Foundations of Classical Electrodynamics
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Lecture-67
Lorentz Gauge, Maxwell's Wave Equation

Hello student to the course for foundation of the classical electro dynamics. Today we have module 4 and lecture 67. And in this lecture, we will discuss about the Lorentz gauge and then we will be going to discuss Maxwell's wave equation.

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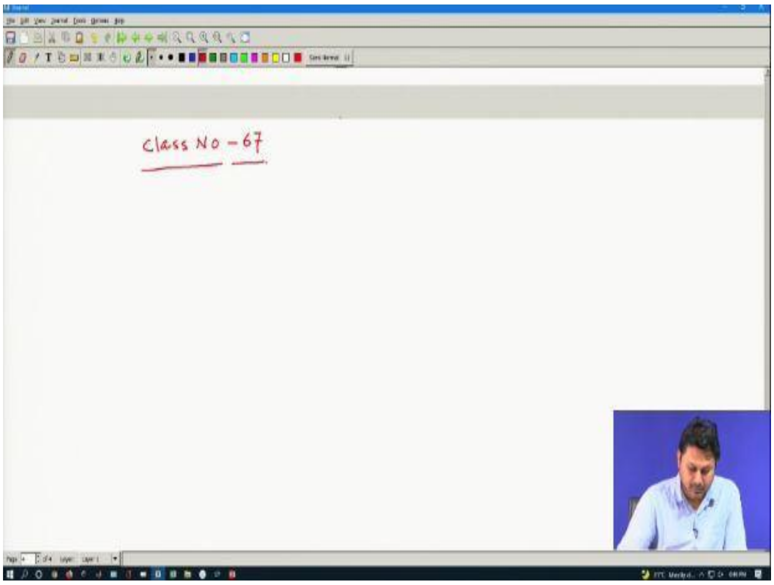

So, we have class number 67**.** So, for whatever we have let me quickly remind that there are 4 Maxwell's equations and we wrote this Maxwell's equation as homogeneous and inhomogeneous form.

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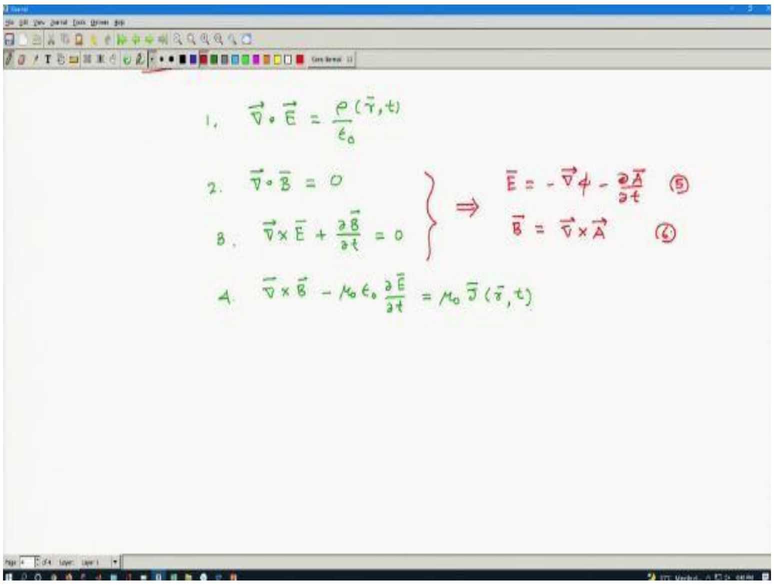

So, that was the first equation having the source term. So, this has to be a non-homogeneous equation. Second equation, which is $\vec{\nabla} \cdot \vec{B}$ that is 0, third equation, $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}$ that was 0. And fourth equation is $\vec{\nabla} \times \vec{B}$ - $\mu_0\, \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ that is $\mu_0\, \vec{J}(\vec{r}, t)$. So, among these 4 equations, these 2 were homogeneous. And from this homogeneous equation we find the relationship like $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$.

So, I can extract this information in terms of a scalar and vector potential, just exploiting these two homogeneous equation and $\vec{B}$ was $\vec{\nabla} \times \vec{A}$. And after that we put this information in 1 and 4 and putting this let me write down this equation, let us write it as 5 and 6.

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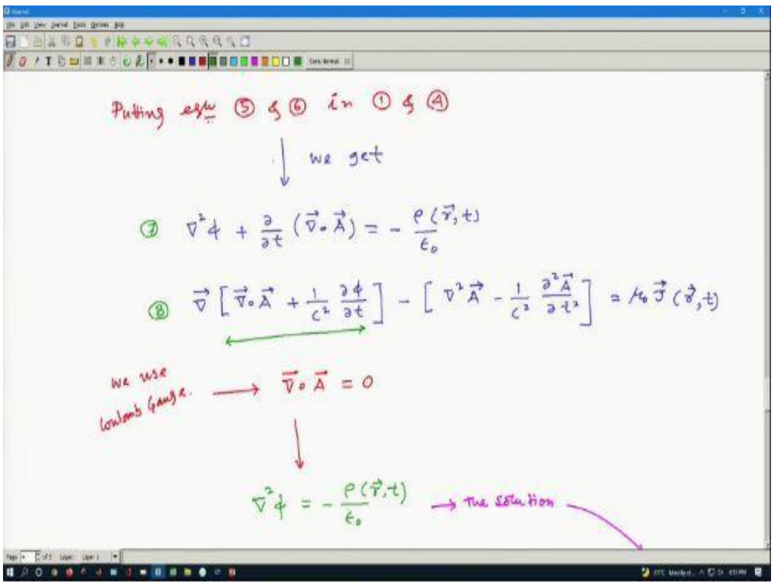

