Foundation of Classical Electronics

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Lecture-65
Maxwell's Equation: A Complete Overview

Hello student to the foundation of classical electrodynamics course under module 4 we have lecture 65 today. And this lecture basically we will try to understand Maxwell's equation once again. Last day we discussed few things but a complete overview I will like to provide to understand this Maxwell's equation in terms of scalar and vector potential, so let us start.

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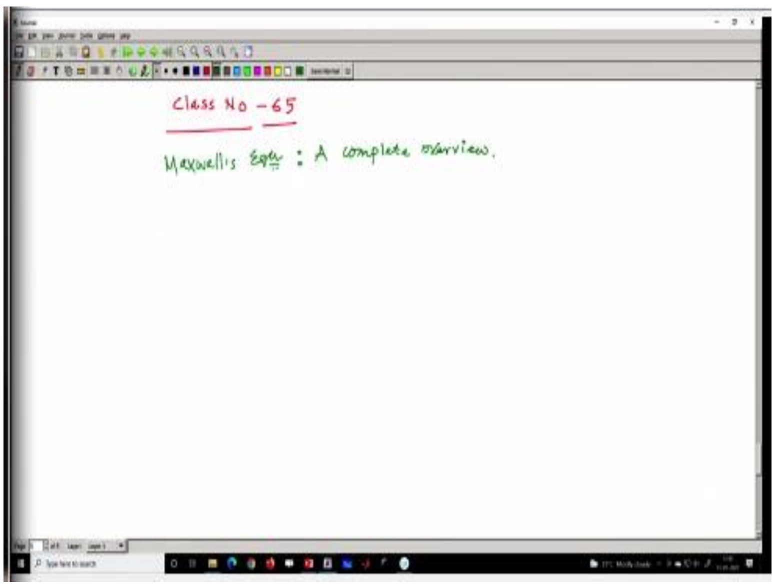

So, we have class number 65. So, quickly recap what we have done, so I should say that this is understanding of the Maxwell's equations. So, Maxwell's equations: a complete overview. So, let us start with the Gauss's law quickly I remind.

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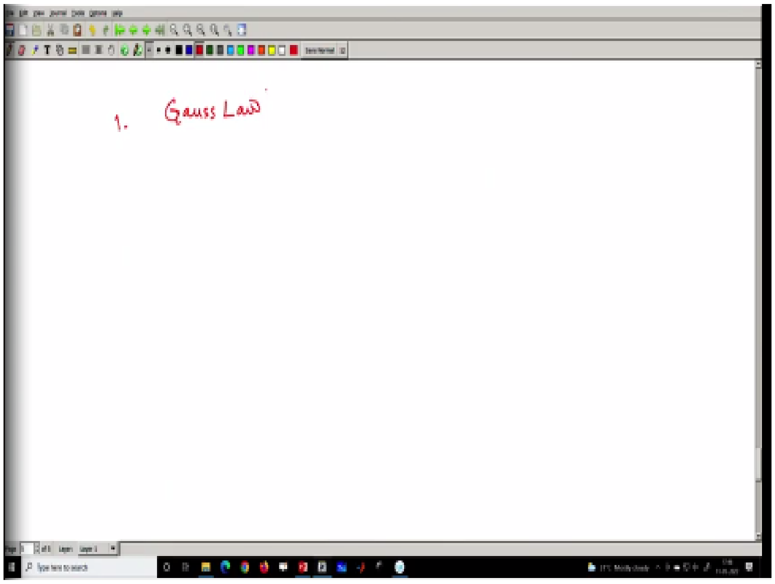

So, what Gauss's law says?

