**Foundations of Classical Electrodynamics**
**Prof. Samudra Roy**
**Department of Physics**
**Indian Institute of Technology – Kharagpur**

**Lecture – 07**
**Line, Surface and Volume Integration**

Hello students. So, welcome to the foundation of classical electrodynamics course. So, today we have lecture 7 and we will be going to learn the line, surface and volume integration.

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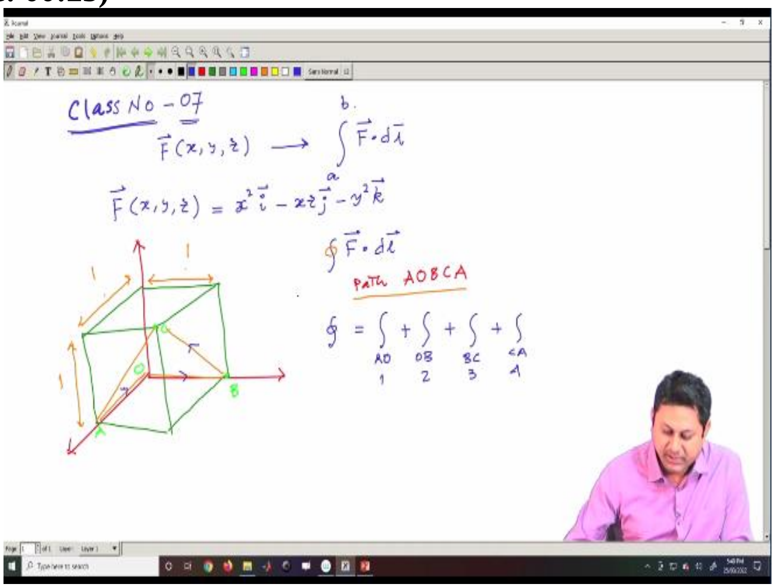


We have today class number 7. So, last day we mentioned about the line integral and if F is a vector field say function of x, y, z some vector field then the corresponding line integral we calculate like F dot dl some point a to b. So, today we will be going to calculate for a given problem. So, the problem is the vector field F it is given us this is a vector field, which is a function of x, y and z and it is given as $x^2\,\hat{\imath}$ - xz $\hat{\jmath}$- $y^2\,\hat{k}$. So, this is a vector field and the problem is this. This is the coordinate system and in this coordinate system I am having a unit box.

So, this point let us call this point O, this point, say A and this point here it is C, this is B. These are the points A, O, C, B. Now, I need to calculate this quantity the line integral I need to calculate. For a path, which is closed, for the path I should mention. The path is AOBCA. So, I can show you the path here. So, the path is I need to go from A to O that is over this path and then O to B then I go from here to here this path and then B to C go from here to here and C to A. So, you can see that this is a closed path. So, that is why I can write a closed circle here.

So, this signifies that this integral is over this given closed path so, my path is this and vector fields is given and also it is given that this length of these section is one, unity, this is one. So, from here to here, so, my drawing is need to be changed to understand properly let me so, from here to here, this length is 1, so this is a unit all the lengths are 1. So, this is the unit length cube with side 1 unit. So, let us now try to calculate this integral.

So, I can now the first path is I can now divide these closed integrals into 3 paths. So, the total integral can be divided into 4 paths rather. So, this is one path is A to O, because my path is AOB, then O to B, then B to C, this is my first path, this is my second path, this is my third path. And finally, C to A this is my fourth path, it is a 3-dimensional path. So, 2-dimensional problem are even easier. So, that is why I took a 3-dimensional path so, that you can understand that how so, finally, I am coming from C point to A point. Now, one by one I just calculate so far that is all.

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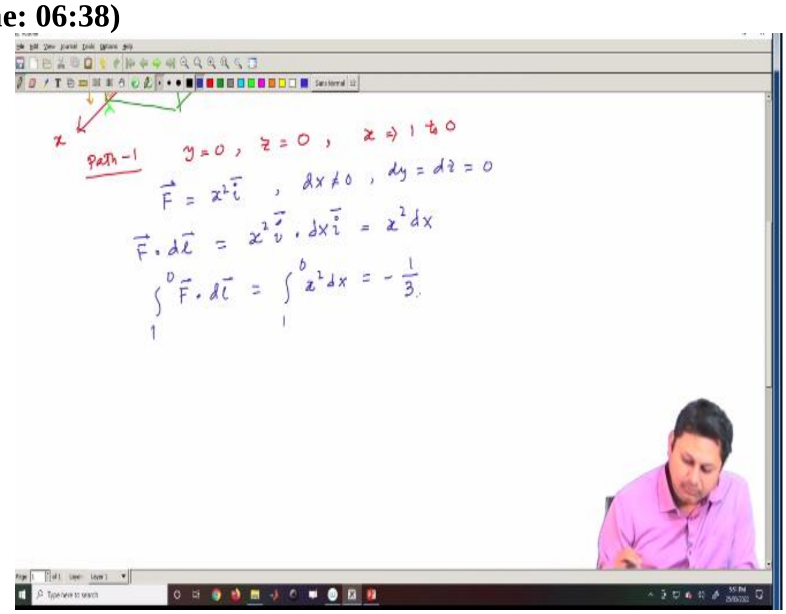


