$ cos φ and what is this quantity $\frac{\partial}{\partial x}$ that I also derived here $\frac{\partial}{\partial x}$ so, this is simply sin φ $\frac{\partial}{\partial \rho}$ + $\frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi}$.

And then finally, I am having the $\hat{k}$ that should be simply replaced by the $\hat{z}$ because that is the way it is and the $\frac{\partial}{\partial z}$ term is simply $\frac{\partial}{\partial z}$.

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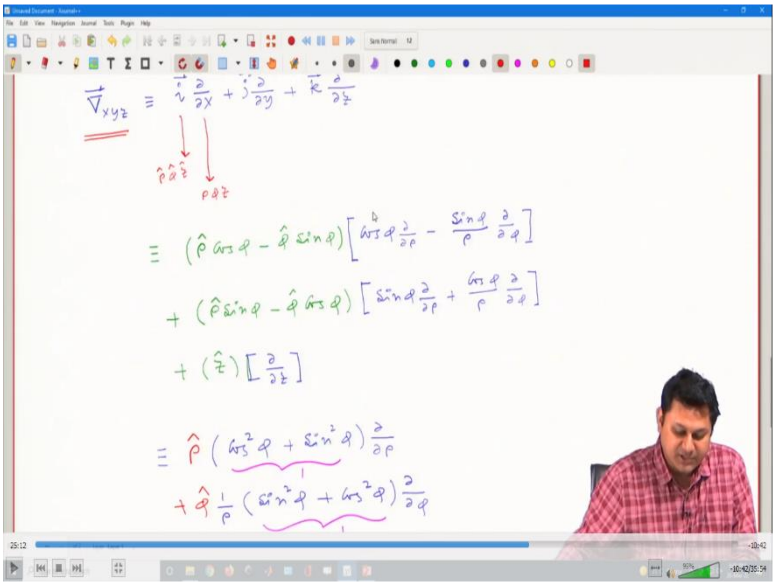

Now, I just try to find out multiply all these things and try to gather everything for $\hat{\rho}$, $\hat{\varphi}$ and $\hat{z}$ that is all and if you do that, for $\hat{\rho}$ what value you gather, you find that you are gathering this one, one is here and another is here. So, you are gathering eventually $(\cos^2 \varphi + \sin^2 \varphi)\frac{\partial}{\partial \rho}$ that you are gathering.

What about the other term, + $\hat{\varphi}$ which term you are gathering, you are gathering if I multiply this, so, one by it seems $\frac{1}{\rho}$ you can take common $\frac{1}{\rho}$ you can take common so, what term you are getting, yor are getting $\sin^2 \varphi + \cos^2 \varphi$ and then $\frac{\partial}{\partial \varphi}$. Finally, the last term where you do not need to do anything it is simply $\hat{z}$ and then $\frac{\partial}{\partial z}$.

Now, if you look carefully, already you figure out that this quantity is 1 and this quantity is also 1. So, eventually the operator we are having is $\hat{\rho}\,\frac{\partial}{\partial \rho} + \hat{\varphi}\,\frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{z}\,\frac{\partial}{\partial z}$, which is nothing but the operator at ρ φ z coordinate system so, you make the transformation, you start it here this is your starting point.

You have ijk and $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ then you just try replace i in terms of the unit vector ρ φ z the way we figured out a few classes ago. And then $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ you can figure out with the relationship with ρ φ z that we know from the beginning and exploiting these 2 expressions, you eventually convert this operator $\vec{\nabla}_{xyz}$ to this operator $\vec{\nabla}_{\rho\varphi z}$.

So, this is the way one can convert, but you can see that it is a very a little bit lengthy calculation and also you can think of that how to change from xyz to r θ φ as I mentioned, you can take it as homework and try to find out this transformation.

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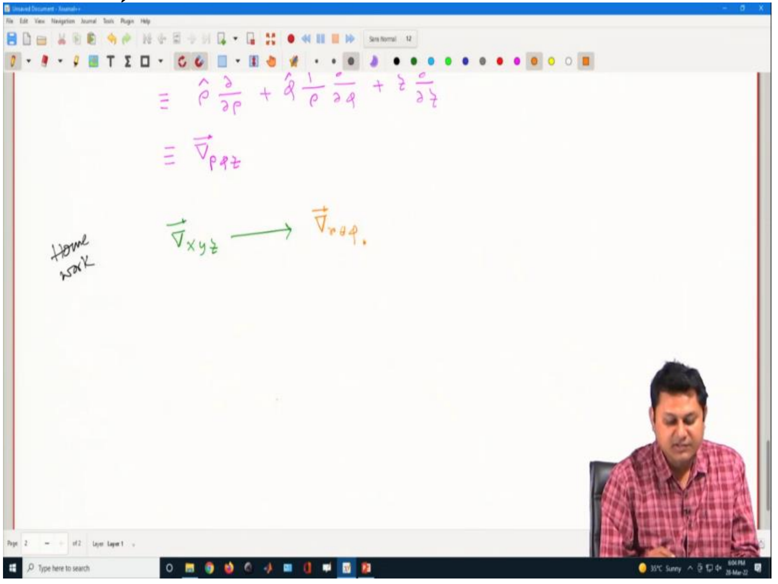

So, homework transformation so, homework is so, I like to make it as homework please work out this transformation that starting form xyz you need to transform, you need to figure out the expression, which is in spherical coordinate system and it is r θ φ, this transformation you need to figure out from here to here and the procedure you should follow the same that I mentioned here. Now, after that we will move forward and the next thing that we will do is called the divergence.