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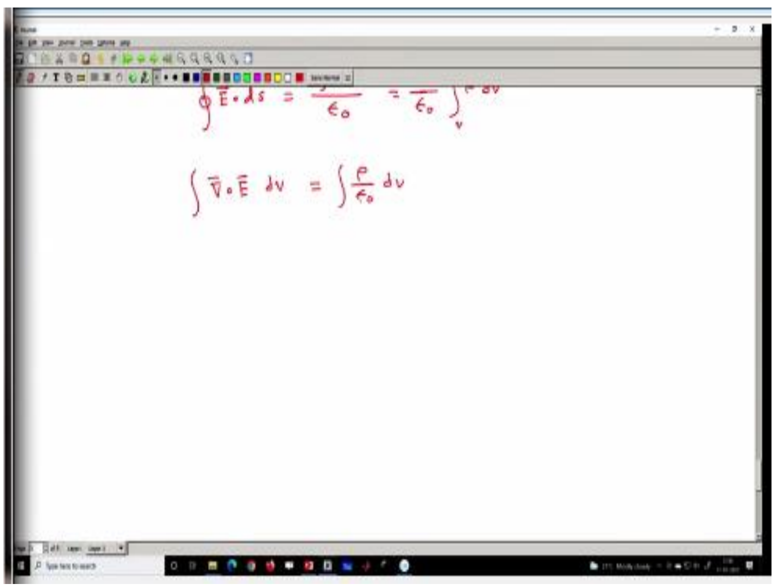

In Gauss's law in electrostatic we had $\vec{E}$ • d$\vec{s}$ is $\frac{Q_{enclose}}{\epsilon_0}$ that is $\frac{1}{\epsilon_0}\int \rho\, dv$ over v. So, the total amount of electric flux is $\frac{Q_{enclose}}{\epsilon_0}$ and in terms of ρ it is this one. And from that we had that $\vec{\nabla}$ • $\vec{E}$ dv Gauss's law = $\int \frac{\rho}{\epsilon_0}\, dv$, 2 for any volume and then eventually we get the equation Maxwell's equation 1.

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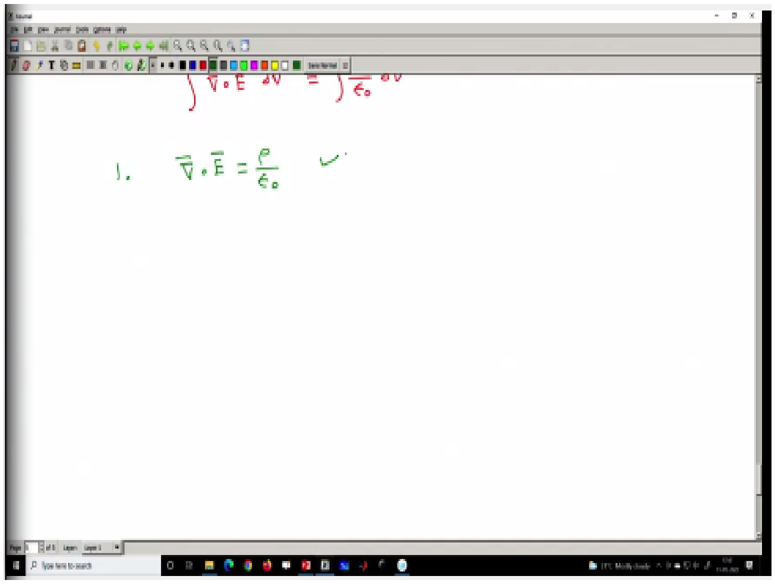

And that is $\vec{\nabla} \bullet \vec{E}$ is $\frac{\rho}{\epsilon_0}$ that is our first equation. What about the second one?

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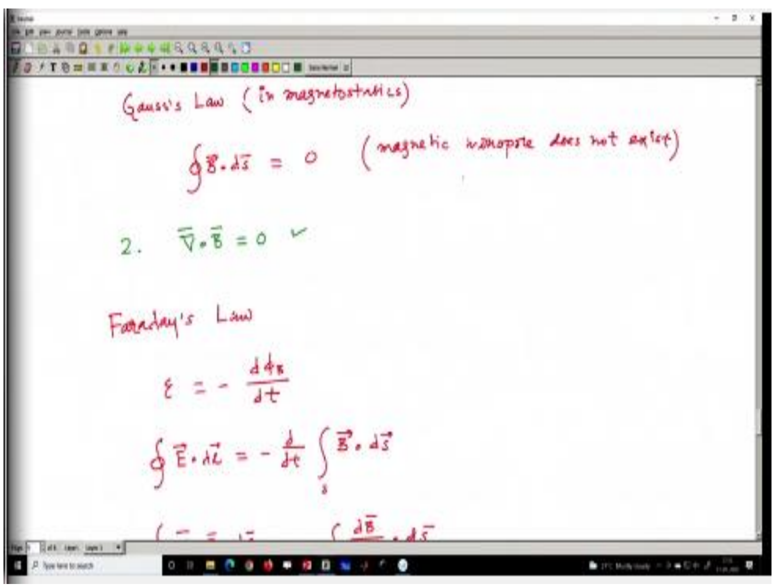