And putting equation 5 and 6 in equation 1 and 4, which are non-homogeneous in nature, we get these 2 equations the following 2. We get $\nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla} \bullet \vec{A}) = -\frac{\rho(\vec{r},t)}{\epsilon_0}$ and another equation is $\vec{\nabla}\,[\vec{\nabla} \bullet \vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}] - [\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}] = \mu_0\,\vec{J}(\vec{r}, t)$.

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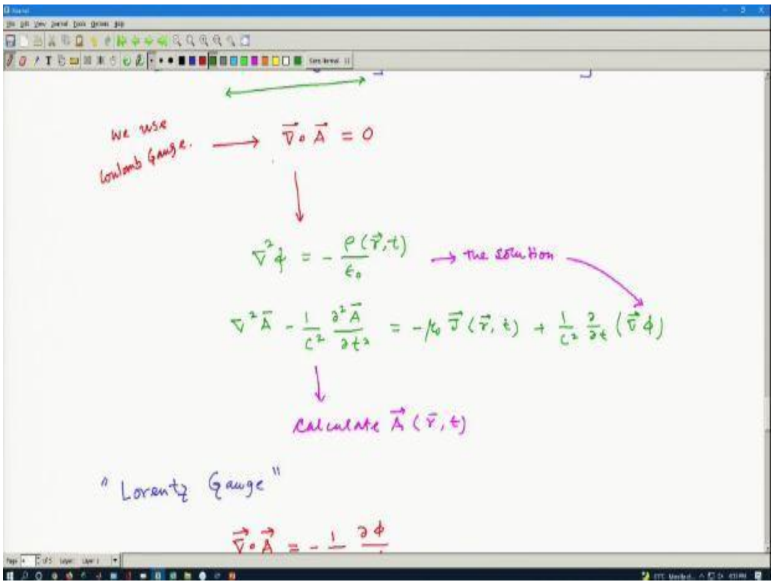

So, after that, what we have? We had the Coulomb gauge. And the condition of the Coulomb gauge is $\vec{\nabla} \bullet \vec{A}$ has to be 0 that is the constant we are using, after using the Coulomb gauge what we get? We get the first equation as this quantity is 0. We get the first equation as this. So, that means, ϕ is now satisfying a simpler Poisson equation. And the next equation, which is not that simple, so, we had $\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$ that quantity is equal to - $\mu_0\,\vec{J}$.

This source has to be given because, this is a known quantity. Without that we cannot calculate the magnetic or electric field. And then we had a term like + $\frac{1}{c^2}$ and then $\frac{\partial}{\partial t}$ and then $\vec{\nabla}\phi$. So, whatever the ϕ we have here, the solution after doing the Poisson equation I plug it here. So, that the right-hand side is known and you can calculate this also. So, from then we can calculate $\vec{A}$, which should be a function of $\vec{r}$ and t.

So, this equation gives me the solution of ϕ, that ϕ I put here. And then I should have a wave equation with a source term and that wave equation I solve with the source term to get my $\vec{A}$. So, that is the recipe when we use the Coulomb gauge. Now in today's class we are going to discuss the Lorentz gauge. So, what is the advantage of the Lorentz gauge? Let us check.

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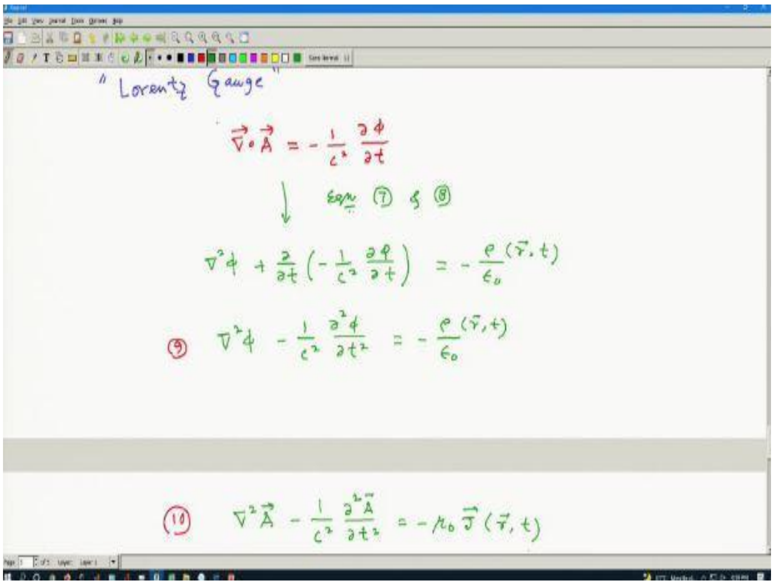

Lorentz gauge: So, in Lorentz gauge, what we make is this instead of taking $\vec{\nabla}\cdot\vec{A} = 0$. We take $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}$, we just take it. Why we take this? Because, if you look carefully we have a disturbing term here. So, if this term is not there I am making a line here. So, I have a term here if this term is somehow 0, then my second equation will be much, much simpler.