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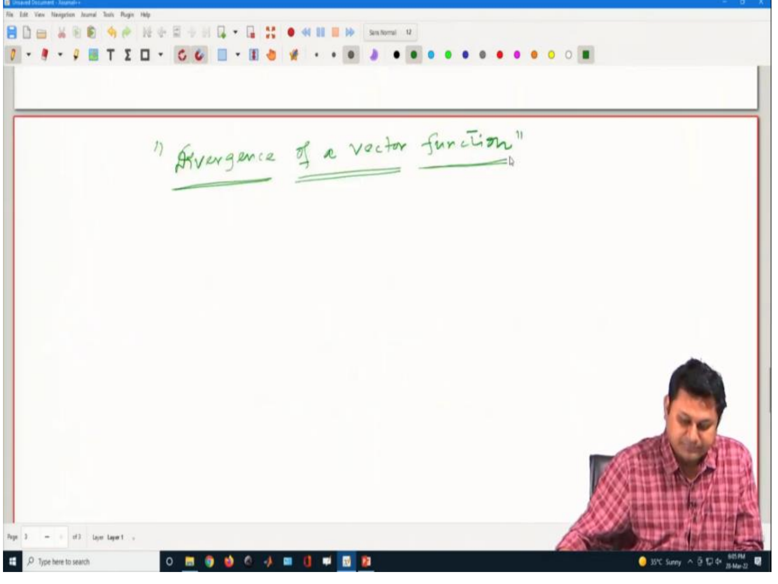

The next very important operator. So, the topic I will go to discuss now is divergence of a vector function. So, what is divergence and how it operates over the vector function that we are going to discuss here in this course. So, divergence of a vector functions.

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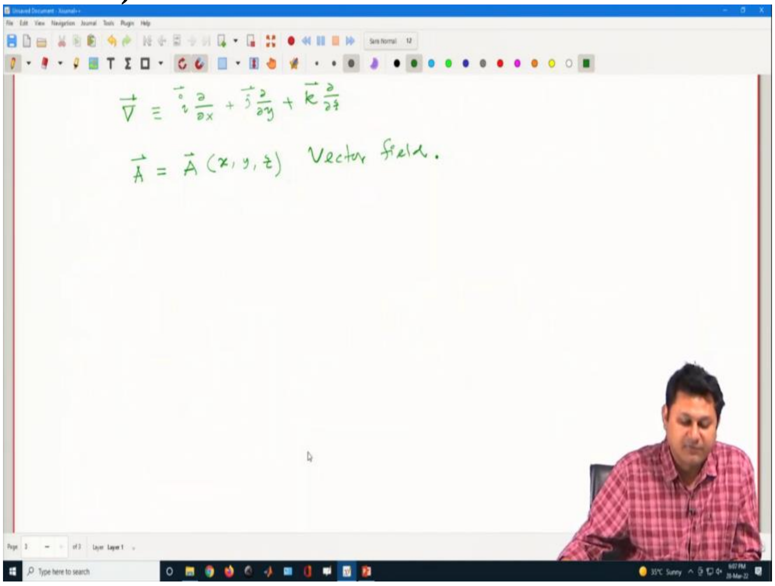

So, what is divergence I will discuss but let us first write once again this $\vec{\nabla}$. So, this $\vec{\nabla}$ is simply $\hat{\imath}\,\frac{\partial}{\partial x}$, + $\hat{\jmath}\,\frac{\partial}{\partial y}$ + $\hat{k}\,\frac{\partial}{\partial z}$. So, this is the ijk unit vector and $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, now, if I operate this over a vector field suppose a vector field is given A is a vector field, is a function of x, y and z. This is a vector field typical vector field, a varying function every point which is a vector that is called a vector field, a function which depends on the space point xyz in Cartesian coordinate system, and each and every point it produces a vector so, maybe in the next class I will show some typical picture of the vector field to make you understand how the vector field in general look in 2D at least in 2D we can show that.

