Foundations of Classical Electrodynamics
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology-Kharagpur

Lecture-70
Maxwell's Equation in Matter (Contd.)

Hello student the foundation of classical electrodynamics course. So, under module 4 we have lecture number 70 today. And we will be going to continue our discussion on Maxwell's equation in matter.

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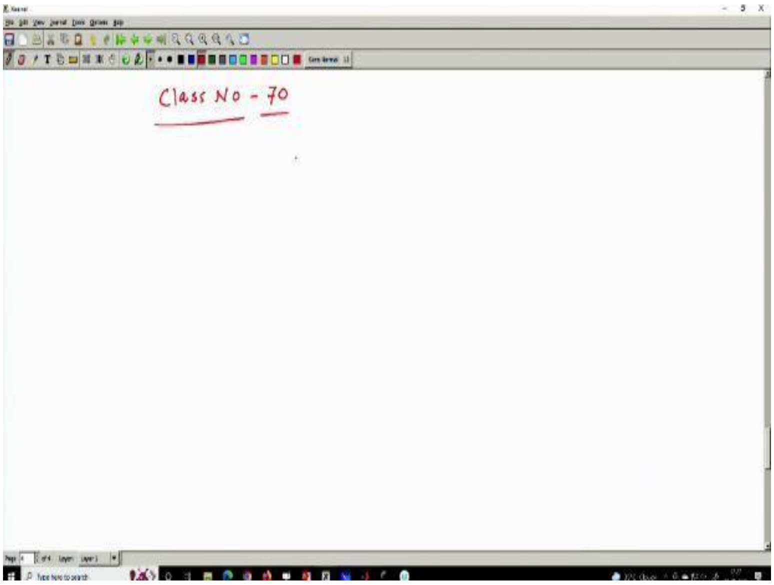

Today I have class number 70. So, in the last class we find that when we have a material when we calculate the Maxwell's equation in a matter.

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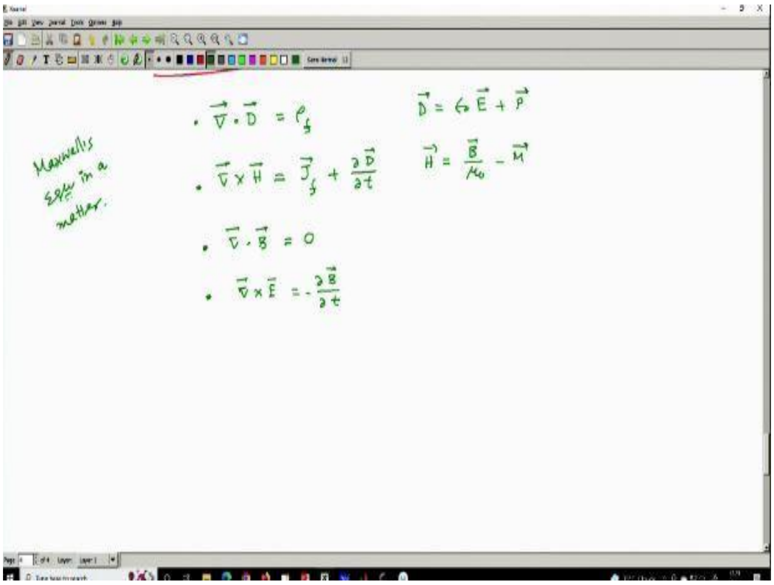

Then the source equation, two equation containing the source is modified, one is $\vec{\nabla} \cdot \vec{D}$ is ρ_f free charge density and that is one modification where $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$. And another equation that we modified is the $\vec{\nabla} \times \vec{H}$ = free current density + the rate of change of displacement current where, $\vec{H}$ is defined like $\frac{\vec{B}}{\mu_0} - \vec{M}$. So, these are the 2 equations that we modified.

But the other two equation these are the equation having source term but the other two equations like $\vec{\nabla} \cdot \vec{B} = 0$ remain same. And also the equation like $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ is also remain intact. There will be no change, only change when we discuss the Maxwell's equation in matter then this the term associated with the charge density and the current density we are going to modify. This is the Maxwell's equations in the matter. Well, let us try to find out what should be the boundary condition then.

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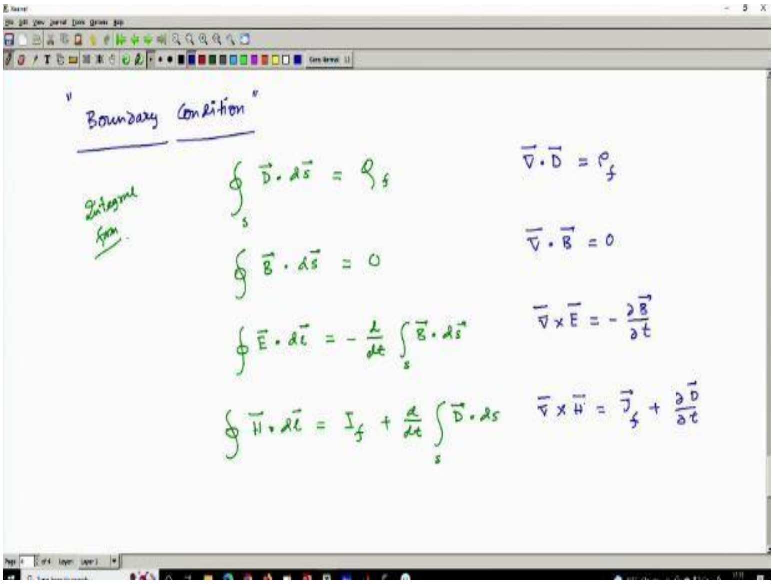

So, the boundary condition for this set of equations. So, now I write the Maxwell's equation in integral form these Maxwell's equation in matter in integral form if I write. So, I should have the first equation in this way, this is the integral form, since this integral form I should not have this operator anymore. So, I have the $\oint \vec{D} \bullet d\vec{s}$ is equal to the total free charge that is first one.

Second one is the magnetic flux is simply 0 that is the consequence of the second equation. Then we have the Faraday's law and if I write in integral form it should be $\vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}$ and then we have the magnetic flux $\vec{B} \bullet d\vec{s}$ over surface integral. And finally we have the $\oint \vec{H} \bullet d\vec{l}$ that is the free current and then plus $\frac{\partial}{\partial t} \int \vec{D} \bullet d\vec{s}$. So, quickly if I in the right-hand side, so, this is corresponds to the equation $\vec{\nabla} \bullet \vec{D} = \rho_f$.

This is the equation corresponds to $\vec{\nabla} \bullet \vec{B} = 0$. This equation corresponds to $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. And finally this equation is $\vec{\nabla} \times \vec{H}$ is free current density + $\frac{\partial \vec{D}}{\partial t}$. So, these are the equations left-hand side we have integral form and right-hand side we have the differential form, the usual form is the differential form. But, you should also appreciate the integral form. In this case for boundary condition we require this integral form. So, let us now do what is the boundary condition.