So, for path 1, if you look carefully, what information I get. For path 1, if this is x, y and z for path 1, you can see that y = 0. What about z, z = 0 so, the expression becomes simpler there is an only change in x. So, x is changing from 1 to 0 this is the variation we are having here. So, for that if I calculate my F first so, F simply it is given like $x^2$, so, z is 0, y is 0, so we simply have my F has $x^2$ i, that is all. So, and also x 0, y 0 mean dx, dy = 0.

So, F dot dl this quantity is $x^2$ i dot dx i, because other terms are 0, y 0 and I should write here explicitly. So, dx is not equal to 0, but dy and dz this is 0. So, this is simply $x^2$ dx. So, now if I integrate it from say 1 to 0, F dot dl, it should be integration of 1 to 0, x square dx, it should be $-\frac{1}{3}$, that is for path 1.

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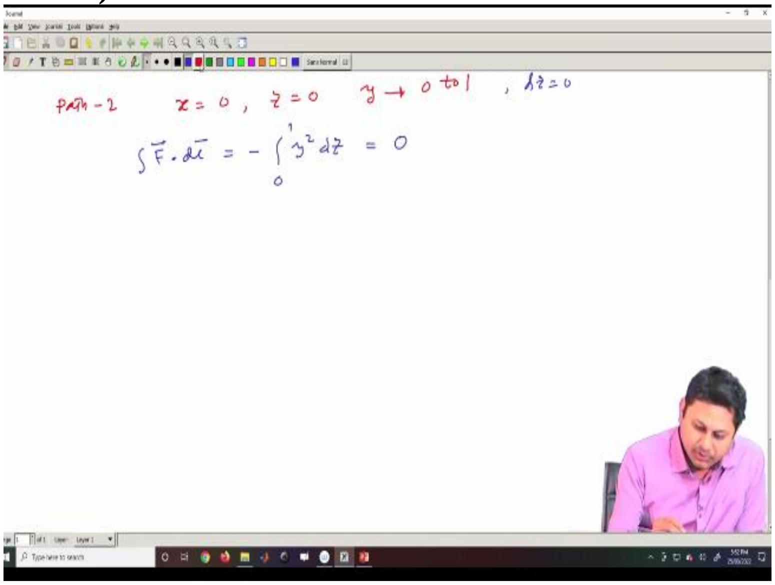


What about path 2? In path 2, if I look carefully path 2 means I am going O to B now, now O to B x will be 0, z will be 0, so, that I should write here first. So, my x = 0, z = 0 and y changes from 0 to 1. So, again I can have my F so, if I go back to the expression my F, y is non-zero, so, my F should be so, this quantity and now I should write directly. So, for path 2, my F dot dl this quantity gives me simply minus of 0 to 1, $y^2$ and here you have to be careful, because y is non-zero.

But when you have F dot dl associated to y we have dz, but here also dz = 0. So, this quantity is eventually when y is not equal to 0, but dz is 0. So, eventually this gives me 0 as dz = 0, because there is no change in z, it is in xy plane over x y this changing. What about path 3? For path 3, what we have I can see that it is going from B to C so, y value is 1 but x and z they are changing.

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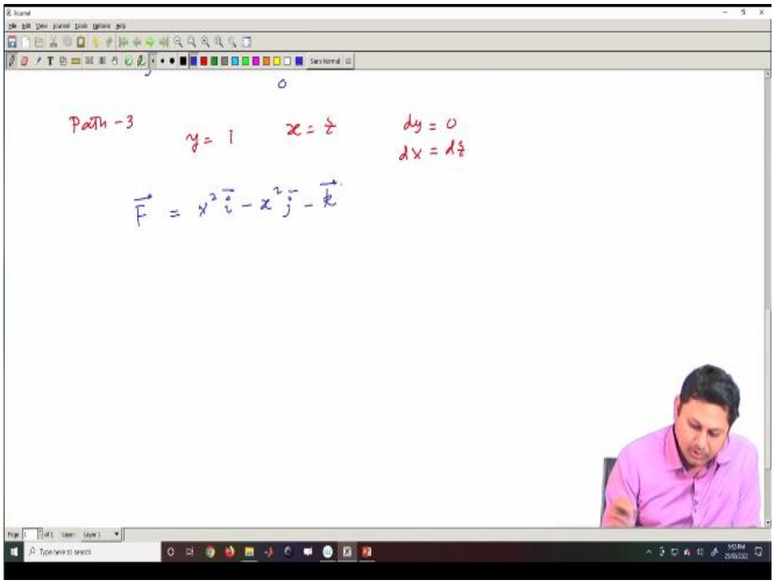


So, for path 3, I have y = 1 and x = z this condition I have. dy obviously, it should be 0 and dx = dz that is the condition. So, that means my F I can write it as $x^2$ i – $x^2$ j - k because my x and z are the same so, I just replace z to x here, so, it is $x^2$ this also becomes $x^2$ and this is 1 so, that is why it is simply this. Now, my dl what is my dl? The length element it should be dx i + dz k, dy is 0 here. So, F dot dl simply comes out to be if I do that simply comes out to be $x^2$ dx and then – dz, but dz and dx are same, so, I can have ($x^2$ – 1) dx as dx = dz.