Second one is Gauss's law in magnetostatic and that basically tells me that the $\oint \vec{B} \bullet d\vec{s}$, that is the flux is 0. And that tells me like that magnetic monopole does not exist. And we had an expression from here that the $\vec{\nabla} \bullet \vec{B}$ should be 0 that is our equation 2.

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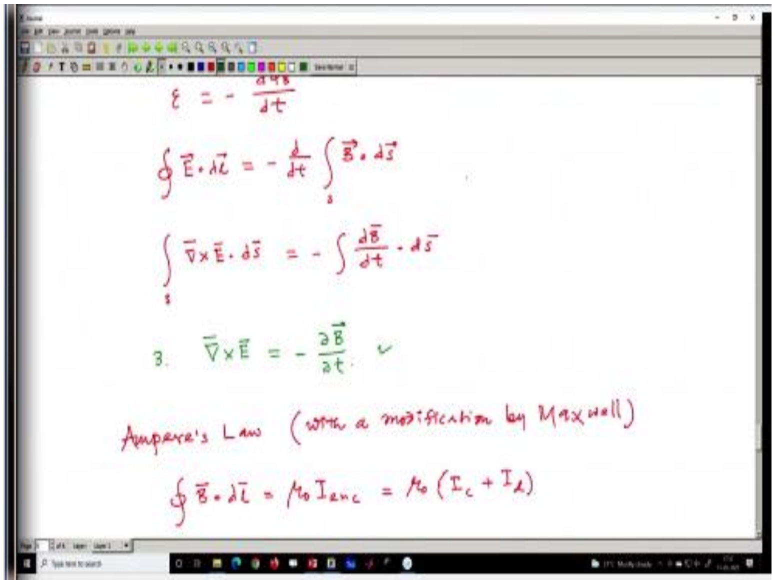

Then we had the Faraday's law, which says that the EMF can be generated due to the rate of change of magnetic flux in a closed loop mathematical statement is this. And if I want to write in an integral form it should be $\vec{E} \bullet d\vec{l} = -\frac{d}{dt}$ and I should have $\vec{B} \bullet d\vec{s}$, where this is a surface integral. So, that eventually $\vec{\nabla} \times \vec{E} \bullet d\vec{s} = -\frac{dB}{dt} \bullet d\vec{s}$ and we had our third equation, which is saying $\vec{\nabla} \times \vec{E} = -\frac{dB}{dt}$, that is our third equation, 3.

What about the final equation, fourth equation? Fourth equation is essentially the Ampere's law modified by Maxwell's. So, we had Ampere's law with a modification by Maxwell and what was that? That was $\vec{B} \bullet d\vec{l}$ in integral form that should be equal to μ_0 $I_{enclose}$, this $I_{enclose}$ is now containing 2 term, so I should write one is $I_{conducting}$ and another is $I_{displacement}$. So, that eventually gives us the fourth equation.