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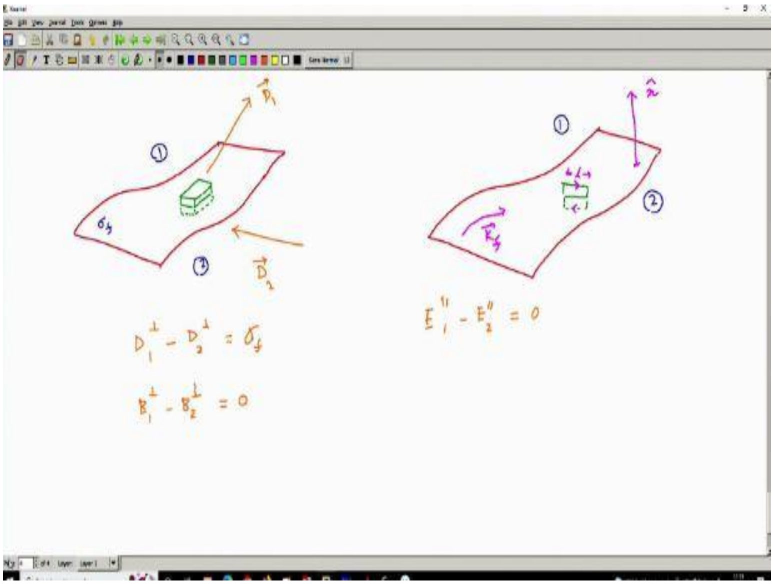

So, before we have a surface like this is the surface we are having and try to find out the boundary condition for $\vec{D}$. So, we had a surface pill box like this below that we have the dotted line. So, surface is over this region. And let us consider σ_f to be the free surface charge density here and if I have 2 $\vec{D}$ so, this is the region 1. So, let me consider this as region 1. Say here we have region 1 and this region is region 2.

And if we have the $\vec{D}$ like in this case, this is my $\vec{D}$ in arbitrary direction like say this is my $\vec{D}_2$. And suppose this is another $\vec{D}$ along this direction, say so this is my $\vec{D}_1$. So, now if I try to find out the components and exploiting the way we did before I am not going to do the detail calculation but, you can understand that. So, $D_1^{\perp}$ - $D_2^{\perp}$ = σ_f that is the first equation you will have.

And for $\vec{B}$ it is 0 so, you should have $B_1^{\perp}$ - $B_2^{\perp}$ = 0. So, the perpendicular component of the displacement vector is discontinuous. It depends on the surface current density. If the free surface current density if there is no free charge here. Then it is the perpendicular component is continuous but, if there is then it should be equivalent to this difference is in fact equivalent to the free surface charge density. And for magnetic field $\vec{B}$ field this perpendicular component is conserved.

Now the parallel component whenever we discuss the parallel component, then I need to use the expression of $\vec{H}$ here and for parallel. So, I am going to use this curl expression that means this line integral I am going to use. So, in first case, when I use the surface integral, then the equation this equation and this equation was considered and that is why $\vec{D}$ and $\vec{B}$ components are written here.

Now we are going to consider the other 2 equations, which is the line integral. And for line integral we need to write down the expression in terms of boundary condition in terms of $\vec{E}$ and $\vec{H}$. So, again let me draw quickly the surface. So, this is the surface. So, this is our surface say and previously we have a surface here but, now we should have a line here closed line. These things we did in earlier class.

And inside below we also have the continuation of this line. And the line direction here is along this. And this length is usually l and the surface current that is flowing here free surface current is $\vec{K}_f$. And this is a perpendicular direction of the surface $\hat{n}$. And this is region 1 and this is region 2. Now here if I calculate like the way we calculated earlier. So, it should be $E_1^{||}$ - $E_2^{||}$ = 0.

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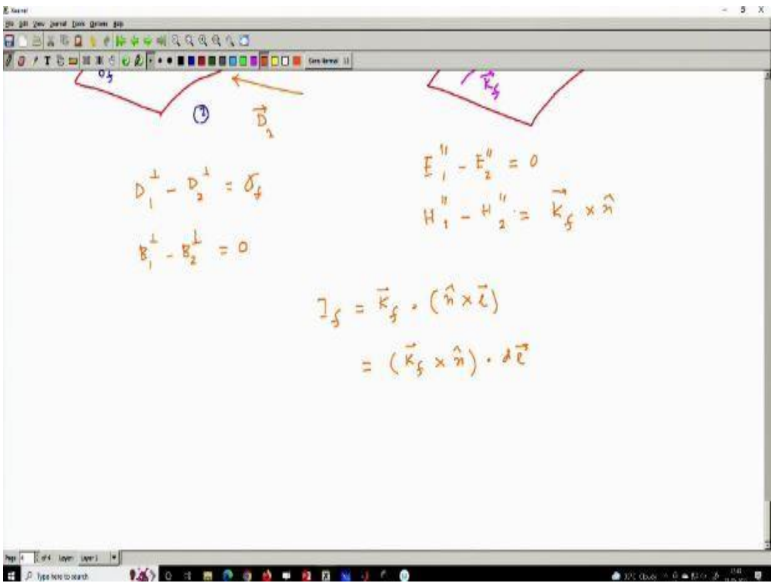

And if you calculate in terms of $\vec{H}$ it should be $H_1^{||}$ - $H_2^{||}$ that should be equivalent to the $\vec{K}_f \times \hat{n}$ because, I_f the free current is simply $\vec{K}_f \cdot (\hat{n} \times \vec{l})$ and that is $(\vec{K}_f \times \hat{n}) \cdot d\vec{l}$. Now if I exploit this equation and this equation considering the line integral here. So, this is the result one can expect and this is precisely the boundary condition in terms of $\vec{D}$ and $\vec{H}$. Now after having these boundary conditions, let us now jog down what we had so far in terms of Maxwell's equation in different medium.

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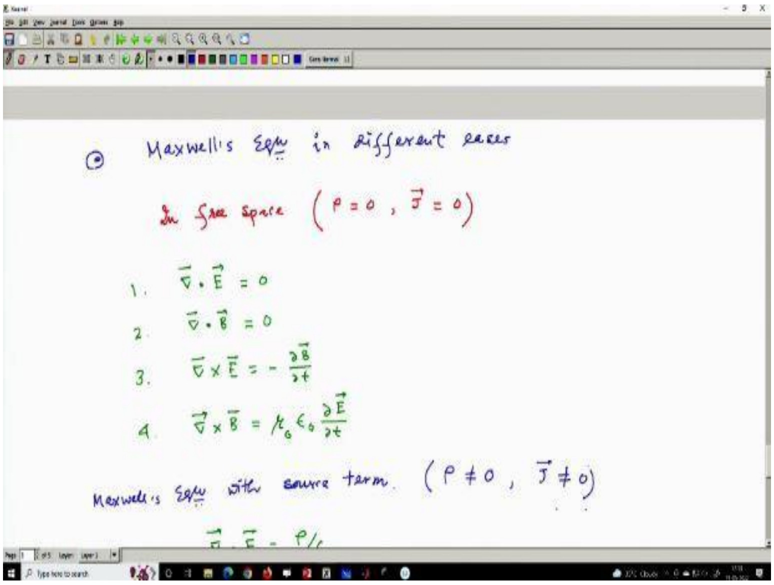

So, here what I am writing is the Maxwell's equations in different mediums, in different cases. So, the first one is the most simple one that we discussed. So, in first case I write in free space or in vacuum. In free space what we had both the source term ρ and $\vec{J}$ should be 0, that makes the equation even simpler and I simply write this equation in this way. So, equation 1 is $\vec{\nabla} \cdot \vec{E}$ is 0. Equation 2 $\vec{\nabla} \cdot \vec{B}$ is 0. Equation 3 $\vec{\nabla} \times \vec{E}$ is $-\frac{\partial \vec{B}}{\partial t}$.