So, in Lorentz gauge what we do, we just take $\vec{A}$ in such a way, that $\vec{A}$ and ϕ in such a way because, we have a liberty to $\vec{A}$ and ϕ that this condition should satisfy. If that is the case, then what we are getting out of that, then these 2 equations whatever the equation I wrote here. So, maybe I can put a name here. So, maybe this is 5, 6 is there. So, 7, 8 these are my fundamental equations.

And now I am putting the gauge to simplify it. So, with this condition equation 7 and 8 become much simpler and that is this. So, we have say equation 9 now. After putting this condition to equation 7 what we get? We get this $\vec{\nabla}\phi$ then we have $+\frac{\partial}{\partial t}$, we have $\vec{\nabla}\cdot\vec{A}$ in place of $\vec{\nabla}\cdot\vec{A}$ I should write at $-\frac{1}{c^2}\frac{\partial\phi}{\partial t}$ that I wrote simply.

Rest of the term is simply $-\frac{\rho(\vec{r},t)}{\epsilon_0}$. Now you can see that this is having a very well-known form. And this well-known form is $\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$. This is nothing but the wave equation with a source term, $-\frac{\rho(\vec{r},t)}{\epsilon_0}$. Let us put equation 9, this one. So, this is the equation we are getting after exploiting the Coulomb gauge. And now I will do the same thing for other. And for other equation it is straight forward because, this term will completely zero.

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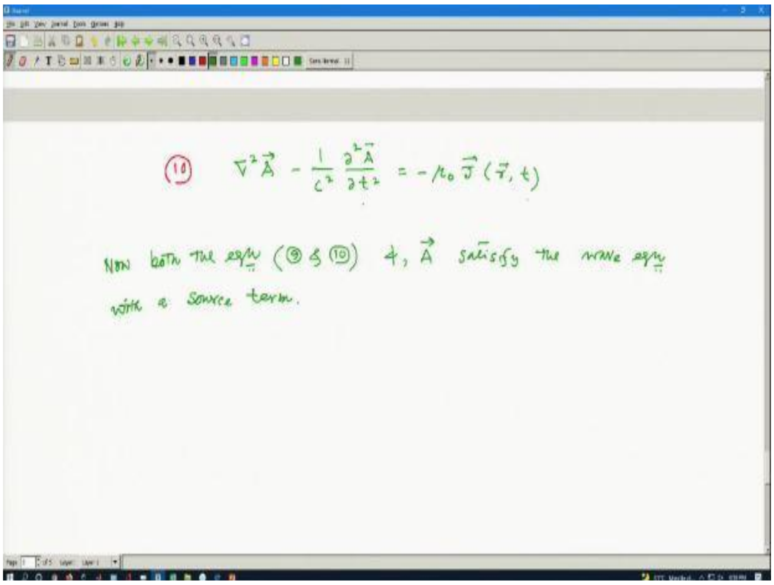

And we have equation 10, which is for the vector potential $\vec{A}$. And the equation is again something like $\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$ = - μ_0 $\vec{J}(\vec{r}, t)$. So, both the equations are satisfying. So, now both the equations, which equation, equation 9 and equation 10 that is equation for both the equation, equation for ϕ and $\vec{A}$ in other word because, in equation 9 we have only ϕ, in equation 10 we only have $\vec{A}$, satisfy the wave equation with a source term.

And we have the method to solve the wave equation with a source term. Now like the before, so, this Lorentz gauge condition I need to fix it because, I consider that $(\vec{\nabla} \cdot \vec{A}) - \frac{1}{c^2}\frac{\partial \phi}{\partial t}$ but for given $\vec{A}$ and ϕ this condition may not be true. This condition may not satisfy. So, then what we do? We need to fix that like we did in the Coulomb gauge.

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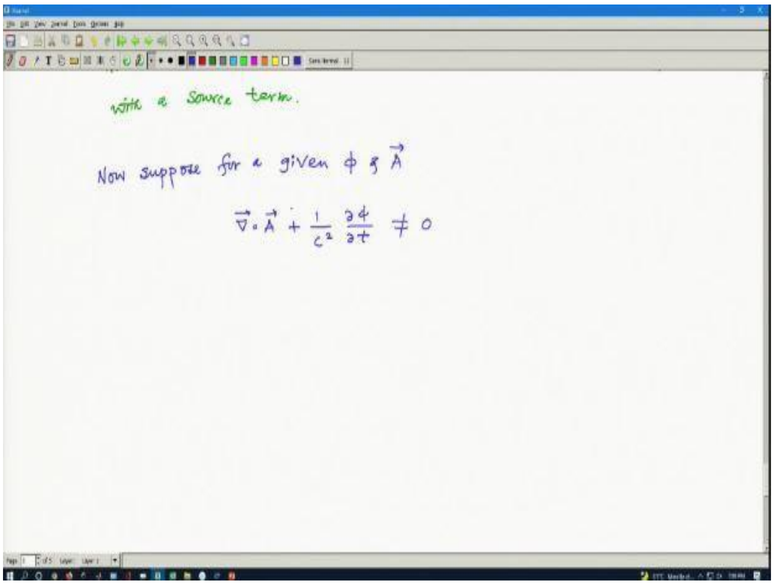

