Foundation of Classical Electronics
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Lecture-66
Maxwell's Equation: A Complete Overview (Contd.)

Hello student for the foundation of classical electrodynamics course under module 4, today we have lecture 66. And we will be going to continue the discussion on the Maxwell's equation that we started in the last class.

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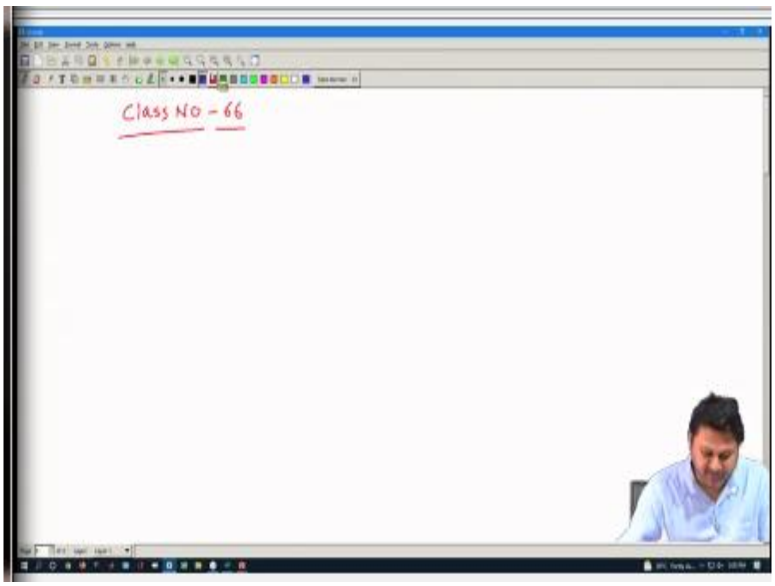

So, today we have class number 66, so what we have in the last class? Let me remind.

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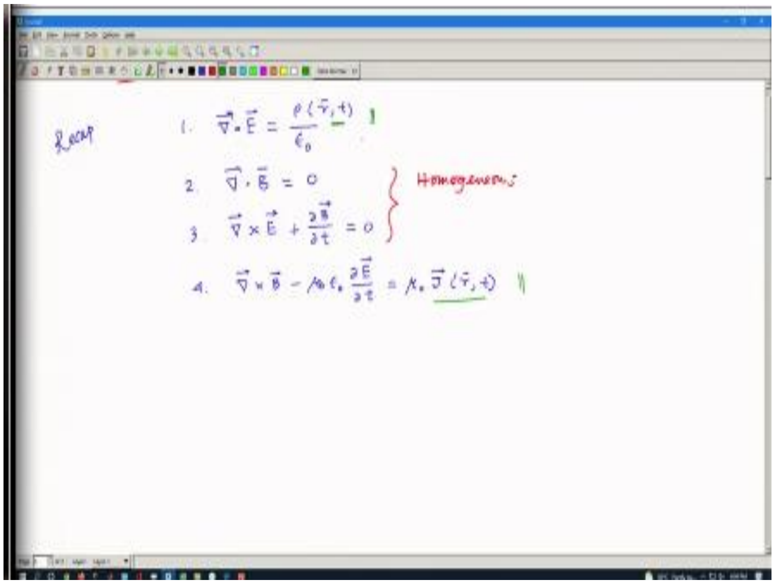

Let here make a small recap here that we wrote the 4 Maxwell's equation in this way, $\vec{\nabla} \cdot \vec{E} = \rho$, which should be a function of $\vec{r}$ and t divided by ϵ_0, this is equation 1. Equation 2 is $\vec{\nabla} \cdot \vec{B} = 0$, equation 3 was $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}$ that was 0. And finally equation 4 was $\vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$ as a function of $\vec{r}$ and t, that was 4 equation. But we had a set of equation, which is homogeneous; these 2 equations was homogeneous, equation 2 and 4 because right-hand side there was no source term.

So, this 2 equation was homogeneous and rest of the 2 equation here we had the source term, this one and this one we had the source term sitting here, one is ρ and another is $\vec{J}$, so these are 2 source terms that is there. So, we extract the information of equation 2 and 3 and we eventually have this equation.