And finally we have $\vec{\nabla} \times \vec{B}$ is $\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$, ρ and $\vec{J}$ 0, so, this is the expression these are the expressions in free space, this is Maxwell's equation in free space. So, now the Maxwell's equation with source term, if there is a source term so how should I write.

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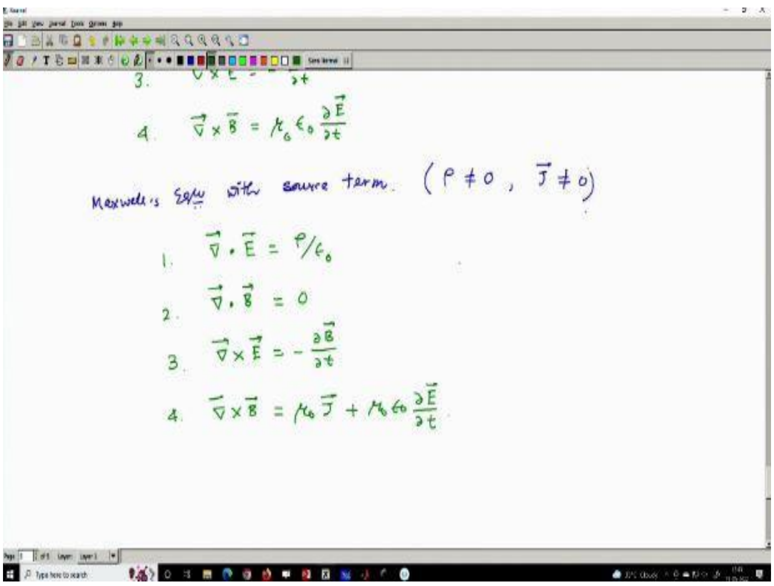

So, we write Maxwell's equation with source term. When we write source term that means, now no longer ρ and $\vec{J}$ are 0, rather they are not zero. And the 4 equations again a similar kind of equations will be there, only equation 1 and 4 will be modified. So, $\vec{\nabla} \cdot \vec{E}$ is now $\frac{\rho}{\epsilon_0}$ then I have $\vec{\nabla} \cdot \vec{B} = 0$. Then we have $\vec{\nabla} \times \vec{E}$ is $-\frac{\partial \vec{B}}{\partial t}$ and then $\vec{\nabla} \times \vec{B}$ is $\mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$.

This is the expression most of the books you will find. This is the expression for Maxwell's equation with source term but not in the medium. We need to introduce this ρ and $\vec{J}$ explicitly in a different way in the medium that I like to write here.

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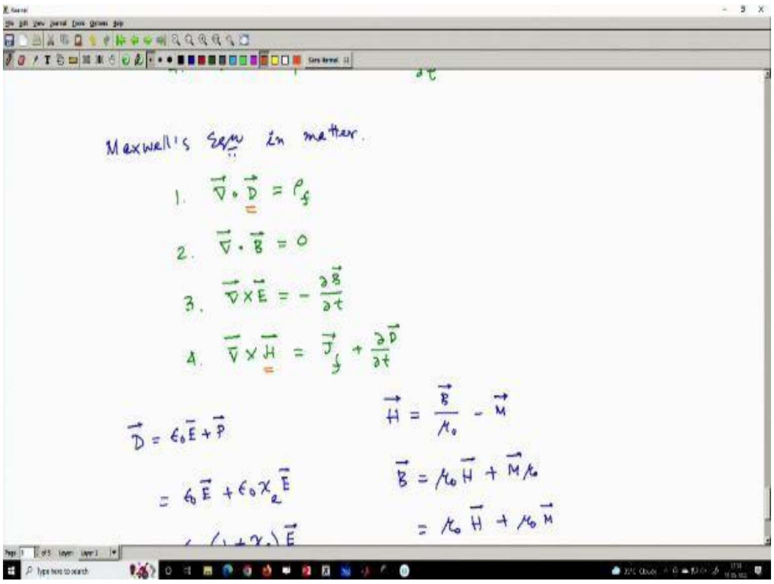

So, now if I want to write Maxwell's equation in matter, then the form of the equation is slightly modified that we discuss but let me write it here. So, the first equation will be modified like this. Instead of $\vec{E}$ I am going to get $\vec{D}$ and here I have ρ_f. Second equation will not want to change because, there is no source term and I mention that the term the Maxwell's equation having the source term is only going to modify, when we are discussing the Maxwell's equation in a matter.

Third equation again will remain unchanged. So, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. And finally I have the equation again that having certain modification it should be $\vec{J}_f + \frac{\partial \vec{D}}{\partial t}$. Here you should remember few things because, I am writing this in terms of $\vec{D}$ and $\vec{H}$. So, the expression of the $\vec{D}$ and $\vec{H}$ should be given.

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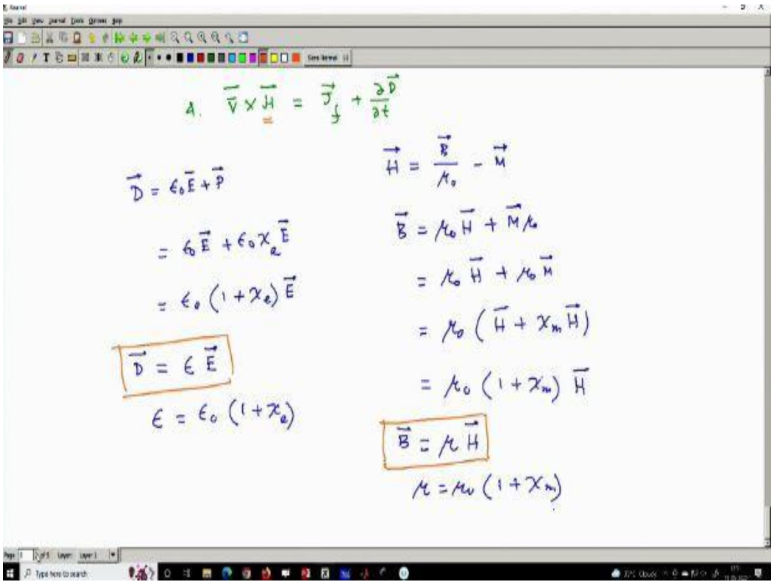

The expression of the $\vec{D}$ is in terms of $\vec{E}$ it is ϵ_0 $\vec{E}$ and the polarization $\vec{P}$ that is one. An expression of the $\vec{H}$ in terms of $\vec{B}$ and magnetization $\vec{M}$ is also there. So, let me write it $-\vec{M}$. So, we can also write few things because, the relationship with $\vec{D}$ and $\vec{E}$ and $\vec{H}$ and $\vec{B}$ can be written in this way. So, here it is ϵ_0 $\vec{E}$ and $\vec{P}$ again I can write in terms of $\vec{E}$ like ϵ_0 χ_e $\vec{E}$.