So, now suppose for a given ϕ and $\vec{A}$ because, this is given and for that there is a possibility that this is not satisfied. That is $\vec{\nabla} \cdot \vec{A} - \frac{1}{c^2}\frac{\partial \phi}{\partial t}$. Maybe I am making a mistake here no no it is fine there is a negative sign. So, this quantity is may not be equal to 0 that means, given ϕ and $\vec{A}$ does not satisfy the Coulomb gauge. But we can make them satisfy with the proper choice of $\vec{A}$ and ϕ the liberty that we have. So, that we are going to exploit.

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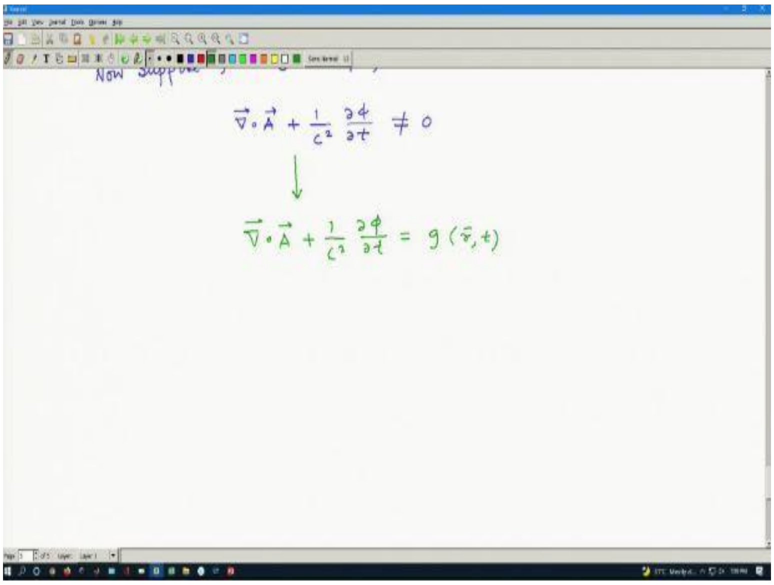

So, if that is the case, if it is not 0 it should have some value. So, let me write this value. So, suppose this is not equal to 0 and that is why I can write that, this is not equal to 0. So, it should be equal to something and this something is say g, which is a function of $\vec{r}$ and t. It has to be a scalar quantity because, left-hand side is a scalar divergence and this is scalar quantity. So, right-hand side should be a scalar field.

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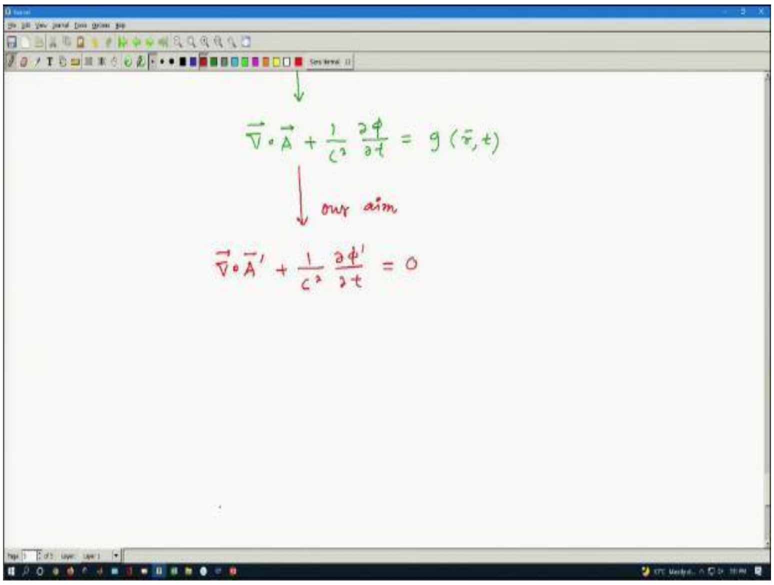

Now our aim is to find $\vec{A}'$ and ϕ', $\vec{A}'$ and ϕ' such that this Lorentz gauge condition satisfied. So, this is nothing but the condition of the Lorentz gauge. So, our aim is to find out a set of $\vec{A}'$ and ϕ' such that this condition satisfied. So, how to do that? Let us check. So, that is our aim.