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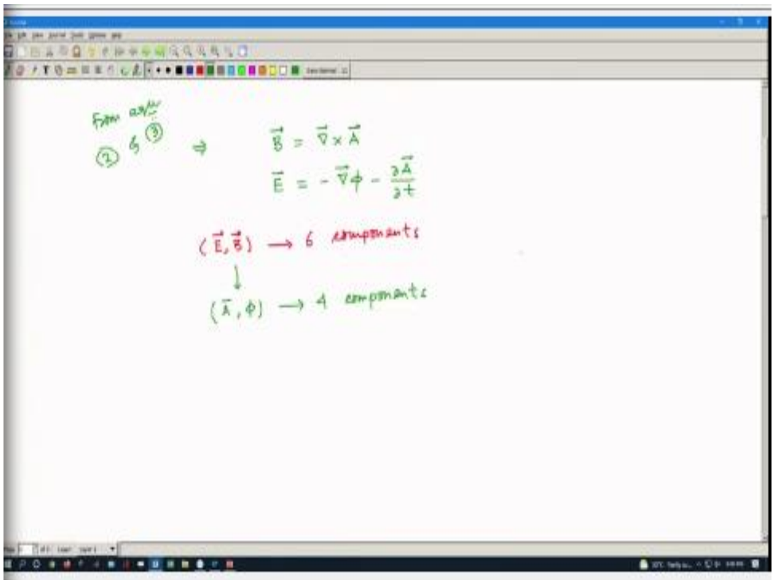

So, what we get from the 2 homogeneous equations 2 and 3? We had this, we had $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$, this we get. So, we extract the information of equation 2 and 3 and eventually get $\vec{B}$ and $\vec{E}$. Now you can see that $\vec{B}$ and $\vec{E}$ is having the 6 component and this 6 component can be extracted from the 4 component of $\vec{A}$ and ϕ, so I should make a note here.

The set $\vec{E}$ and $\vec{B}$, which eventually try to solve these 2 vectors they are having 6 components and this information I extracted in terms of $\vec{A}$ and ϕ. Now ϕ is a scalar quantity that makes the total number of component 4, so we now have 4 components. So, all the information that we had here we can have the $\vec{E}$ and $\vec{B}$ the 6 component from this information of the 4 components, we can reduce the number of components.

Now the next thing is how to solve $\vec{A}$ and ϕ, should we have equation for $\vec{A}$ and ϕ? And in order to do that, last day we mentioned that we will put this information $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$ into this non-homogeneous equation that is in equation 1 and 4, so next step is this.

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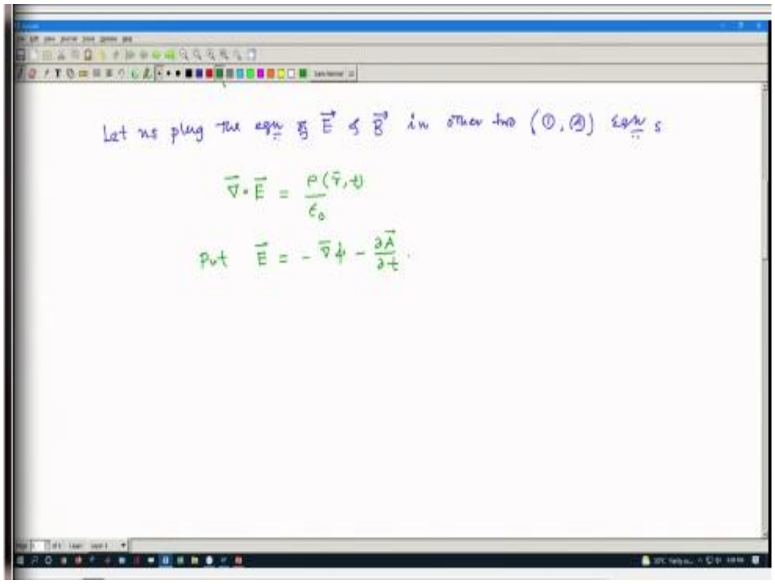

Now let us plug the equation of $\vec{E}$ and $\vec{B}$ these 2 equation we are having right now in terms of $\vec{A}$ and ϕ, $\vec{E}$ and $\vec{B}$ in other 2 which is equation 1 and equation 4, 2 equations, they are non-homogeneous equations. So, let me first write down this first non-homogeneous equation, so this is $\vec{\nabla} \cdot \vec{E}$ was equal to ρ function of $\vec{r}$, t, this function of space and time ϵ_0. Now in this equation I put $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$.

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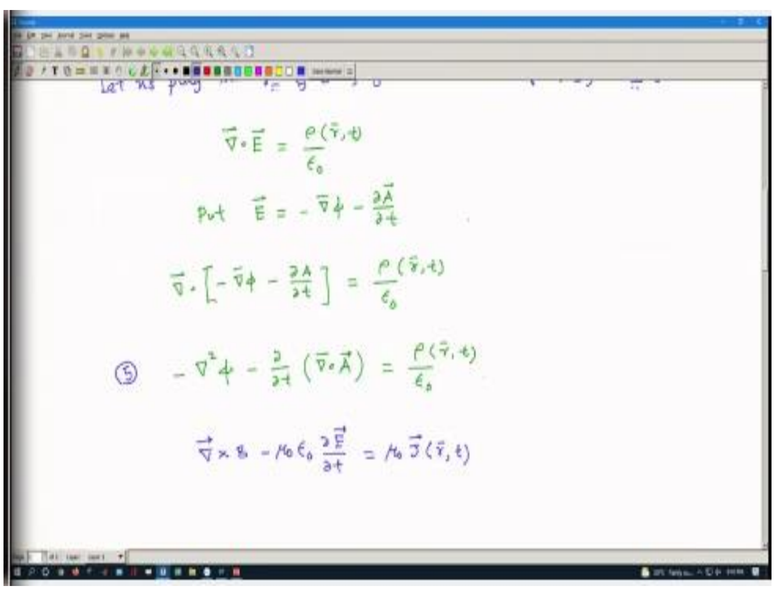