So, that makes me ϵ_0 $(1 + \chi_e)$ $\vec{E}$. And that is simply $\vec{D} = \epsilon\ \vec{E}$ where, ϵ is ϵ_0 $(1 + \chi_e)$. This is the way we define the relationship between $\vec{E}$ and $\vec{D}$ in terms of ϵ the permittivity**.** Similar way I can have my $\vec{B}$ here. So, $\vec{B}$ is simply μ_0 $\vec{H} + \vec{M}$. So, here I write μ_0 $\vec{H}$, there is a relationship between $\vec{M}$ and $\vec{H}$ and I have μ_0 multiplied here. So, it should be μ_0 $\vec{M}$ here. And let us take μ_0 common first μ_0.

Then I have $\vec{H}$ plus then I have magnetic susceptibility multiplied by $\vec{H}$ that is the relationship between $\vec{M}$ and $\vec{H}$. So, it should be $\vec{H}$ here. So, I can write it as μ_0 $(1 + \chi_m)$ $\vec{H}$ or simply $\vec{B} = \mu$ $\vec{H}$. My $\mu = \mu_0$ $(1 + \chi_m)$. So, this is the way we can represent the Maxwell's equation in different cases. In free space the first one, second one is when we have a source term and then when we have a Maxwell's equation inside a matter.

Inside the matter only thing that you need to consider is now it is written in the first equation and the last equation that is the equation having the source term is modified. It is written in terms of $\vec{D}$, which is the displacement vector and $\vec{H}$. The magnetic field is now in written in terms of $\vec{H}$. And the relationship between the $\vec{B}$ and $\vec{E}$ with $\vec{D}$ and $\vec{H}$ is shown here. So, the relationship is this.

This is the relationship between $\vec{D}$ and $\vec{H}$. And this is the relationship between μ $\vec{B}$ and $\vec{H}$, $\vec{D}$ and $\vec{E}$ and μ and $\vec{H}$**.** So, now we have an idea that how to write down the Maxwell's equation in different system. So, now we will be going to discuss about the electromagnetic wave.

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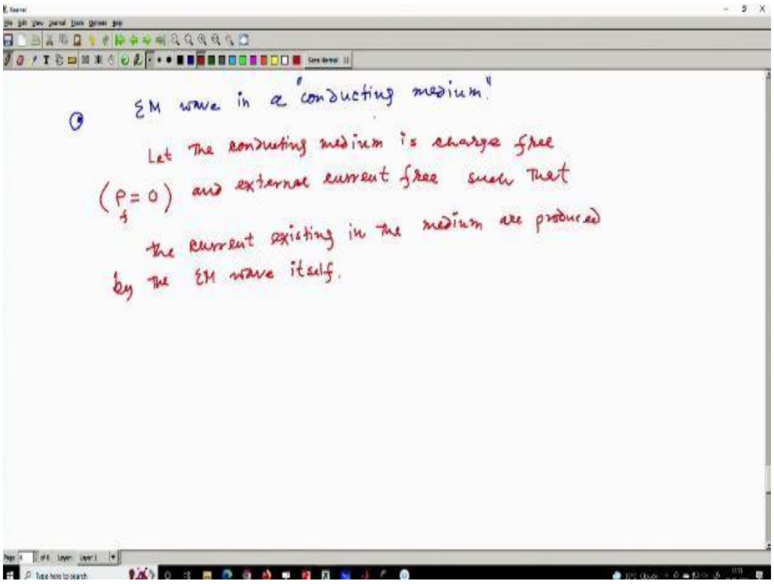

So, our next topic is to understand the EM wave in a conducting medium. So, far we are dealing with the dielectric medium and write down the Maxwell's equation. Now the situation is different we are having a conducting medium, dielectric medium or magnetic material we deal with the Maxwell's equation. But, now we are dealing with the conducting medium and in conducting medium the Maxwell's equation going to modify.

So, as the corresponding wave equation that we are going to find out and that should be our final topic in this module. In module 4 we have a final topic, this is the final topic. So, eventually this is our last class but we will be going to discuss few tutorial problem later. So, in the conducting medium, so, let us write in this way. Let the conducting medium is charge free. That means, we consider there is no free charge ρ is 0 here and external current free also.

External current free means, there is no free current charge density. So, this is f is 0 and also there is no $\vec{J}_f$ and this is such that. The current existing in the medium is produced by the EM wave itself. So, what is the meaning of that? That we are having a medium where the free charge density is not there and also the external current is free. There is no such external current. Only the current that because, it is a conducting medium. The only current that we have in the system is due to the electromagnetic wave itself. So, what current we are talking about here, we are simply talking about the current that is driven by the external electric field.

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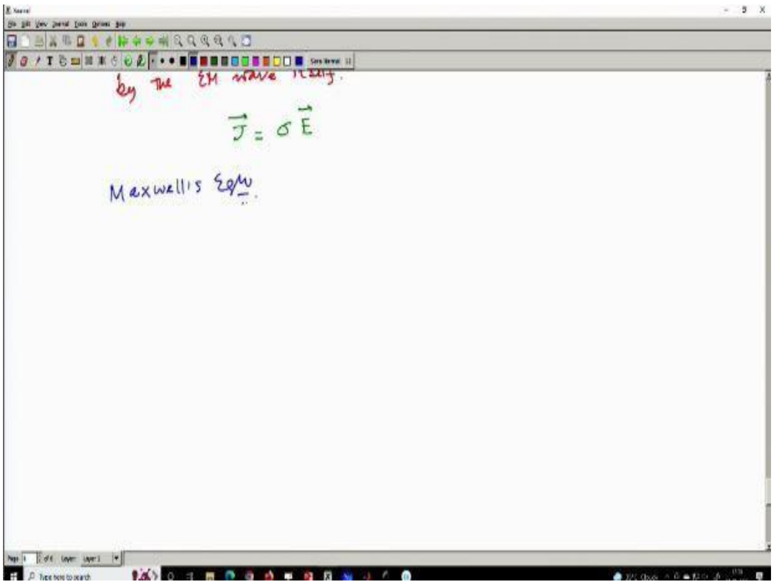

And I can write an expression like $\vec{J}$ is equal to then $\sigma\ \vec{E}$. So, that is a well-known equation. So, that is the relationship with the current that is governed by the electromagnetic wave, electric field is there. So, $\vec{J}$ should be $\sigma\ \vec{E}$ where, σ is conductivity.