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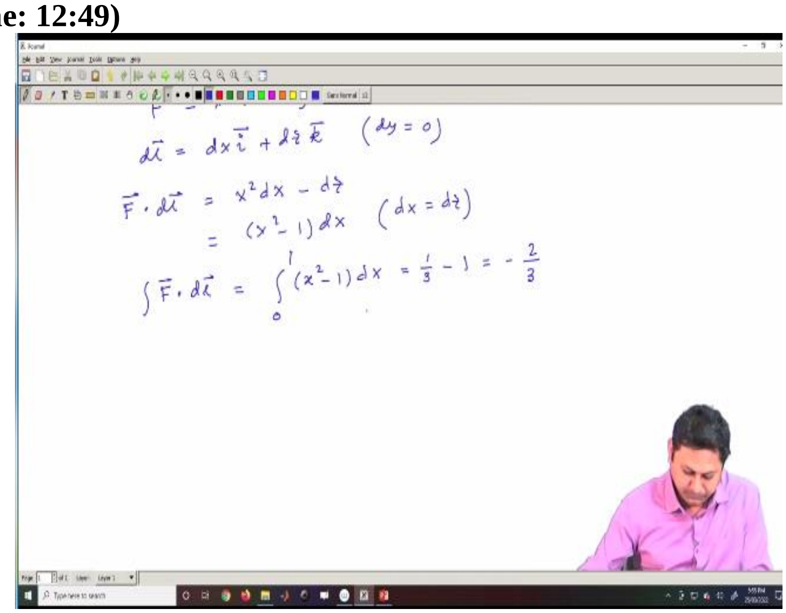


Now, I am in a position to integrate because I know what are the integrals. So, F dot dl for this path 3 should be equal to x is changing from 0 to 1, so I just put 0 to 1, integral is everything is in terms of x, so I should not have any issue here. And the result is $\frac{1}{3}$ -1, it should be $-\frac{2}{3}$ and finally, for path 4.

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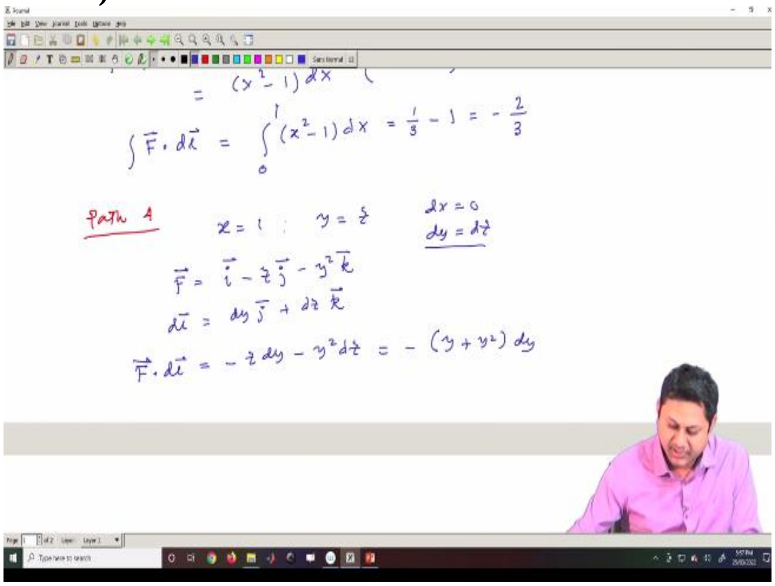


For path 4 what we are having for path 4, if I check here for path 4, I am going C to A and when I coming C to A, x value is fixed it is 1 and y and z are changing. And then I should write that here, my x = 1 but y = z that means dx = 0 and we have dy = dz that is the condition and putting everything into F, I can have F = i (because x is 1) - z j - y square k. And what is my dl? dx is 0. So, I can have dy j + dz k. Now dy and dz are same, they are varying in this way only.

So, eventually I have F dot dl = - z dy – $y^2$ dz, which is simply – (y + $y^2$) dy, because y and z are the same, so I just replace z = y and dz = dy and then I am going to get this. The final thing is to execute this integration.