So, if I put there then I simply have $\vec{\nabla} \cdot (-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}) = \frac{\rho(\vec{r},t)}{\epsilon_0}$. So, this thing is simply $-\nabla^2\phi$ - $\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A})$, I just put this divergence inside the operator $\frac{\partial \vec{A}}{\partial t}$. And then in the right-hand side I should have $\frac{\rho}{\epsilon_0}$, ρ being a function of position and time, that is one equation we are having, that is all, okay.

So, this is say equation 5. What is the other equation that I like to exploit here? The other equation is this one. So, we had $\vec{\nabla} \times \vec{B}$ minus equation fourth Maxwell's equation $\frac{\partial \vec{E}}{\partial t}$ that is equal to $\mu_0 \vec{J}(\vec{r}, t)$. Now you know the $\vec{B}$ because $\vec{B}$ is $\vec{\nabla} \times \vec{A}$.

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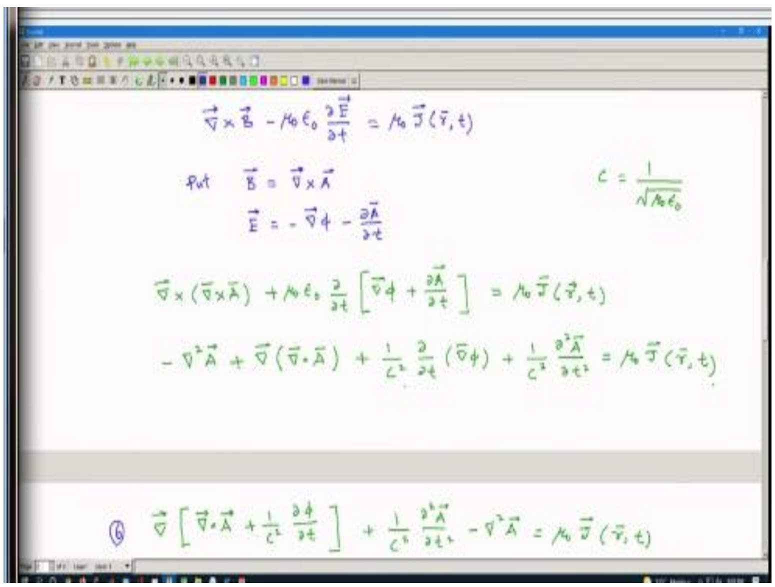

So, I put here again $\vec{B} = \vec{\nabla} \times \vec{A}$ and not only that I need to put $\vec{E}$ also here because $\vec{E}$ is there, $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$. These 2 I put here in this equation and let us see what we get. So, we get something like $\vec{\nabla} \times \vec{B}$. So, we should get like this $\vec{\nabla} \times \vec{A}$, I just replace $\vec{B}$ to $\vec{\nabla} \times \vec{A}$ - $\mu_0 \epsilon_0$ and then I have $\frac{\partial}{\partial t}$ of this quantity. So, here we had a minus I can put it as plus and then it should be $\vec{\nabla}\phi + \frac{\partial \vec{A}}{\partial t}$ and that is equal to the source term $\mu_0 \vec{J}(\vec{r}, t)$, we get this.

So, this term if I want to simplify then we know the famous identity it should be $\nabla^2\vec{A}$, then + $\vec{\nabla}(\vec{\nabla} \cdot \vec{A})$ and then we have, so $\frac{1}{\mu_0\epsilon_0}$, I now write like $\frac{1}{c^2}$. Because we will see later that c, which is the velocity of light can be represented in terms of μ_0 and ϵ_0 in this way. So, let us write in this way only, we will be going to discuss this part again in next class maybe.

And then we have $\frac{\partial}{\partial t}(\vec{\nabla}\phi)$ and then we have $+\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}(\vec{r}, t)$ that we get. So, we have an equation where many terms are there but anyway, so let us combine few terms here and let us do in this way. Let us write this equation like gradient of because we have 2 gradients, so gradient of let us make like $\vec{\nabla} \cdot \vec{A}$.

And then $+\frac{1}{c^2}$ then we have $\frac{\partial \phi}{\partial t}$, I just extract the gradient out of this, this is 1 equation, one part of this, so I write this part. And another part is like $\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$ and $-\nabla^2\vec{A}$ that is $\mu_0 \vec{J}(\vec{r}, t)$. So, these are the 2 equations, so I cannot do much after that, so we need to put certain condition here, so that we are going to discuss, so this is say equation 6. So, let me jot down what we get so far.