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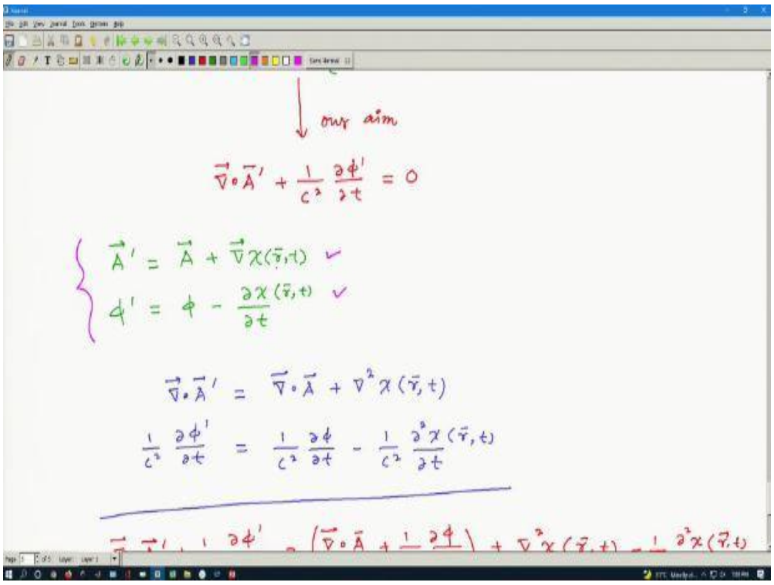

Now we know that $\vec{A}'$ can always be written like given $\vec{A}$ plus the gauge function this χ is our gauge function. And ϕ' I can write as ϕ - $\frac{\partial \chi}{\partial t}$, χ should be a function of $\vec{r}$ and t. This is $\vec{r}$ and t. We just need to fix this function we need to know what function this is. If I put a suitable function then I can have my $\vec{A}'$ and ϕ' such that it follows the Lorentz gauge.

It satisfy the Lorentz gauge and my equation becomes simpler. Now next is if I make a $\vec{\nabla} \cdot \vec{A}'$ right-hand side it should be $\vec{\nabla} \cdot \vec{A}$ plus Laplacian of this function. And left-hand side what we do I multiplied $\frac{1}{c^2}$ and make a time $\vec{\nabla}\phi'$ and right-hand side it is simply $\frac{1}{c^2}\frac{\partial\phi}{\partial t}$ - $\frac{1}{c^2}$ $d^2\chi$, which is a function of $\vec{r}$ and t. This now I simply add both the side because left-hand side I want something.

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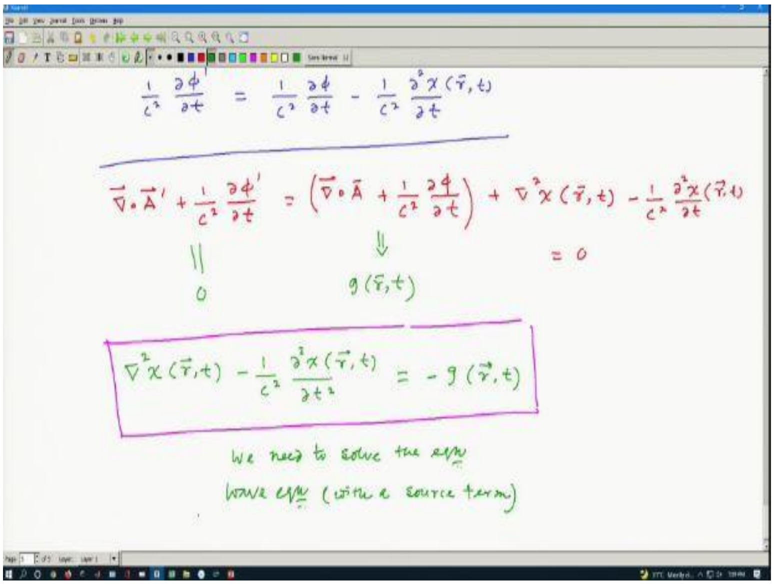

So, if I add then the left-hand side becomes $\vec{\nabla} \cdot \vec{A}' + \frac{1}{c^2}\frac{d}{dt}$ that and this term is $\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}$. This quantity plus I have $\vec{r}$, t $-\frac{1}{c^2}\frac{\partial^2\chi(\vec{r},t)}{\partial t}$. Now we make this quantity equal to 0. So, that means, so this has to be 0 because, this is our so we want this to be 0. So, this quantity has to be equal to 0. And what value we have here, this is the given value this is not equal to 0 and this value is suppose I mention here, this is equal to something called g.

So, I now have this value as g. So, if that is the case then I can have the equation and that equation simply tells me that gradient Laplacian of this scalar field $\chi - \frac{1}{c^2}\frac{\partial^2\chi(\vec{r},t)}{\partial t}$ = - $g(\vec{r}, t)$. So, that means if I choose my χ in such a way that satisfies this equation, which is again a wave equation having a source term. Then eventually I can form my $\vec{A}'$ and ϕ' exploiting this expression such that it satisfies the Coulomb gauge condition.