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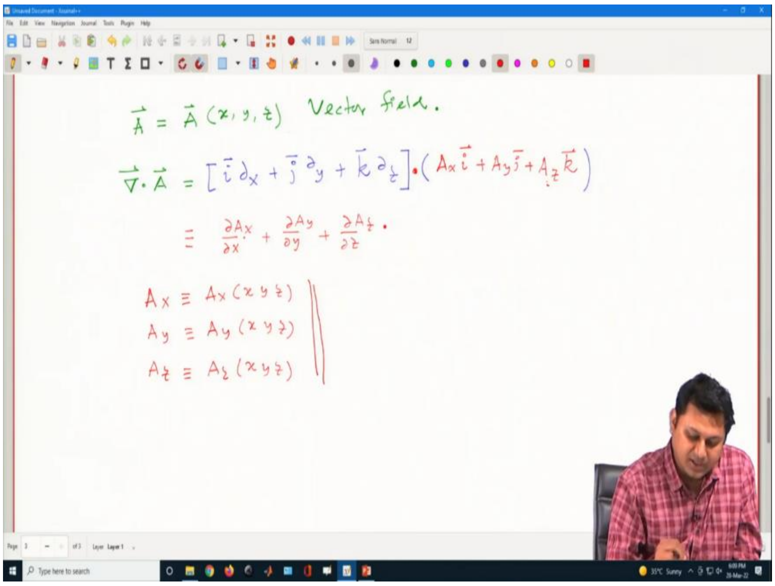

So, if I now want to operate these things over A, I will simply operate this operator i ∂_x in shorthand notation I am just writing j ∂_y + k ∂_z this is a partial derivative I am writing it will be going to operate over this A vector and this A vector is simply A_x i + A_y j + A_z k. So, these things so, A_z k bracket close. So, what we are getting here, so, this operator is related to a dot product.

So, a dot should be placed here, a very important term is dot, so the way we do the dot product here also we are going to do that. So, these things will be equivalent to like $\frac{\partial}{\partial x}$ will operate over A_x only, because i and i is here the way we make the dot product, then $\frac{\partial}{\partial y}$ will be going to operate over A_y and finally, $\frac{\partial}{\partial z}$ will operate over A_z. You can see that these are A_x, A_y, A_z are the components. But A_x can be a function of x y z, A_y can be a function of x y z and A_z can be a function of x y z in general

And now, the operator is operating that it will make a partial derivative with respect to x for A_x, it will make a partial derivative with respect to y over A_y and partial derivative over z.

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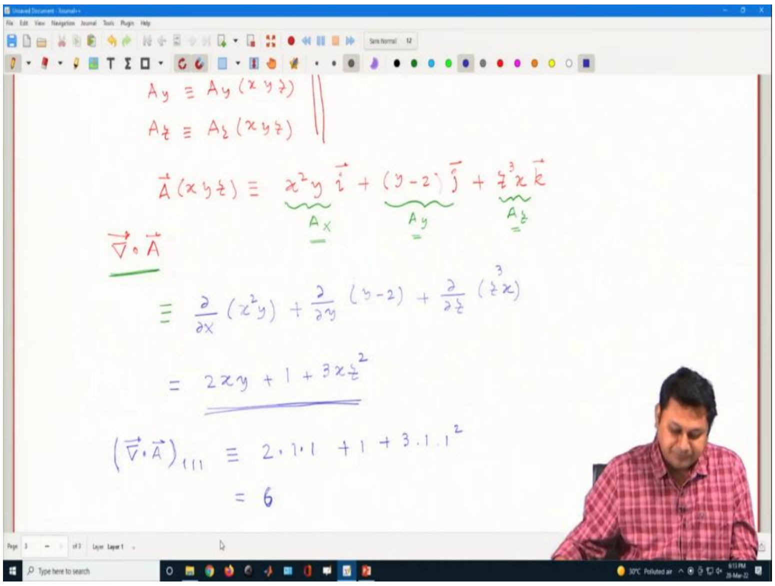

So, quickly I am showing one very simple example suppose an arbitrary example, an arbitrary say x y z a vector field is defined like x^2y i + (y – 2) j + z^3x k, an arbitrary completely arbitrary vector field. Now I am operating this stuff over there, I am operating this over A. So, as per my rule this $\frac{\partial}{\partial x}$ will be over A_x and this is my A_x. So, let me identify which is my A_x here.

So, this is my A_x, this is my A_y and this is my A_z. So, A_x, A_y, A_z as I mentioned these are in general function of xyz. Now, I am trying to find this quantity and that means I will operate $\frac{\partial}{\partial x}$ over A_x, which is x^2y then I will operate $\frac{\partial}{\partial y}$ over A_y, which is (y – 2) and then $\frac{\partial}{\partial z}$ over A_z, which is z^3x as simpler as that. Now I am making a partial derivative.

So, that means this will be 2xy + 1 + 3 xz^2 when I am making the partial derivative with respect to z, x will behave like a constant and I will just make a derivative, so, this should be the result at any x y z point. Now, if somebody asks find out the value of this quantity at some given point say 1 1 1. What should be the value of this quantity? So, that quantity for this given field is simply 2 multiplied by 1 multiplied by 1 because x and y is 1 1 + 1 + 3 1.1^2, so it should be simply 2 + 1 + 3. So, it should be simply 6.

So, that is the value I am having here. So, the next thing that we; are going to understand that what geometrically it means. So today, I am not having that much of time to discuss in detail the geometrical meaning. So, in the next class, maybe I can find out what is that I can do in detail, that what is the meaning of the divergence, and try to prove one very important theorem, which is called the divergence theorem. So, with that note, I do like to conclude my class here. Thank you very much for your attention, and see you in the next class.