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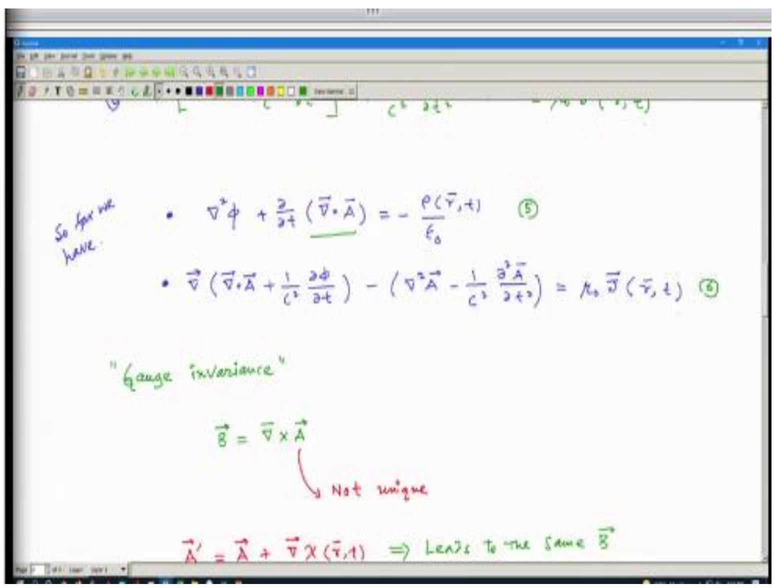

So far what we have? We have one equation, which is $\nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A})$ that is equal to how much? That is equal to minus of because I take the negative sign, this negative sign I take, so it should be $-\frac{\rho(\vec{r},t)}{\epsilon_0}$ that is one equation I get as equation 5. And equation 6 what I get is this one. If we write side by side then we will find something that $\vec{\nabla}(\vec{\nabla} \cdot \vec{A})$ and then $\frac{1}{c^2}$, so I have $\frac{\partial \phi}{\partial t}$ and then $-(\nabla^2\vec{A}$

$-\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$) that is equal to $\mu_0 \vec{J}(\vec{r}, t)$. So, this 2 equation we have, now after that we need to put something to make this equation little bit simpler and the way we do is called the gauge.

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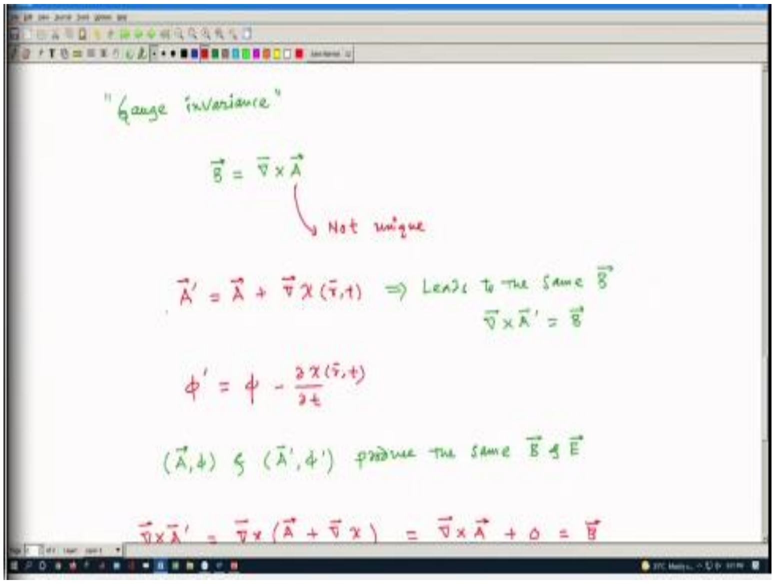

So, now here we will do something called Gauge invariance. The idea of the Gauge is not new because we have already introduced that in earlier class, so we will exploit this once again. What do you mean by Gauge invariance here? What do we mean? So, $\vec{B} = \vec{\nabla} \times \vec{A}$ and we find that this $\vec{A}$ is not unique, so it is not unique. So, we can choose as per our convenience. And **we** for example I can have $\vec{A}'$, which is the given $\vec{A}$ + the gradient of another function, which may be the scalar function time which may be a function of space and time.

And that eventually gives the same $\vec{B}$; it leads to the same $\vec{B}$. That means if I do $\vec{\nabla} \times \vec{A}'$ I will return back whatever the $\vec{B}$ is there, that is 1. So, I can manipulate my $\vec{A}$ through this function χ_i**.** So, I have a liberty to take my $\vec{A}$. Now not only that I can also have liberty to have the ϕ, this is a vector potential, so the scalar potential ϕ' I can write in this way whatever the ϕ is given and minus if I do this quantity, if I do this thing.