So, step by step I derive everything. So, this basically tells us, that if I put certain constant certain gauge like Coulomb gauge or Lorentz gauge. Then my equation becomes simpler. So, at the end of the day what equation we are getting here, we are getting equation 9 and 10 to solve. And once you solve because ρ and $\vec{J}$ is given. If ρ and $\vec{J}$ is given we can solve equation 9 and 10 and we can get ϕ and $\vec{A}$. As soon as we get ϕ and $\vec{A}$, we know what is the value of electric field and magnetic field because, they have a relation.

Now how I get equation 9 and 10 exploiting the Coulomb gauge. And how I exploit the Coulomb gauge, if the condition $\vec{\nabla} \bullet \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}$ is not equal to 0. Suppose it is a g, then I exploit the relation $\vec{A}'$ and ϕ'. Considering the fact that $\vec{A}'$, ϕ' will produce the same $\vec{E}$ and $\vec{B}$. So, since $\vec{A}'$, ϕ' are producing same electric field and same magnetic field instead of using $\vec{A}$ and ϕ.

Now I will be going to use my new $\vec{A}'$ and ϕ'. This new $\vec{A}'$ and ϕ' can be constructed from the given $\vec{A}$ and ϕ and also an arbitrary function χ, which I can fix, which I can put it is my choice. Now I choose my χ in such a way that it satisfy the Lorentz gauge condition to make the equation very simple. And we find that the χ I choose in such a way that it satisfy this equation.

This is again a wave equation with source term. So, we need to solve this equation to find my χ. So, this is my hand, so, what I will fix. So, basically we need to solve this is a wave equation with a source term. And this source term is again given because, when $\vec{A}$ and ϕ is given, then I can find that the $\vec{\nabla} \bullet \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}$ that basically g. So, I know that. This is not equal to 0 that is the initial condition.

So, that initial condition I am going to exploit here to find out my χ. So, this is roughly the discussion of the Coulomb gauge just primitive discussion. So, that you can have an idea what is the meaning of Coulomb gauge, what is the meaning of Lorentz gauge? So, this is Lorentz gauge, by the way. So, then you understand that how exploiting the Coulomb and Lorentz gauge.

You can simplify the equation in terms of the vector potential and scalar potential. And then you use this vector potential and scalar potential to find out your final equations. So, the Maxwell's 4 equation, which deal with the value of electric field and magnetic field. The information is gradually, we extracted this information from in the form of $\vec{A}$ and ϕ and eventually we solve this $\vec{A}$ and ϕ.

So, now after that we will do a very important thing and that is called the Maxwell's wave equation. So, now we had an idea about the wave equation because, wave equation we had a special separate lecture. So, we will be going to find out how the Maxwell's 4 equation leads to the wave equation. So, let me write down here first.

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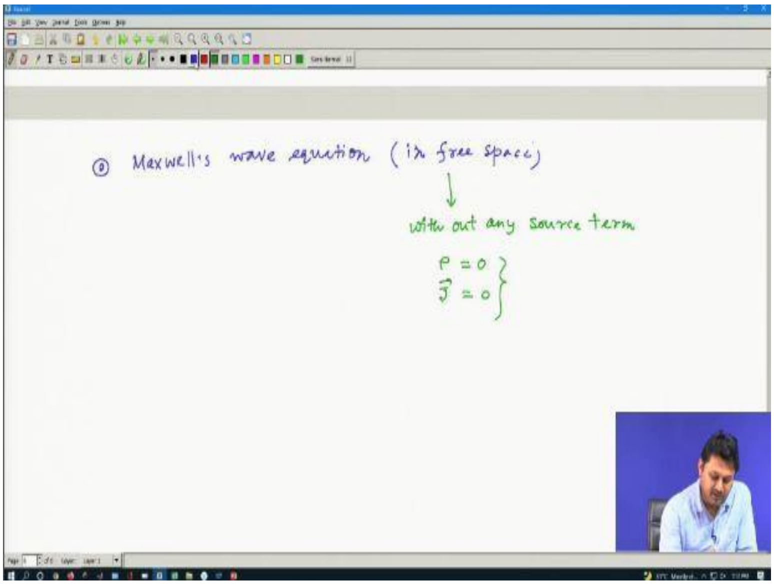

So, the next topic is Maxwell's wave equation, say in free space let us start with in free space. So, in free space means eventually, without any source term that means, $\rho = 0$ and $\vec{J}$ is always also equal to 0. So, if I write down the Maxwell's equation under this condition the equation become very simple.