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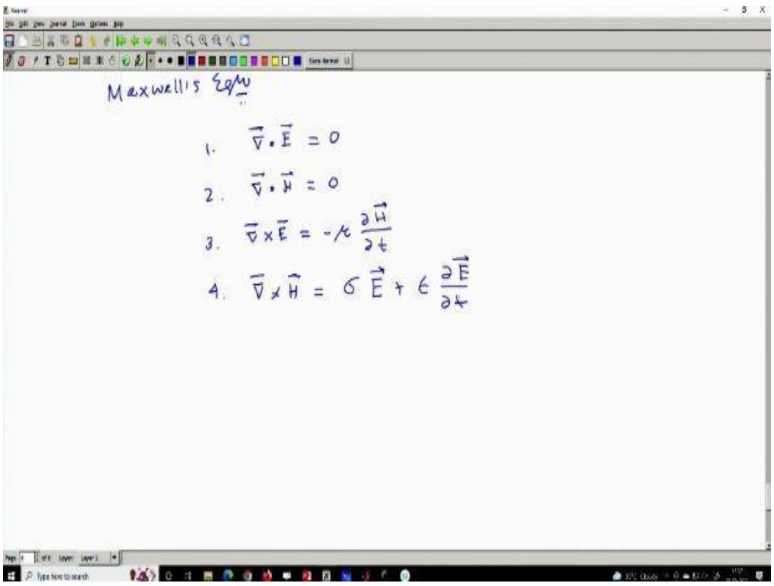

So, the Maxwell's equation now under that condition, the Maxwell's equation we can write in this way. For this case, the first equation is $\vec{\nabla} \cdot \vec{E} = 0$ because, there is no charge density. Second equation $\vec{\nabla} \cdot \vec{B}$ or $\vec{\nabla} \cdot \vec{H}$ let us write in terms of $\vec{H}$ is 0. Third equation is $\vec{\nabla} \times \vec{E}$ is $-\frac{\partial \vec{B}}{\partial t}$ let us write in terms of $\vec{H}$. So, I should write it $-\mu\frac{\partial \vec{H}}{\partial t}$.

And the fourth equation is $\vec{\nabla} \times \vec{H}$ is $\sigma\vec{E}$ because, there is a $\vec{J}$. And that $\vec{J}$ should be $\sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}$. So, these are the 4 equations we are having. Now if I want to find out the Maxwell's wave equation.

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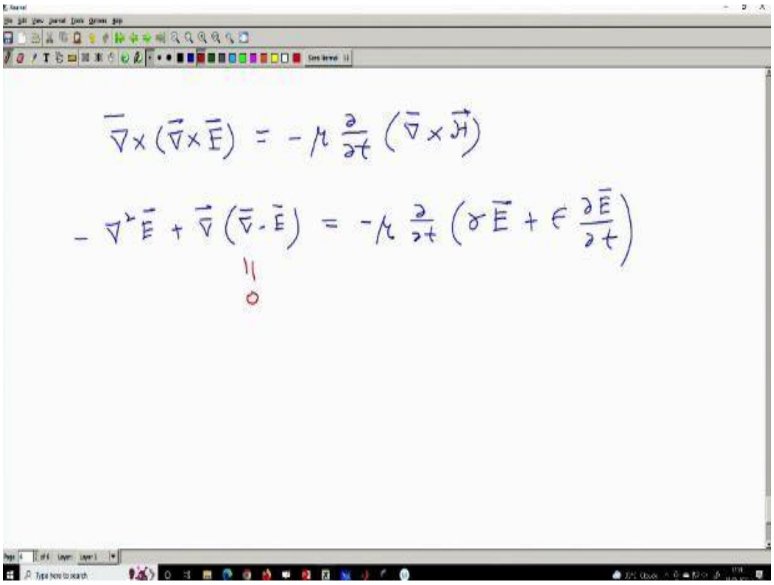

So, I will should make a curl over the equation third equation $\vec{\nabla} \times (\vec{\nabla} \times \vec{E})$ the standard procedure that we followed. That should be equal to $-\mu \frac{\partial}{\partial t}$ and then I have $\vec{\nabla} \times \vec{H}$. So, left-hand side the famous identity that $\nabla^2\vec{E} + \vec{\nabla}(\vec{\nabla} \cdot \vec{E})$. Right-hand side what I write is simply $-\mu \frac{\partial}{\partial t}$ then $(\sigma\vec{E} + \epsilon\frac{\partial \vec{E}}{\partial t})$ that I find. Now this is equal to 0.

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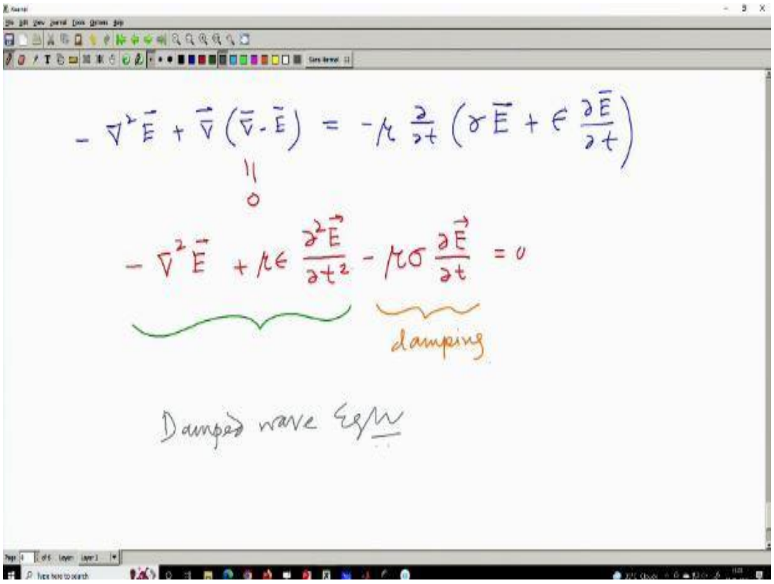

So, what I get we get simply $-\nabla^2\vec{E}$ and then + $\mu\epsilon\,\frac{\partial^2 \vec{E}}{\partial t^2}$. Now - $\mu\,\sigma$ then we have $\frac{\partial \vec{E}}{\partial t}$ that is equal to 0. So, we have a first order derivative term with respect to time, first order derivative of $\vec{E}$ with respect to time. So, that gives me, so, that is the wave equations up to these we can identify this to a wave equation.

But, this additional part is putting some kind of damping. So, this is a damping part. So, this equation is overall a damped wave equation. We are having a damped wave equation here in our hand. When we deal with the here I think I have this sign will be plus. So, we have a damped wave equation in our hand. And let us now consider the plane wave solution because, we know the solution for the wave equation is plane wave.