Then $\vec{A}$ and the set $\vec{A}'$ ϕ' will going to produce the same $\vec{B}$ and same $\vec{E}$, let us check that first whether they are producing the same $\vec{B}$ or same, same $\vec{B}$ we know that but same $\vec{E}$ whether they are going to produce or not, let us check. So, first I am saying that $\vec{A}$, ϕ, the set and $\vec{A}'$, ϕ' produce the same $\vec{B}$ and $\vec{E}$, so let us check that. So, if I make $\vec{\nabla} \times \vec{A}$, so that we already did earlier, so let me.

So, if I make $\vec{\nabla} \times \vec{A}'$ it should be $\vec{\nabla} \times \vec{A}$ + gradient of this and that is eventually $\vec{\nabla} \times \vec{A} + 0$ because gradient of something if I took a curl over that it should be 0 and that is nothing but my $\vec{B}$. So, $\vec{A}'$ is producing the old $\vec{B}$, what about the $\vec{E}$?

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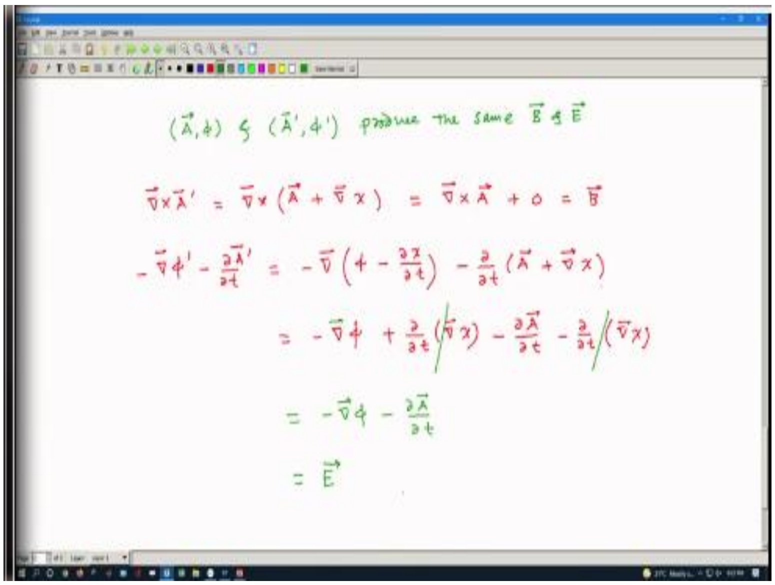

Because $\vec{E}$ is, so if I want to find out this quantity $\phi' - \frac{\partial \vec{A}'}{\partial t}$, so this ideally should give us the same $\vec{E}$, so let us check whether it is giving the same $\vec{E}$ or not. So, I have this and then it should ϕ' is my ϕ minus that is that quantity, it should be $\phi - \frac{\partial}{\partial t}$ and $-\frac{\partial}{\partial t}$, $\vec{A}'$ is $\vec{A}$ plus of this. So, what I am getting here? Minus of this, plus of let us put this inside, so I have now $\vec{\nabla}\chi$.

Then $-\frac{\partial \vec{A}}{\partial t}$ and then $-\frac{\partial}{\partial t}$ of this quantity. Note that this will going to cancel out, this quantity and this quantity will be going to cancel out. And eventually what I left with is this $-\frac{\partial \vec{A}}{\partial t}$, which is equivalent to my old $\vec{E}$. So, that means $\vec{A}'$ and ϕ' will produce the same $\vec{E}$ and $\vec{B}$. So, but I can manipulate my $\vec{A}'$ and $\vec{B}'$.

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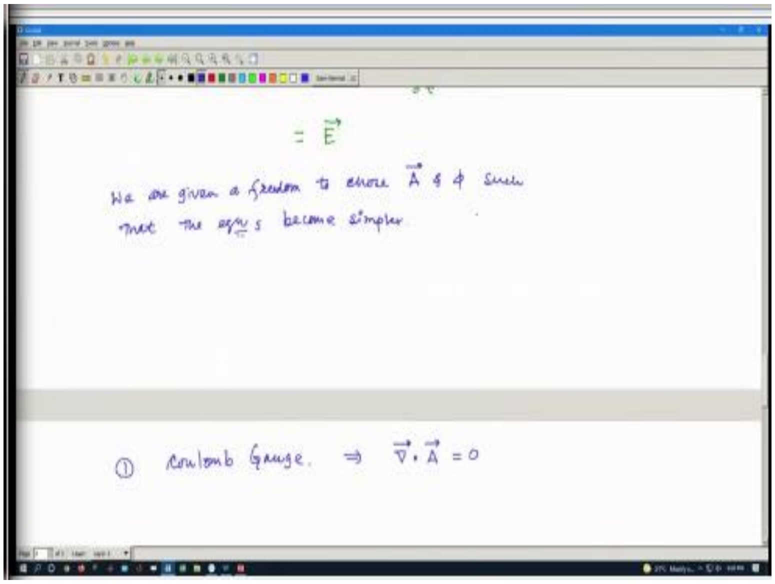