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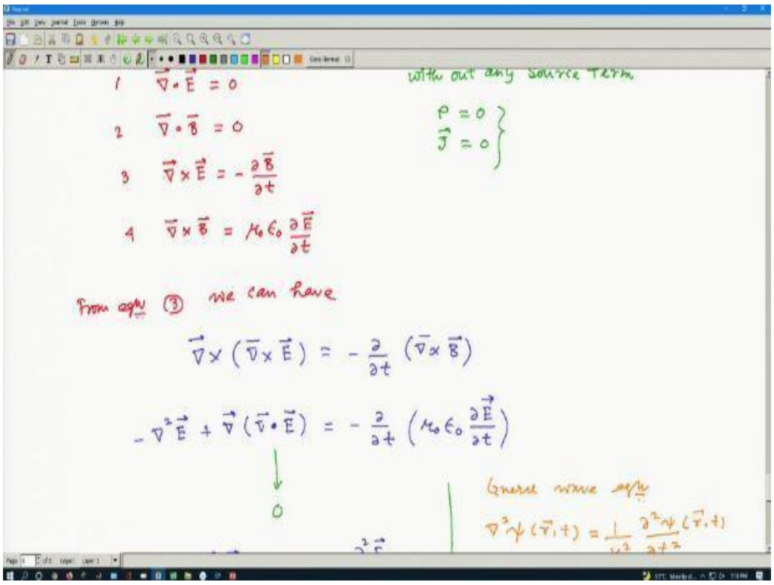

And this will be the form, $\vec{\nabla} \cdot \vec{E}$ should be equal to 0, $\vec{\nabla} \cdot \vec{B}$ will be equal to 0, $\vec{\nabla} \times \vec{E}$ will be $-\frac{\partial \vec{B}}{\partial t}$, this is part of the simplest form of the Maxwell's equation without any source term and $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. So, in principle they are all the 4 equations are homogeneous equations. Now we will be going to exploit this equation. So, this is equation 1, this is equation 2, this is equation 3 and 4.

So, from equation 3, we can have so, what we do from equation 3 if we take the curl of both side. So, far we are dealing with the divergence of both side and check that the left-hand side and right-hand side is consistent or not but, now what we do that we take the curl on both the side. So, if I do that $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$. Now $\vec{\nabla} \times (\vec{\nabla} \times \vec{E})$ again this is a famous identity that is minus $\nabla^2\vec{E} + \vec{\nabla}(\vec{\nabla} \cdot \vec{E})$.

This identity we use several times. So, you should remember this identity. Then $-\frac{\partial}{\partial t}$ and in the right-hand side, we have $\vec{\nabla} \times \vec{B}$ and $\vec{\nabla} \times \vec{B}$ in exploiting the equation 4 I can simply write it is $\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. Now in free space, this quantity is 0. This quantity is 0 because, there is no source term in the free space and $\vec{\nabla} \cdot \vec{E}$ should be 0.

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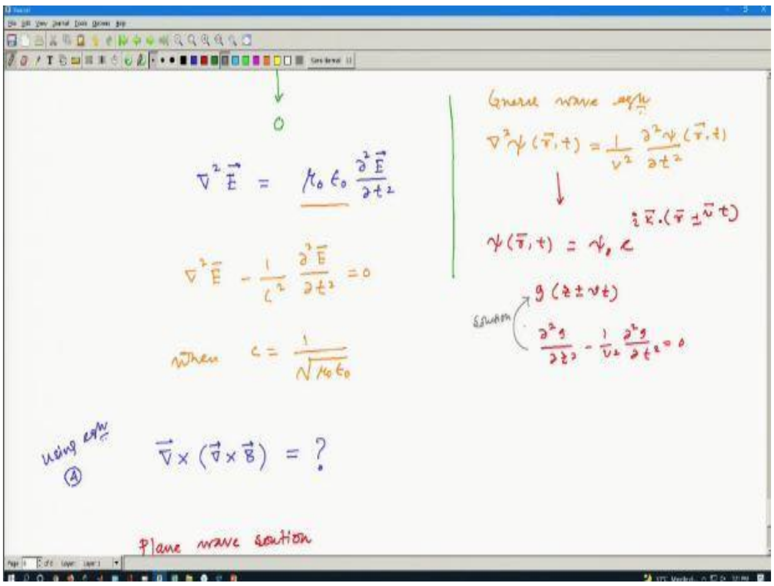

If we make this term equal to 0, the rest of the term is like $-\vec{\nabla} \cdot \vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$. So, I can make it plus both the side. So, this equation exactly is like a wave equation. So, let me write down the wave equation this side. So, the general wave equation was $\nabla^2\Psi$ as a function of $\vec{r}$ and t = 1 divided by velocity square $\frac{1}{v^2}$ and then $\frac{\partial^2\Psi(\vec{r},t)}{\partial t^2}$.

That was the equation wave equation we discussed earlier where, v was the velocity. So, now if I look carefully these 2 equations then we can see that $\vec{E}$ is also following. So, let us remove this from both the side. So, $\vec{E}$ is satisfying the wave equation. Under the condition, that this can be written in terms of the velocity, so how I write this in terms of velocity. So, I can simply write $\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$, when c is nothing but $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$.

So, during the calculation when Maxwell's did this calculation he find that μ_0 ϵ_0 is sitting in the place of $\frac{1}{c^2}$. And he calculated μ_0 ϵ_0 and he find that this value is very close to the velocity of light c and then realized that light is nothing but an electromagnetic wave. That was the revolutionary findings, so here we are doing the same thing. Now if you do the same thing for $\vec{B}$ like, this is a homework. So, please try to do the same thing for $\vec{B}$.