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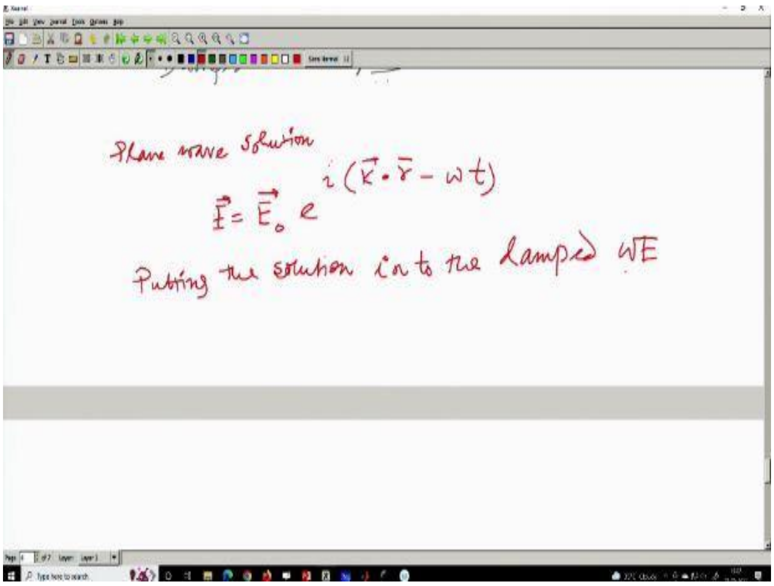

So, the plane wave solution, if I consider plane wave solution that is $\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ that is the way we have our plane wave. Now if I put this plane wave here in this equation. So, what we get is like putting the solution in to the damped wave equation then what we have is this.

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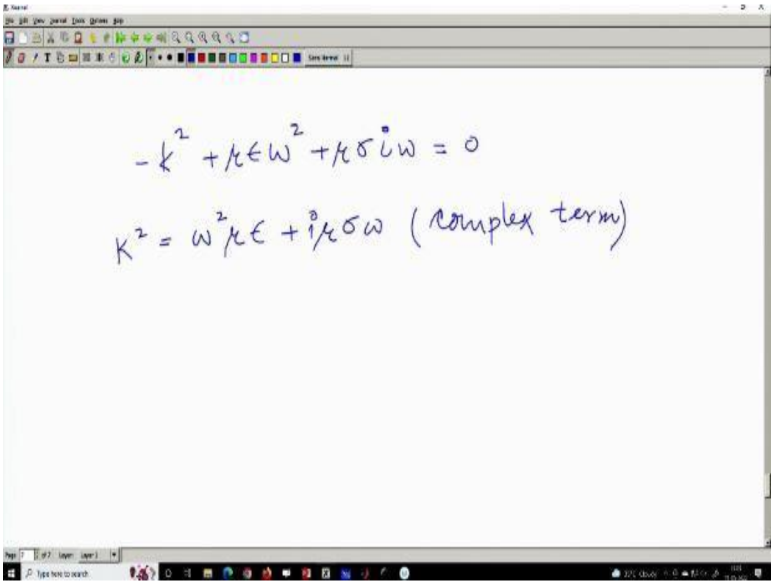

We have $-k^2 + \mu\,\epsilon\,\omega^2 + \mu\,\sigma\,i\,\omega$, just make the derivative and you will get all these things. Now k^2 is simply $\omega^2\,\mu\,\epsilon + i\,\mu\,\sigma$ and ω. Now this is a complex term. You can see that k is now a complex term, which is having a term associated with i, which is here sitting.

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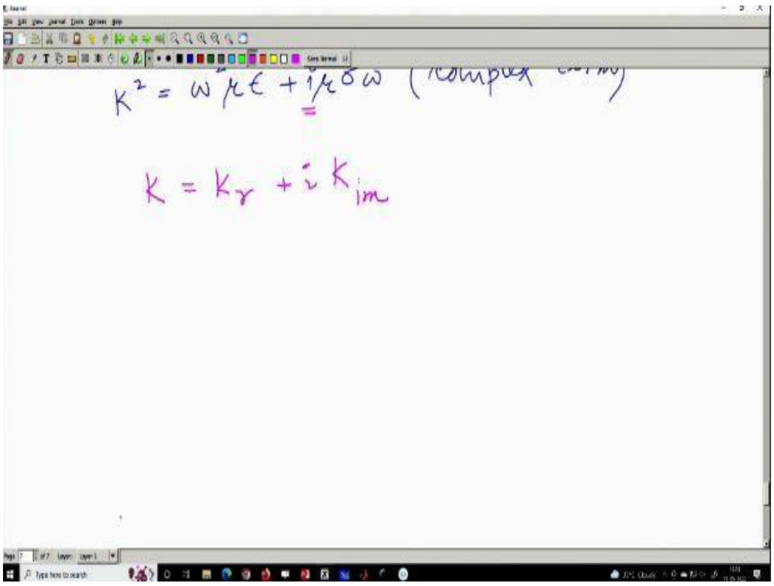

Now I can write k in 2 forms. So, say k I can write the real part k is a propagation constant. So, I can write this propagation constant real part and an imaginary part like this $k_r + i\,k_m$.

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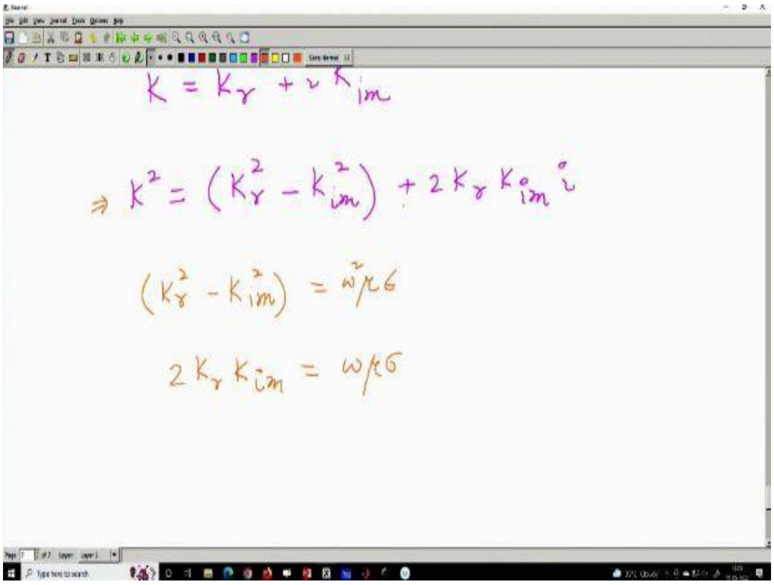

Now k^2 is simply k_r^2 - k_{im}^2 + 2 k_r k_{im} and there should be a i term or you make a square of that. So, if I tally with this equation whatever we get here, this one from, this one and this one. Then I can write it like k_r^2 - k_{im}^2 that is simply ω^2 and then μ ϵ that is one term. And another term is 2 k_r k_{im} is ω μ and σ. So, from those 2 equations I can have this relationship. And if you solve that it is not a complicated task to solve with 2 equations, 2 unvariables, 2 unknowns.