So, that means the outcome is we are giving a freedom to choose $\vec{A}$ and ϕ such that the equations become simpler, that is the goal. So, whatever the equation we had here, so these 2 equation written here these 2 equation this one and this one I need to in fact solve to find out $\vec{A}$ and ϕ and once we solve for $\vec{A}$ and ϕ then I can use this 2 equation again to find out my $\vec{E}$ and $\vec{B}$, that is the strategy.

But in order to solve these 2 equations we need to, this is not very easy to solve but we can make some mechanism through which we can make this equation simpler and this mechanism is basically the Gauge invariance. So, I can choose my $\vec{A}$ and ϕ suitably to make these 2 equations simpler. Now the question is how to make this equation simpler? So, one way is the coulomb Gauge that we discuss earlier and the Coulomb Gauge is suggesting that is the condition of the Coulomb Gauge.

So, first we will discuss about the Coulomb Gauge and this Coulomb Gauge is suggesting that this is we put a constraint is that my $\vec{A}$ should be such that I can take my $\vec{A}$ in such a way that the $\vec{\nabla} \bullet \vec{A}$ should be 0.

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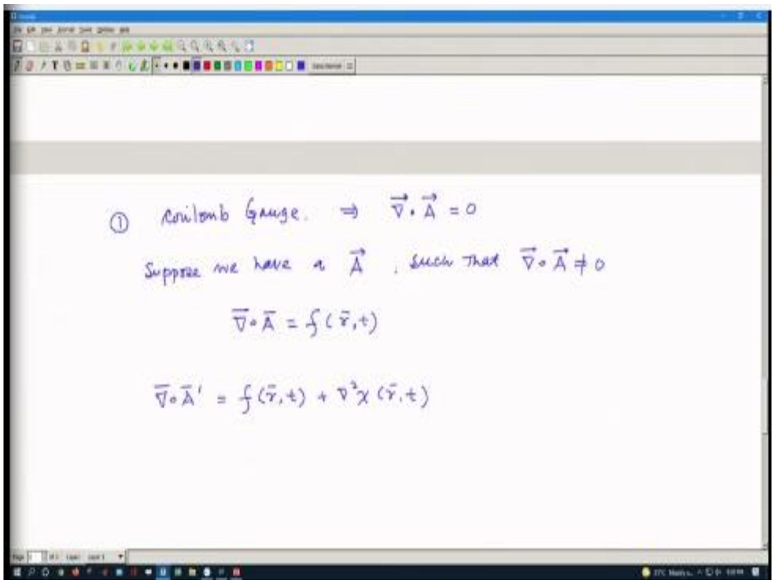

Now the question is suppose a $\vec{A}$ is given to me suppose we have suppose we have a $\vec{A}$ such that the $\vec{\nabla} \bullet \vec{A}$ is not equal to 0 but we know the value and this is not equal to 0. And if this is not equal to 0 then we say if this value is say some scalar field $f(\vec{r}, t)$. Now we can always make a $\vec{A}'$ such that, in this way I can make $\vec{A}'$ plus, so this is the strategy. Because I can have my $\vec{A}$ in this way, so now I make a divergence both the side of this I can always have.

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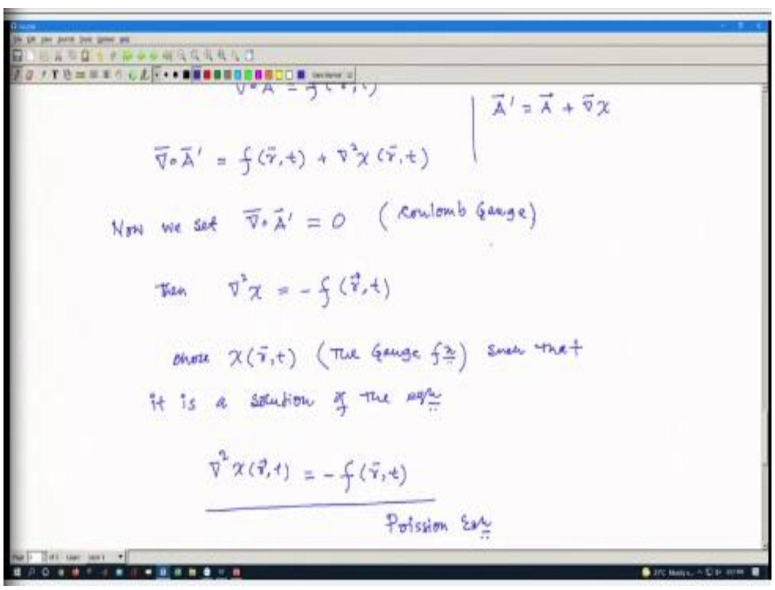