I exploit equation 3 by taking a $\vec{\nabla} \times \vec{E}$ and you can do the same thing by taking the curl of. So, try to do this and you will find that you will be going to get the same result for $\vec{B}$ as well with the same velocity. So, if you do what we get. So, using equation 4 and you will get the similar expression like you get for $\vec{E}$. That is homework for you, you just check it. Now let us consider that what should be the solution that we are going to consider quickly.

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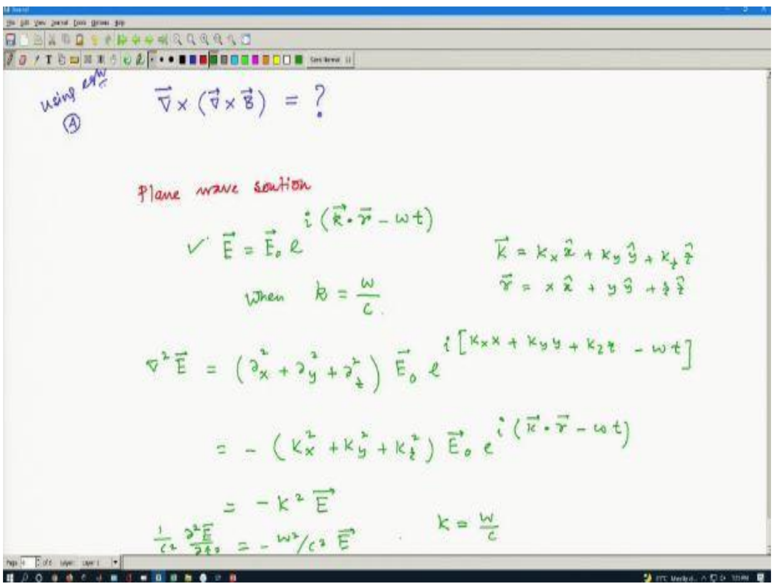

So, it is called the plane wave solution very special kind of solution because, we know that the wave equation has a specific kind of solution. So, let me write down here also. So, when we have this equation and we mentioned that the solution should have a specific form. And the specific form if I write it should be like Ψ_0 and say e_i. So, something like $\vec{k}$ and $\vec{k} \cdot \vec{r}$ - like v t it is like az - vt kind of form.

So, one should expect this kind of form for plus, minus. This is the general form because the general solution if you remember it should be this is a plane wave solution. But the general solution in 1 dimension if you remember it was z plus minus vt that was the form of the solution of the wave equation. When g was the function. So, in 1 dimension that was the equation and we find that this was the equation and we find that if this equation then it is the solution.

So, here we will be going to get a plane wave solution. And this plane wave solution of the Maxwell's wave equation is of the form like $\vec{E} = \vec{E}_0\ e^{i(\vec{k} \cdot \vec{r} - \omega t)}$ that is the solution. When $k = \frac{\omega}{c}$. So, under this condition if you plug this solution here, plug this expression $\vec{E}$ to this equation here, you will be going to see that this is basically the solution of that.

So, we can check it quickly. So, what is $\nabla^2\vec{E}$ because, I need to first calculate this quantity. So, this is if I do that in cartesian coordinate, so, it is $d^2x + d^2y + d^2z$ and it should be operated on $\vec{E}$. And $\vec{E}$ is $\vec{E}_0$ I should put a vector here, vector then exponential i and here we have $k_x\ x + k_y\ y + k_z\ z$ I just expand the $\vec{k}$. So, my $\vec{k}$ here is $k_x\ \hat{x} + k_y\ \hat{y} + k_z\ \hat{z}$.

And my $\vec{r}$ is $\hat{x}$, $\hat{y}$ and $\hat{z}$, so, then -ωt bracket close. And if I operate these things then what we get is this one. So, this will be going to operate twice and every time it operates, so, ik will come out. In this case, when we operate over operated by ∂_y. So, ik_y will come out and so on. So, eventually we have this, we have $k_x^2 + k_y^2 + k_z^2$ and rest of the term will remain same.

So, I just simply write my is $\vec{E}_0\ e^{i(\vec{k} \cdot \vec{r} - \omega t)}$. So, this is simply $- k^2\ \vec{E}$. This is the left-hand side, what about the right-hand side? The right-hand side I can write here, the right-hand side is $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$. So, if I write $\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$, so it should be simply $-\frac{\omega^2}{c^2}$ and this $\vec{E}$. And we already mentioned that k should be equal to $\frac{\omega}{c}$ for which the solution is there.

So, we can see that the left-hand side is equal to right-hand side. So, that means this is a solution for Maxwell’s wave equation, I am going to use this solution, this is a special property. Why we called a plane wave we may discuss in the next class. So, I do not have much time to today to discuss further. So, I like to stop here. So, in the next class maybe we will be going to find out more about the solution and the relative direction between the $\vec{E}$ and $\vec{B}$ because $\vec{B}$ will also follow the same equation.

So, the solution of the $\vec{B}$ will also be a plane wave solution. And then we will discuss that how to find out the relative direction of $\vec{E}$, $\vec{B}$ the value etcetera by just exploiting the Maxwell’s other equation, which is the Gauss's law and others law. So, with that note I like to conclude. Thank you for your attention and see you in the next class.