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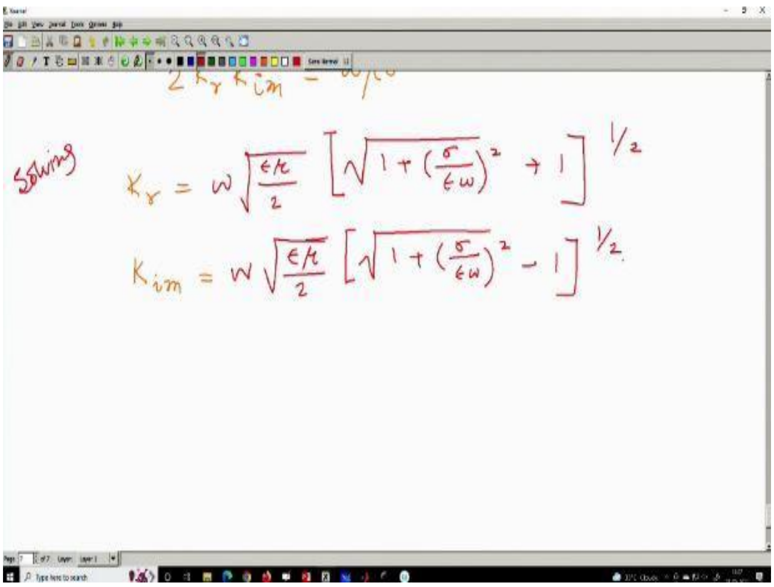

Then one can have k_r solving I am just writing the solving you have k_r = ω $\sqrt{\frac{\epsilon\mu}{2}}$ and then we have [$\sqrt{1 + (\frac{\sigma}{\epsilon\omega})^2}$ + 1]$^{1/2}$ that is my real part of the k. And what is the imaginary part of a considering the damping? Imaginary = ω $\sqrt{\frac{\epsilon\mu}{2}}$ [$\sqrt{1 + (\frac{\sigma}{\epsilon\omega})^2}$ - 1]$^{1/2}$ that is the solution for k_r and k_{im}.

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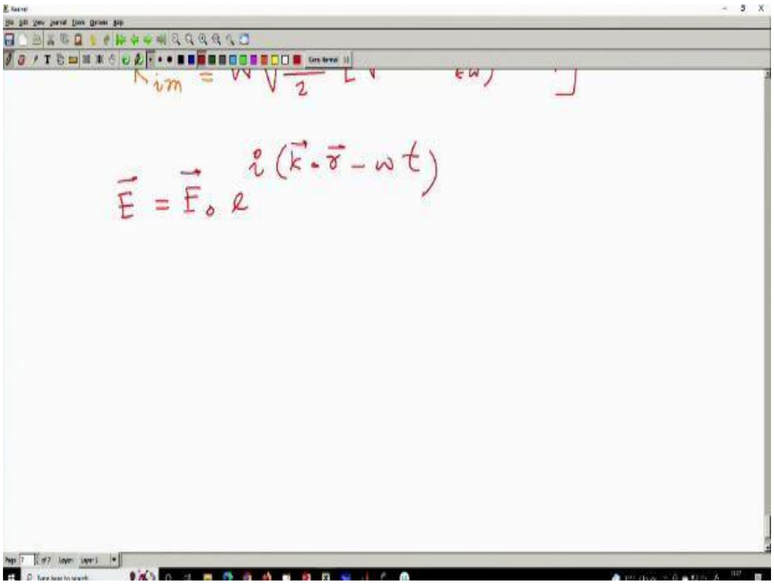

Now the solution if I go back to the solution it was $\vec{E} = \vec{E}_0 \ e^{i(\vec{k} \bullet \vec{r} - \omega t)}$ that is the form of the solution.

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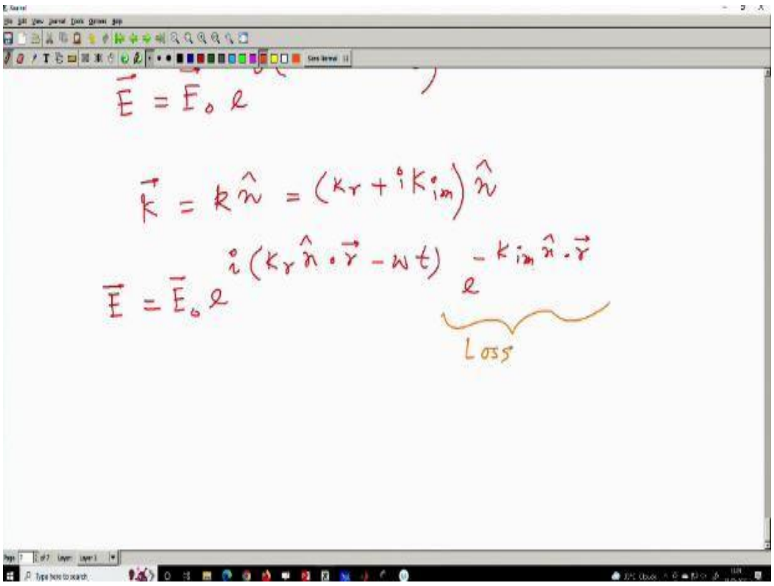

Now k is now a quantity, which is having the imaginary part. So, I should have write k say a direction $\hat{n}$ and this is having 2 parts the amplitude it is k_r + i k_{im} and the vector $\hat{n}$. So, if I put in the solution one can readily find that $\vec{E}$ is simply $\vec{E}_0 \ e^{i(k_r \hat{n} \bullet \vec{r} - \omega t)}$ that is a sinusoidal kind of term. But, we have an exponential term as well.

So, let me use this part. We have an exponential term, which is a decaying part governed by the imaginary part $\hat{n} \bullet \vec{r}$. So, this is this term is the attenuation we are having. This is the loss term. This is exponentially decayed. So, few things we can note.