So, let me write it here I can always have my $\vec{A}'$ as $\vec{A}$ plus this because this will produce the same $\vec{B}$. Now I put $\vec{A}$ divergence over this $\vec{A}'$, when you put the divergence over this $\vec{A}'$ then $\vec{\nabla} \bullet \vec{A}$

already f it is given, so I put this $\vec{A}$ and $\vec{\nabla} \bullet \vec{\nabla}\chi$ is simply Laplacian $\nabla^2\chi$. So, now we demand, now we set that this quantity must be 0, we set that, so that is our Coulomb Gauge condition.

Our coulomb Gauge condition is the $\vec{\nabla} \bullet \vec{A}$ has to be 0, so that we set. If we set that then I can have an equation then the χ has to satisfy equation like this. And I have a freedom to choose my χ mind it, χ this is my hand, so we can choose my scalar field χ, which we call the Gauge function such that it is a solution of the equation this, which equation? It is a solution of this equation.

And we know that what is this equation, this is Poisson equation and we know that how to solve this, this is our Poisson equation. So, that means under Coulomb Gauge I can always make I can choose a $\vec{A}'$ in such a way that I can always have the divergence 0. And this is the recipe to get this $\vec{\nabla} \bullet \vec{A} = 0$ because I have a function χ, which satisfy the Poisson equation and that.

Now the question is what should we get after making that? Because these 2 equation if you now see carefully these 2 equation. So, under Coulomb Gauge this term will go to vanish. And if this term vanishes then ϕ should have a relatively simpler equation, we should have a relatively simpler equation for ϕ, that is the advantage we are getting. And that ϕ if I put in the next equation we should get a solution, so let us do that part as well.

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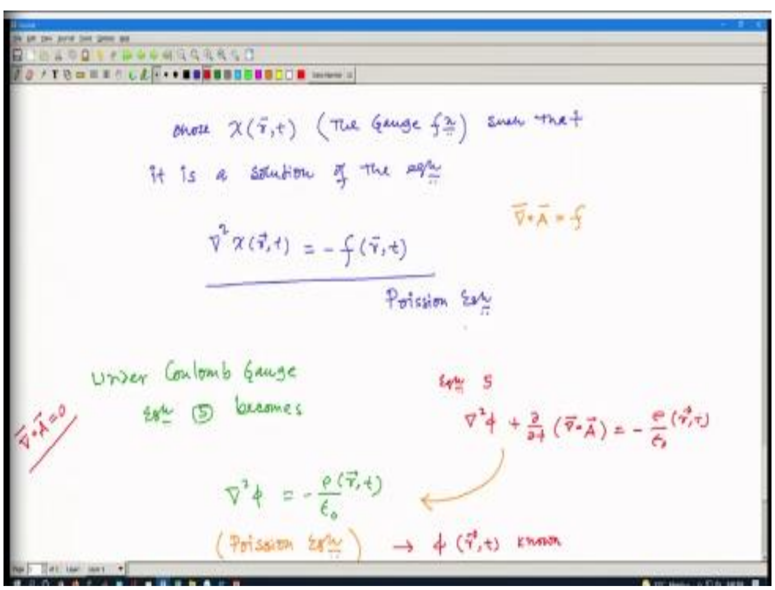

So, under Coulomb Gauge, so whatever the equation we had so let me put a name here this equation say equation, already I had this name, so equation 5 I think, yeah equation 5 and 6. So, this is equation 5 and this is equation 6, I rewrite that again here. So, under Coulomb Gauge equation 5 becomes this $\nabla^2\phi = -\frac{\rho(\vec{r},t)}{\epsilon_0}$, what was equation 5? So, equation 5 is this, better I should write equation 5 because I need to go up every time, so better I just.

So, equation 5 becomes, so let me write equation 5, so what is equation 5? Equation 5 was this $\vec{\nabla}\phi$ and then I guess + $\frac{\partial}{\partial t}$ and then $\vec{\nabla} \bullet \vec{A}$ and then I have equal to $-\frac{\rho}{\epsilon_0}$, ρ should be function of $(\vec{r}, t)$, like this, so that was my equation 5. And under Coulomb Gauge means I should put these equal to 0 that is the condition of the Coulomb Gauge. So, this equation 5 becomes this, so now this is a Poisson equation again.

And in principle we have a solution for the Poisson equation. So, I can readily have my ϕ, so what is the strategy step by step let me go back. I had this equation 5 and for which maybe my Gauge condition $\vec{\nabla} \bullet \vec{A}$ is not fulfilled. But I have a liberty to choose another $\vec{A}'$ such that this $\vec{A}'$ can satisfy that. And I can have an equation for Coulomb Gauge. This is the condition I need to satisfy.