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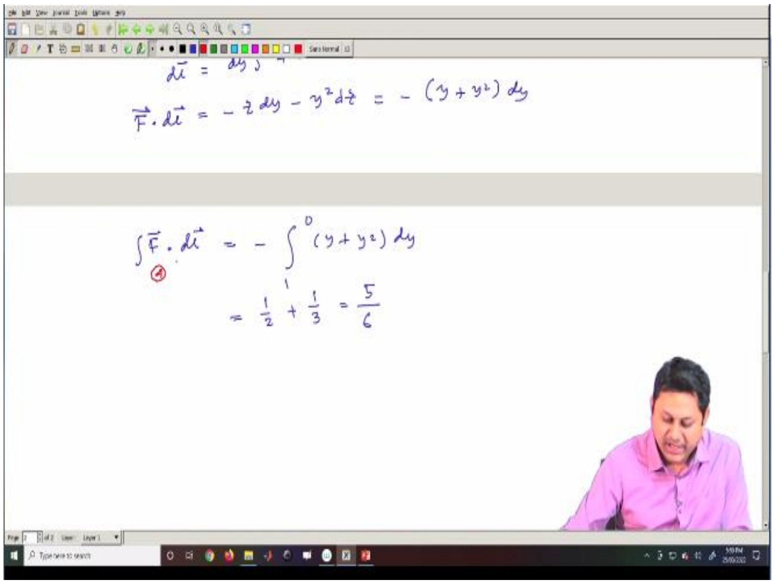

So, for that, I have F dot dl for that path, I should have the variation of y, if you look carefully, I am going from up to down that means the variation of y should be 1 to 0. So, that variation I will put so variation is 1 to 0 and negative sign I already put here and the function is $(y + y^2)$ dy, if you execute it should be $\frac{1}{2} + \frac{1}{3}$ absorbing this negative sign. So, it seems it is $\frac{5}{6}$. Now, finally, you need to add all these paths. So, let us do that. So, this is for path 1, this is for path 2, this is for path 3 and finally this is for path 4.

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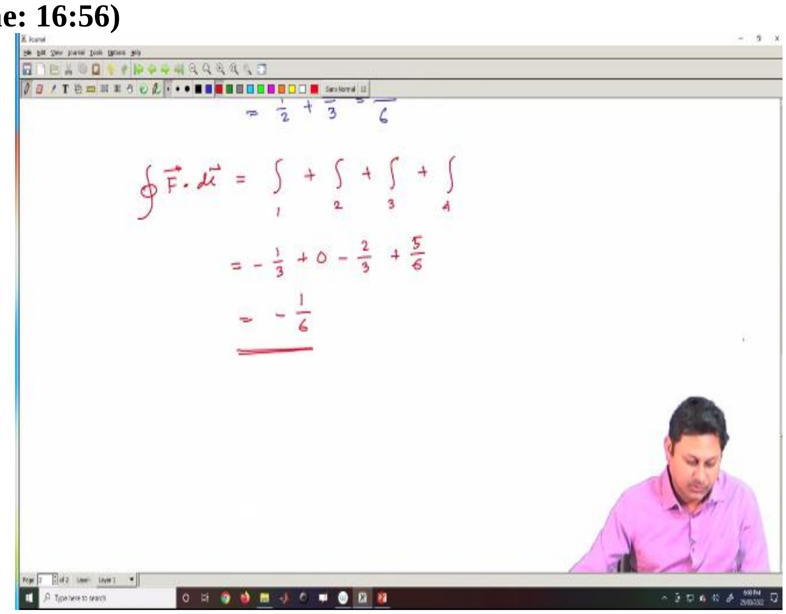

So, when you calculate the total integration, it should be the integration for path 1, plus the value of integration for path 2, the value of integration of path 3 and 4 and now I am going to put the value, so for 1, it is $-\frac{1}{3}$, then plus 0, then $-\frac{2}{3}$ and then $\frac{5}{6}$, these are the 4 values I figured out and the value should be $\frac{1}{6}$. So, the closed line integral value is eventually I find it is $-\frac{1}{6}$. So, this is the way students so I am giving you an elaborate example with a relatively complicated path.

But this is the way you should do you just need to find out what is the path first and then try to understand what is the relationship between x, y, z so that you can integrate it and a vector field is given F and vector field means at any at each x, y, z points the value of the field is given a vector at that particular point it is a direction of a vector and it changes and it forms a vector field. So, this vector field is given and you need to integrate over a closed path and this closed path is given here with this figure and one by one you calculate and you figure out the value.