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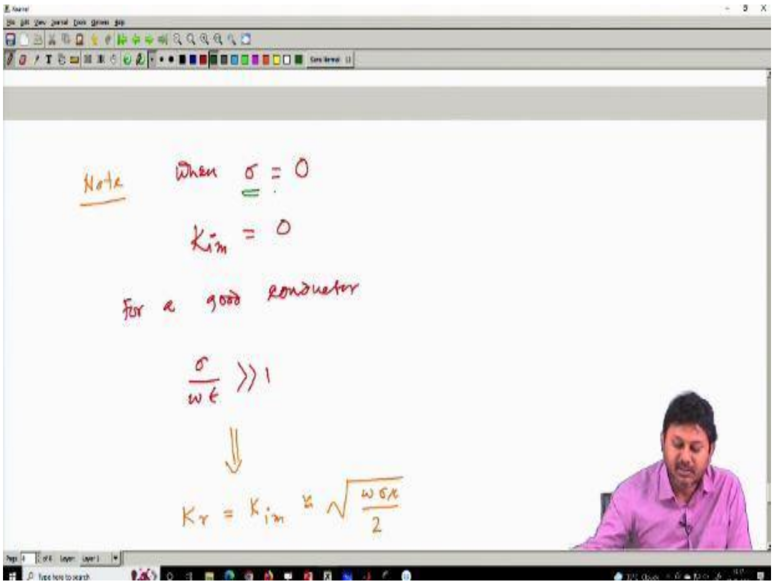

So, let us note this facts that. When note. I am making σ is 0, then k_{im} whatever we have k_{im} if I go back to the equation here. If I put σ = 0, then you can see that k_{im} simply vanishes. So, k_{im} is 0 and so there is no attenuation. And this attenuation is coming only because, of the non-vanishing sigma. This conductivity basically gives some sort of attenuation here in electromagnetic wave. Now for a good conductor, we have the condition.

For a good conductor, what we have is σ divided by that quantity $\frac{\sigma}{\omega\epsilon}$ should be very, very greater than 1. And in that case, we can have $k_r = k_{im}$, which is n equal to ω and then $\frac{\sigma\mu}{2}$. So, that we can put this condition you can put here in the solutions and if you put this in the solutions then we simply have k_r k_{im} n l equal to this.

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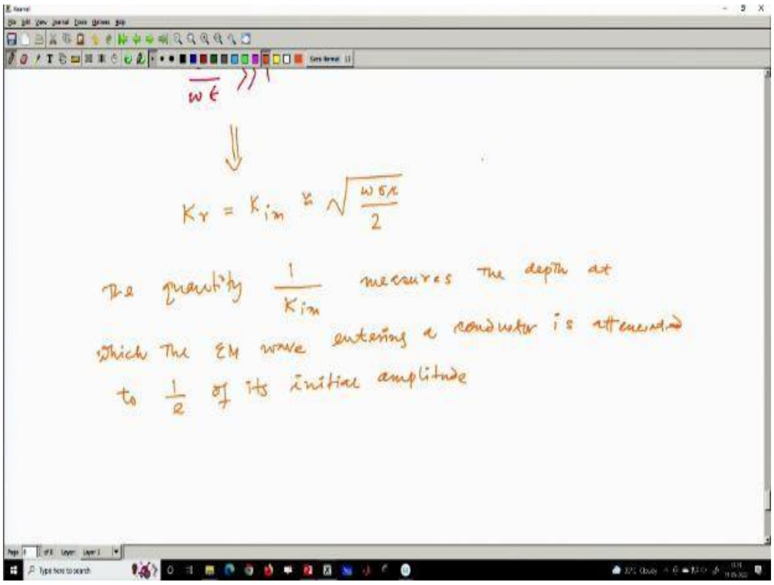

Now, the quantity $\frac{1}{k_{im}}$ that is the reciprocal part of the imaginary part of the propagation constant is generally measures the depth at which the electromagnetic waves entering a conductor is attenuated at the new it to $\frac{1}{e}$ of it is initial amplitude, which is straight forward here. So, k_{im} is here. So, when k_{im} if I have $\frac{1}{k_{im}}$ then this term should be to the power -1.

So, if I compensate if I put this r in such a way that it should be $\frac{1}{k_{im}}$. Then this quantity, which is loss it should be $\frac{1}{e}$. So, the amplitude will be reduced to $\frac{1}{e}$ term. So, that quantity so, whatever the attenuate you know its initial amplitude at the surface.

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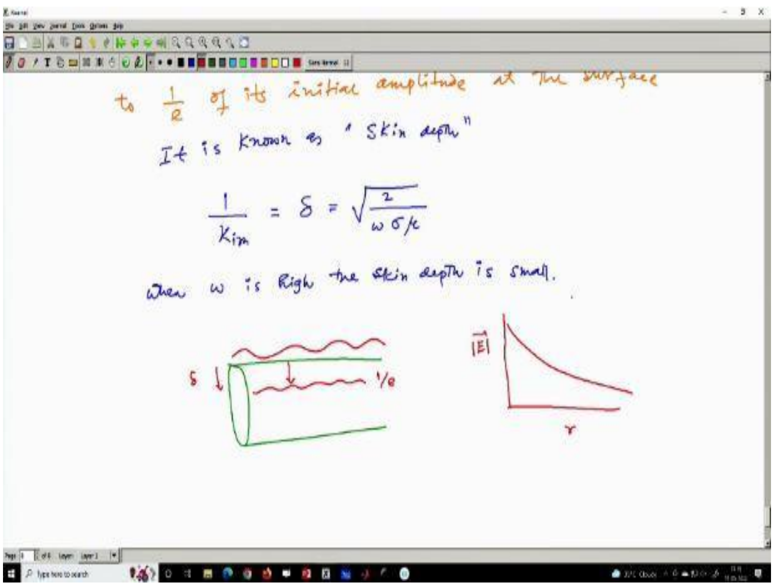

So, whatever the surface we have so the amplitude is decay $\frac{1}{e}$ time. So, it is known as skin depth. So, there is a specific name. So, it is known as skin depth. So, $\frac{1}{k_{im}}$ = δ. The skin depth and can be defined by $\frac{2}{\omega\sigma\mu}$. So, if you can see that for a given conductor.

When ω the frequency is high the skin depth is small. So, very high frequencies suppose, we have a conductor here, so, when the electromagnetic wave is propagating in this conductor. Suppose, this electromagnetic wave is propagating throughout this conductor. So, when it goes inside the conductor there is attenuation huge attenuation of the amplitude. So, this is exponential decay. So, it when it goes here is amplitude will decay down.

So, this is say $\frac{1}{e}$ amplitude and from here to here the depth we call the skin depth see δ. So, now if the frequency is very high, then the skin depth is very small. So, most of the electromagnetic wave for high frequency electromagnetic wave is moving through a conducting material through the surface. So, over the surface this is the $\vec{E}$ value that is attenuating with respect to $\vec{r}$.

So, most of the electromagnetic wave the entire part of the electromagnetic wave is containing over the surface of the conductor. Because, if it enters into the conductor then there should be decay and this decay is due to the imaginary part of the propagation constant $\vec{k}$. And that is arising due to this quantity σ that is the conductivity. The conductivity of the material, in the conductivity of the conductor basically attenuates the electromagnetic wave. And restrict the electromagnetic wave to stay mostly in the surface of the conductor.

So, with that note I like to conclude here in today's class. So, today is essentially the last class of the technical discussion. So, we should have a couple of more classes where, we are going to discuss about few problems, tutorial problems, mainly which will be helpful for you. And you can solve a different kind of problems, but these problems in the tutorial will be some typical type of problem, which generally you face in the exam. So, with that note I will like to conclude my discussion here. Thank you very much for your attention and see you in the next class.