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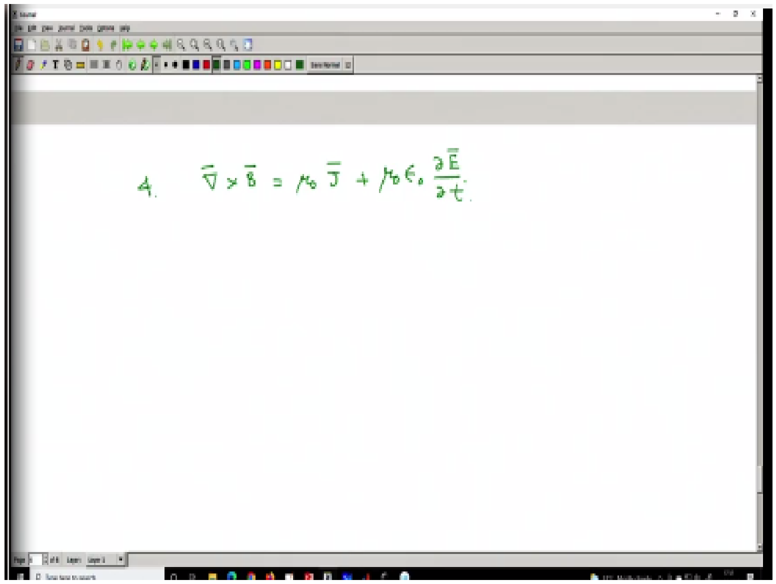

And it was $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ plus that modification $\mu_0\, \epsilon_0 \frac{\partial \vec{E}}{\partial t}$, that was the expression we get. So, these are the 4 equation written in green colour are the Maxwell's equation starting from the fundamental laws. Well, now let me write these 4 equations together, so if I do then again I should write Maxwell's equation.

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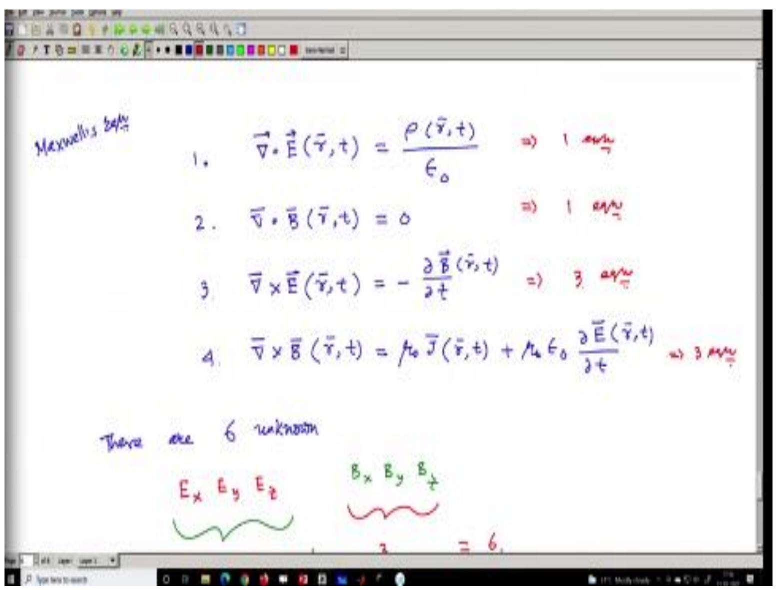

So, first one is $\vec{E}$, which should be a function of $\vec{r}$ and t, that is $\frac{\rho(\vec{r},t)}{\epsilon_0}$. Second equation dot $\vec{B}$, now $\vec{E}$ and $\vec{B}$ I just put a space time dependency, this depends on space and time. Third is $\vec{\nabla} \times \vec{E}$ is

$-\frac{\partial \vec{B}}{\partial t}$ it is a function of r and t and finally $\vec{\nabla} \times \vec{B}$ is $\mu_0 \vec{J}$, which is a function of r and t, this is source term $+\mu_0 \epsilon_0 \frac{\partial \vec{E}(\vec{r},t)}{\partial t}$.

So, here in this equation if you note carefully these are the vector equation, all these fours are vector equation and this vector equation should have some component and component wise if I write, so the unknown here is 6 because if I write in component wise then it should be E_x, E_y, E_z, B_x, B_y, B_z. So, unknown, there are 6 unknown in this set of equations, 6 unknown. And these 6 unknowns are essentially E_x, E_y, E_z that is one set and another is B_x, B_y, B_z, this 3, so this is 3 + this is 3, so total 6 unknowns are there.

Now the number of equation if you look there are 8 equations. So, 8 equations are there to solve 6 unknowns that means there are certain equations, which are not independent. So, how many equations are there?