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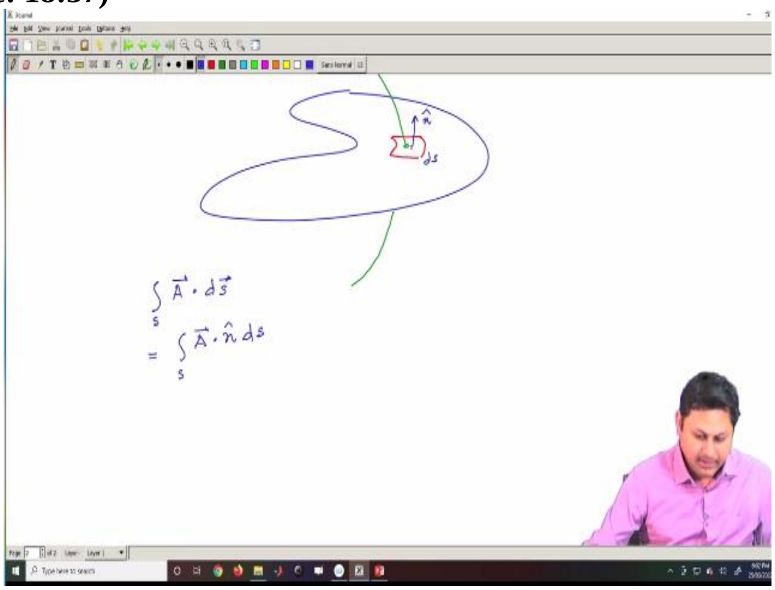


The next thing is the surface integration. So, in surface integration, what we have? Suppose, we are having a surface here like this and a section of the surface tiny section of the surface we can cut and a vector field is moving through the surface suppose, this is a vector field say A, which is moving through this surface. Now, this surface should have a direction this is n unit direction of the surface and this is my ds the small amount of surface.

The surface integral is essentially, if I now write mathematically it is A dot ds this is the value I want to find out this is my surface integral and normally we write s or double integral. This is essentially the surface integral s, A dot the unit vector that we have ds. So, this is the way also you can represent because ds is a surface element with the unit and n is unit vector so, this is the way you can.

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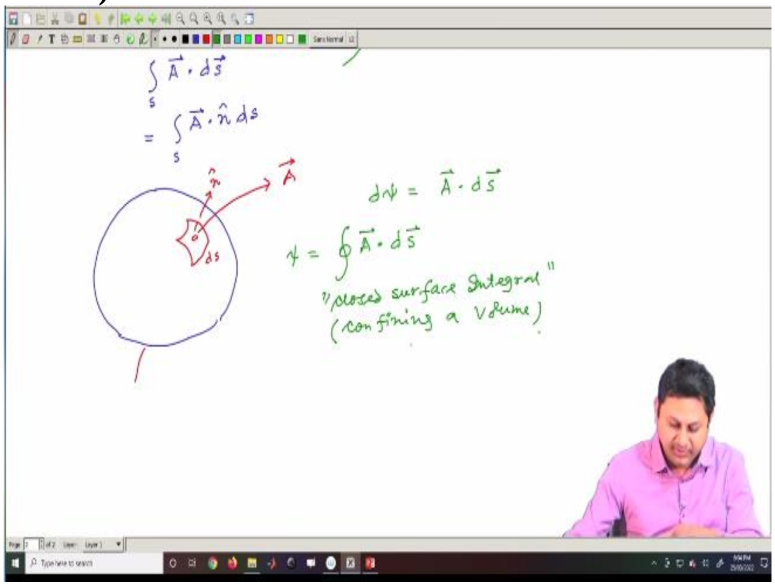


Now, if the surface is closed for example, we are having a closed surface like this like a ball and if I take a section out of this ball like this where you know the field is moving like this and this is ds and this direction is the n unit vector of this surface. Now, for this case the surface element the integration of this thing is say this integration say d$\psi$ is A dot ds. Now, if I want to find out the entire integration for the closed surface, then $\psi$ will be simply closed surface that is why I put a close sign here A dot ds.

So, I will integrate the entire surface, which is closing some volume. So, this is called the closed surface integral. Closed means it is confining a volume, this confining a volume, so, the entire integration should be the over this surface and so, now, let us consider a problem to understand how to calculate this close surface integral. So, maybe I define a problem and then it will be easier.

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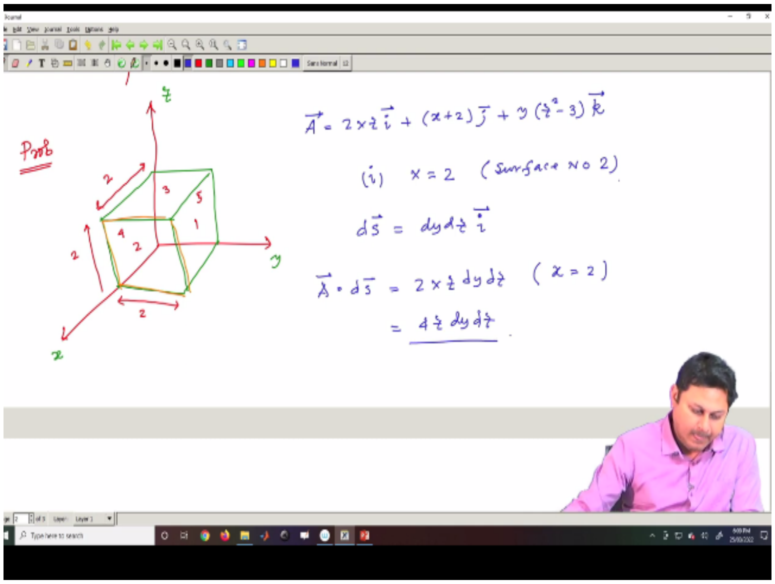