But even if this condition is not satisfied I can have a recipe to make another $\vec{A}'$ here which can satisfy the Coulomb Gauge. And for that I need to choose a function χ a scalar field, which is again following this Poisson equation. Where f if it is not satisfying, so this value is given and I will going to exploit this value here to find out my χ and put it back there to find $\vec{A}'$. Now Coulomb Gauge is satisfied and if the Coulomb Gauge is satisfied then I can have my equation of the potential ϕ in a simpler form and that is my Poisson equation, that part is done.

So, I can modify my equation 5 in this way, what about the 6? Because another equation I need to solve simultaneously. So, let me write down the equation 6, so what about equation 6?

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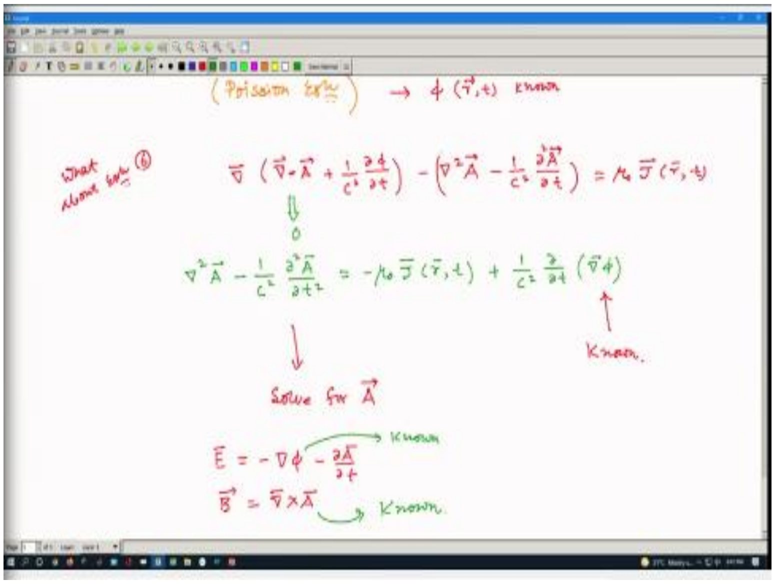

Let me write first, the equation 6 was $\vec{\nabla} \bullet \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}$ what was that? Let me go up and check, $-\vec{A}$ and this is forming something wave equation will come to this part again because I have already discuss about the wave equation. So, now if you look carefully this part of the equation this is nothing but the wave equation we will discuss later maybe in the next class, so that is the equation we had.

Now under Coulomb Gauge condition what we get is this term is 0 here and that is all. So, if this term is 0 then we have the equation which in this way, so we eventually have the equation like the $\vec{\nabla} \bullet \vec{A}$, I am writing this part $-\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$ that is there. And in the right-hand side I should have say -$\mu_0\, \vec{J}(\vec{r}, t)$ and then I have $+\frac{1}{c^2}\frac{\partial}{\partial t}(\vec{\nabla}\phi)$, that we are having. ,So how do I solve that? Would it be possible to solve?

Because in this case, okay, I understand this is a Poisson equation and I can solve ϕ, what about this equation? After Coulomb Gauge I have a little bit simpler form of equation 6 but still this looks very cumbersome but in principle we can solve it because this is a source term that should be given and now this portion is known because you have already solved that. So, you solve that and you know what is my ϕ.

So, ϕ as a function of $\vec{r}$ and t is known now because you know how to solve this Poisson equation and this value should be known and now this known value if you put in the right-hand side, so it becomes right-hand side is completely known and you have a wave equation with a source term and in principle you can solve it. But still there is a cumbersome process to solve this wave equation with a source term. So, maybe we can have a better Gauge.

So, this is a Coulomb Gauge for which you will be going to get an in principle in Coulomb Gauge you can simplify the entire equation in this way. You can calculate your ϕ by Poisson equation then you put this ϕ here in this equation 6 and then you figure out, then you solve for $\vec{A}$. And now your $\vec{A}$ and ϕ is known, once your $\vec{A}$ and ϕ is known then you can exploit this $\vec{A}$ and ϕ to find out your $\vec{E}$, which is $-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$ and your $\vec{B}$ is $\vec{\nabla} \times \vec{A}$.

So, both you can calculate because now this is known and also $\vec{A}$ is known. So, in principle you can calculate your $\vec{E}$ and $\vec{B}$. Well, this is the way to solve $\vec{A}$ and $\vec{B}$ by using this Gauge thing. In the next class, today I do not have time in the next class we will discuss another Gauge, which is called the Lorentz Gauge and in Lorentz Gauge you find that the expression become little bit simpler and it will be much useful. So, with this note I like to conclude here, thank you very much for your attention and see you in the next class.