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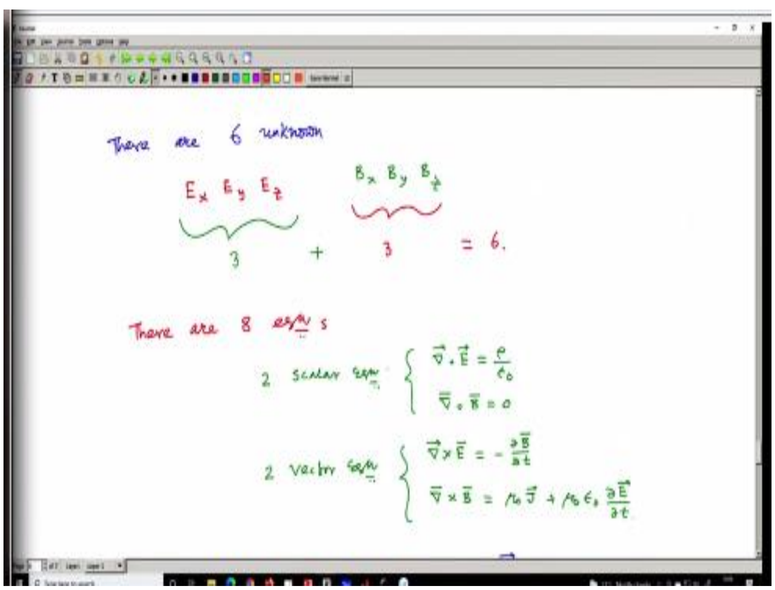

So, there are in general in component form there are 8 scalar equations, there are 8 equations. So, how you get these 8 equations? Because this is a scalar equation if you look carefully, so this is one equation, this is a scalar equation, this is one equation, this is a vector equation, so component wise if you write there should be 3 equations and this is also vector equation.

If I write component wise B_x, B_y, B_z if I try to write in this component form there should be 3 equations. So, 2 1 + 1, 2, 3 + 3, 6, 6 + 2 there should be 8 equations. So, among 8 I should write there are 2 scalar equations; these 2 scalar equations are this and this. If I expand this it should leads to only one equation, if I expand this it should leads to only one equation, so only 2 equations are there.

And there are 2 vectors equation, which leads to 6 equations, which vector equations? $\vec{\nabla} \times \vec{E}$ is $-\frac{\partial \vec{B}}{\partial t}$, left-hand side is a vector, right-hand side is also a vector, so component wise there will be 3 components, so 3 equations are there. Here we have $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$, so this belongs to 1 equation and this belongs to 3, this belongs to 3, so overall 8 equations are there.

So, now the question is 8 equations and 6 variables. So, that means there are some equations, which are related to each other. So, there is a connection, so you should note that. So, there is a connection between this 2 source term here ρ and $\vec{J}$, they are not independent and they are related to the well-known continuity equation. So, they satisfy this equation, so that means they are related, they are not independent, these 2 are not independent.

Since we have a relationship between ρ and $\vec{J}$, so this 8 equation now reduces to 7 equation because 1 constant equation is there, 1 constant is there. So, now still we have 1, equation that need to be correlated. So, 8 equation and then we have 1 constant it becomes 7 equation but I need exactly 6 equation because we have 6 unknowns. So, the other constant is something quite interesting and that is we are having a continuity equation that we get from the concept of the electrostatic problems.

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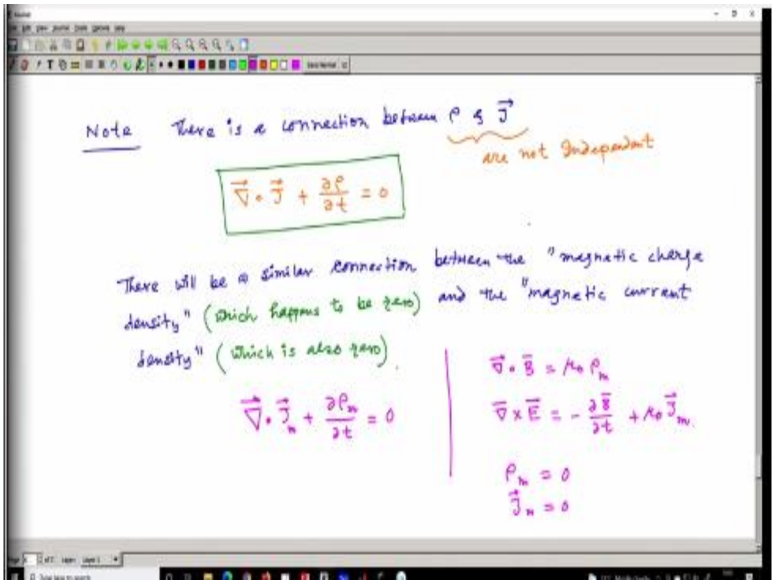