So, the problem is like the previous one the problem is suppose I am giving a surface, a surface is given like this, a block, maybe you can use these colors, this is a surface, this is my x direction, this is my y direction, this is mind z direction and a vector field is given like the previous case the vector field A suppose is given and the vector field is 2xz, an arbitrary vector field I am taking like this plus say (x + 2) j + y ($z^2$ – 3) k. And now, the problem is you need to find out the surface integral for this given vector field for 5 different surfaces.

So, the 5 surface except the bottom this is a problem given I think it is given in Griffiths book. So, I am doing that. So, I can have 5 surfaces, one is here and another is here, another is here, another is here and another is that side these are the 5 surfaces and you need to integrate. So, there are 6 surfaces but you need to integrate the entire surface integral for these 5 surfaces. So, I am not going to do the calculation of all the 5 surfaces, it will be lengthy calculation, but I can show you the technique to find the surface integral for a given surface and I think that is enough.

And also, the information that is given is this length is from here to here, this is a block and this length is 2. From here to here, this length is 2. So, this is a block of length 2. So quickly, I will do the surface integral among 5, I will do the surface integral for the plane where x = 2 so, which plane I am talking about x = 2 that means, I am talking about this surface 2 what as per the figure so the surface number 2. So, for this, this is the surface, where x = 2, I am talking about this one.

So, let me show you clearly so, I am talking about the surface this one, this is the surface and I want to integrate this vector field over this surface. Now, what is ds? The surface element first I need to calculate. ds the surface element is since it is x = 2, the surface element should be dy dz and the unit vector is along x so, unit vector should be i. Now, this is a vector quantity. Now, A dot ds If I calculate, it should be simply 2xz and dy dz. Now, x = 2 that we know. So, eventually this value is 4z dy dz. This is the value we were having. Now, you are in a position to integrate so, now I need to integrate it over this surface.

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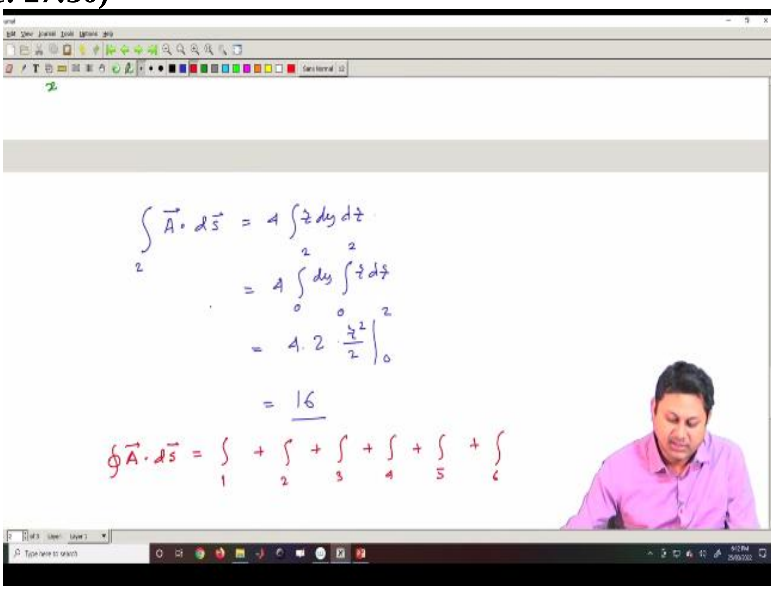

So, for surface 2 that is why I like it here 2 A dot ds will be simply 4 what we get 4 integral of z dy dz. So, I can write 4 integral of dy, dy is changing from 0 to 2 and it is z dz. z is also changing from 0 to 2, because over the surface x and y, y and z both are changing from 0 to 2. And if you execute, you will find the value as 4 into this value seems to be 2 and this value is $\frac{z^2}{2}$, 0 to 2. So, eventually it seems you are getting 16. So, this is the value of one surface.

Now, you are asked to find out the value of the other surface also 5 surface and also if you include the lower one, this surface and integrate everything, so you will be going to get the closed surface integral. So, suppose you are having this surface as well and you calculate the entire so, how you calculate the close surface integral? Let me give you A dot ds is a vector field and you are asked to calculate the closed surface integral the surface is given this is simply a block.

So, one by one you should calculate the surface, there will be 6 surfaces so you calculate 1, then 2, then 3, then 4, 5 and 6 then all the results you sum and whatever the value you get that should be the closed surface integral for this vector field where the surface is defined in this way. So, this is the way you can calculate the closed surface. And also, the other surfaces I am just showing one example of one surface, which is surface number 2, but you can do the surface number 1, in surface number 1 you will have y = 2 and then you go on x and y is changing and this is exactly the similar process you should follow.