Now once one can expect, now there will be in principle there will be a similar connection between the something called magnetic charge and magnetic, not magnetic charge it is called magnetic charge density like the electric charge density like magnetic charge density. Obviously which happens to be 0, we know that there is no magnetic charge, which happens to be 0 because we know that there is no magnetic charge.

But if it is there then there should be a similar connection magnetic charge density and the magnetic current density, that is which is also 0. Because we know that there is no magnetic current. So, that means I suppose to have an expression like $\vec{\nabla} \bullet \vec{J}_{\text{magnetic}}$ + this is the magnetic current density and then I have $\frac{\partial \rho_{magnetic}}{\partial t}$ = 0 but these are not there, this is not, both are 0.

So, in the equation then how the equation looks like? Then the equation was something like $\vec{\nabla} \bullet \vec{B}$, now it is 0 but if I have that it should look like this and another thing divergence, so another thing is $\vec{J}$ here. So, I should have $\vec{\nabla} \times \vec{E}$, which is now $-\frac{\partial \vec{B}}{\partial t}$ then maybe we can have another term like $\mu_0 \vec{J}_m$. Like the way we had here $\mu_0 \vec{J}$ we should have but these 2 ρ_m and J_m both are not there, it is not present in reality.

But still we have an equation 0 = 0, so that makes an another constant. So, the first constant is the connection between ρ and $\vec{J}$, which is very much there, this is the electric current density and electric charge density but the magnetic current density and magnetic charge density should also follow the similar relationship. They should also follow a continuity equation but in reality they are not there, so in real world they are not there, so this is 0.

But this also makes another constant 0 = 0 that also makes another constant. So, there are second constant, so this is the first one and we can say this is another kind of constant one can expect one can have 0 = 0 by considering the magnetic current density and the magnetic charge density, which is not there in the real world. But this still I can consider as a constant, constant number 2. And that basically makes the total number of variable equal to the total number of equation, total number of unknown variable equal to the total number of equation from these 8 equations.

So, we had 8 equation but 2 constant if I remove then we should have 8 equations in our hand. Now again go back to the Maxwell's equation and writing this Maxwell's equation in a homogeneous and non-homogeneous form.

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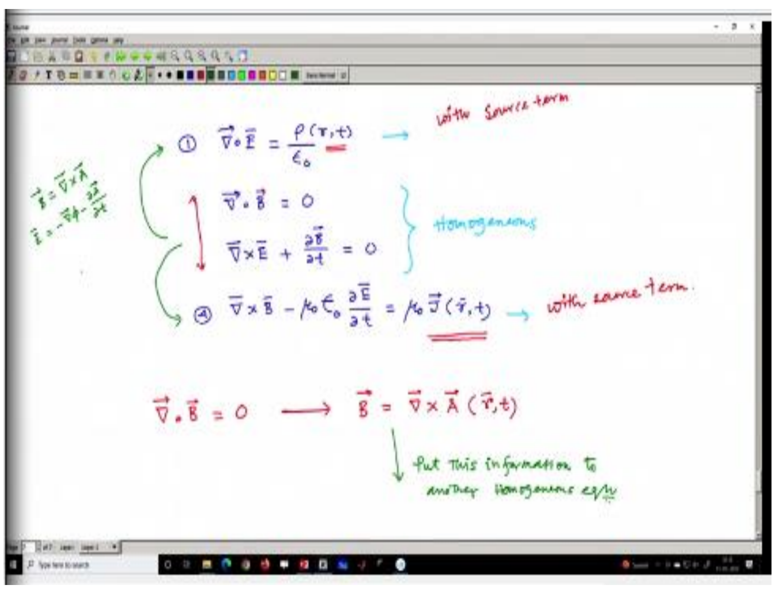