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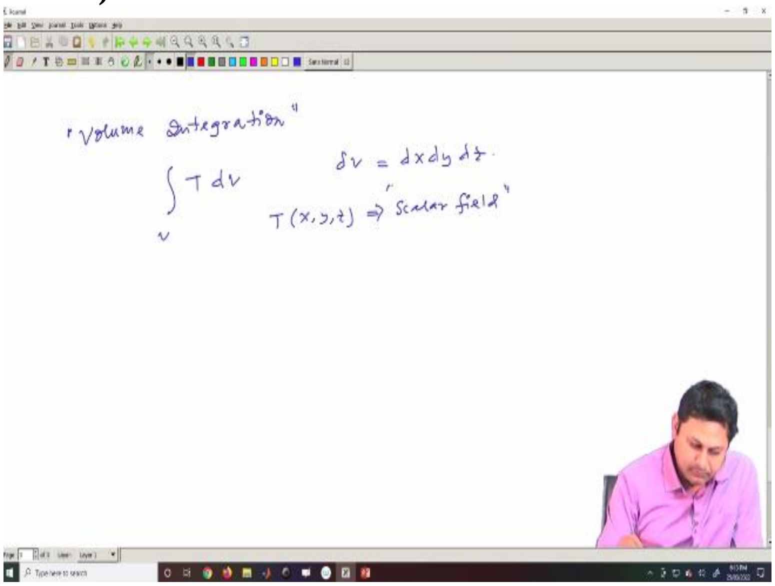


Now, finally we have the volume integration, so, volume integration what you do, I can have a scalar field that can be integrated over a given volume. In Cartesian coordinate, my dv is the volume element and we know that this is simply dx dy dz and T is a function of x, y, z and here in this case this is a scalar field I can show 1 example.

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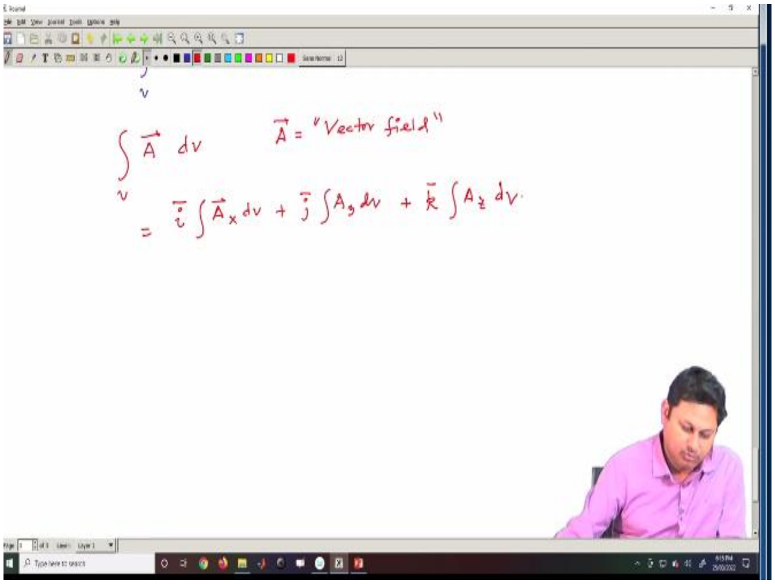


For example, I can so the volume and $\rho$ (x, y, z). $\rho$ is a scalar field, this is the say density of a system, which is changing with x, y, z and you are going to find out what is the total value over a given volume and whatever you are getting here is nothing but the mass. So, this is a very simple example, where you are doing the volume integral and this is your scalar field, which is density here, but in electrodynamics normally we have the vector field. So, this volume integral can also be treated as a vector field.

So, A dv because volume is a scalar quantity, but here important thing is A is a vector field and we can integrate this and the integration if I do I need to you know because A is a vector quantity, so, the integration is essentially $A_x$ then dv + $A_y$ dv + $A_z$ dv you can integrate it in this way. So, quickly we calculate one problem.

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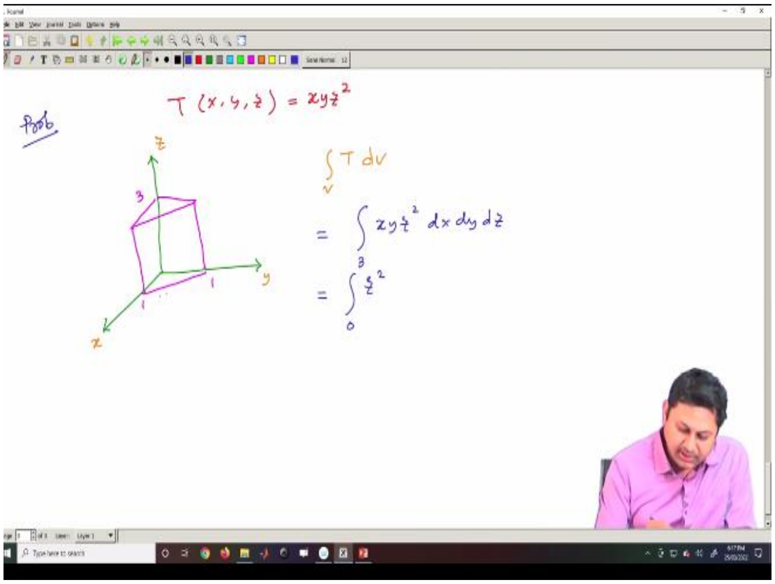


So, the problem, so, we need to calculate the volume integral for a given scalar field. So, scalar field is given like x, y, z is equal to say $xyz^2$, this is also given in Griffiths book. So, now, I need to calculate this over a given volume that is interesting, there is a trick here, so be careful. So, this volume element is like this prism kind of thing. So, the length here is 3, from here to here it is 1 and 1. This is x, this is y and this is z. So, over this given volume I need to calculate this.

So, I need to calculate T dv here for this given volume. So, let us start how to calculate? So, it is $xyz^2$ and the volume element I write this is dx dy dz. Then I just write in this way. So, z is changing 0 to 3. So, I can extract these $z^2$ and also, I have 2 integration here and but x and y are related. You can see that if y changes, then x also changes. So, I need to first find out what is the dependency of x and y.

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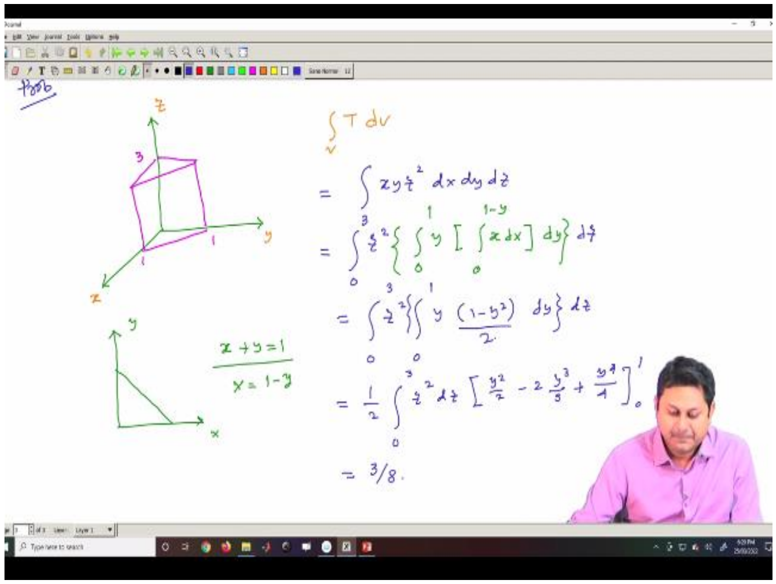


So, if I draw here once again this stuff then you can see there is a relationship going on and that relationship is this is y and this is x. And if I consider this, so one by one cutting is going on. So, that means x + y = 1, this is the relationship we are having here. So, that relationship we are going to use and we use in this way. So, integration 0 to 1 for y, y is there and x in place of x, I just integrate it like 0 to (1 – y) because x from here I can have (1 – y) is changing like that.

So, that change I include here it should be x dx, I first integral make integration over this and then dy bracket close and then over I will have dz. So, this is the way you should integrate let us do the integration then. So, 0 to 3 $z^2$, 0 to 1 then y and this integration becomes (1 – $y^2$) with the $\frac{1}{2}$ term and I have dy that I execute first and then I have dz. So, these things if I do, I will have let us put this $\frac{1}{2}$ outside integration 0 to 3 z square and that term because z and y are independent so, I can have like.

After integrating this seems to be, I can directly integrate it I do not need to put the integration sign because it is a simple function. So, it seems it will be simply the first integration will be $\frac{y^2}{2}$ and then $-\frac{2y^3}{3} + \frac{y^4}{4}$ and 0 to 1. So, if we execute all these things, you will eventually find so, this is $\frac{1}{2}$, this is $\frac{2}{3}$, this is $\frac{1}{4}$ and then whatever $z^3$ by 3. So, eventually, we will get the value $\frac{3}{8}$. So, that should be the value of the volume integral of this scalar field.

So, we have a scalar field, we are integrating these over this structure, closed volume, volume means it is close, there is no point to say close volume over this volume element and we will get the result like this. So, this is the way you should calculate the volume, but you have to be careful enough that what is the relationship between for example, here x and y is changing it is these things these tricky you need to consider when you calculate that.

So, I insist you to practice with different books there are several books available where you can calculate, where you can find this surface integral, line integral and volume integral practice that with this trick and then you will find this easy to calculate it is not that if the volume element is given in a regular form, it is much easy to calculate. With this note I like to conclude thank you for your attention see you in the next class.