So, say I should write $\vec{\nabla} \bullet \vec{E}$ is $\frac{\rho}{\epsilon_0}$, $\vec{\nabla} \bullet \vec{B}$ is 0 and then $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}$ that is 0 and $\vec{\nabla} \times \vec{B}$ - μ_0 ϵ_0 $\frac{\partial \vec{E}}{\partial t}$ that is μ_0 $\vec{J}$ with function $\vec{r}$, t. Here also I should write this is a function of $\vec{r}$, t. So, I just rewrite the Maxwell's equation once again but here I can see that this 2 equations, which is sitting in the middle are homogeneous and the equation in the top, this one and this one are non-homogeneous.

Because it is containing the source term, here we have one source term $\vec{J}$ and here we have another source term ρ. So, this is with source term and this is also with source term, so they are in principle non-homogeneous. Now let us exhaust the content of the homogeneous equation because the homogeneous equations are easy to deal with. So, let us try to understand what we can get from these 2 homogeneous equations. Already this treatment we did earlier but let us do once again.

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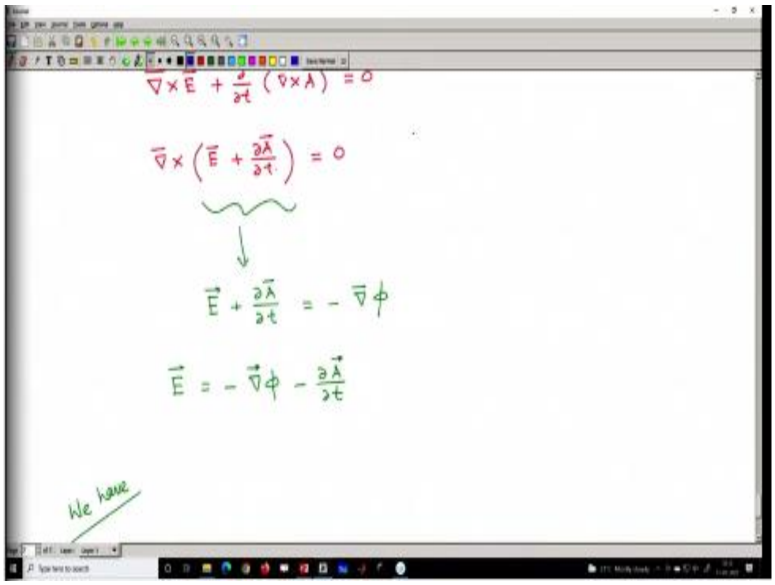

So, I can have the $\vec{\nabla} \bullet \vec{B}$ to be 0, so that readily allow us to write the equation that $\vec{B} = \vec{\nabla} \times \vec{A}$, vector potential $\vec{A}$. This is not new, we had done this earlier but again I am doing. So, that means I can have this equation, so this equation now transferred to another equation and that equation is this one. So, the information what we had is $\vec{\nabla} \times \vec{B} = 0$, I can transfer this equation in terms of $\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{A}$, where we have a liberty to choose $\vec{A}$ we also discuss this part.

Now what we do that? I will put this information $\vec{B} = \vec{\nabla} \times \vec{A}$ into the second homogeneous equation. And if you do then let us put this information, so what I do this? Put this piece of information to another homogeneous equation, so I start with 1 homogeneous equation and I get something and then I put this information to another homogeneous equation and then if I do I can have $\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}$, in place of $\vec{B}$ I just simply write $\vec{\nabla} \times \vec{A} = 0$.

So, this is again a not a very unknown equation because we did it earlier. So, I should write this is $\vec{\nabla} \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0$. Now once we have a curl of some vector field equal to 0, we are in a position to write this vector field to a gradient of a scalar and I can write it this vector field = gradient of some scalar and I have ϕ here, where the ϕ is a scalar quantity. So, my $\vec{E}$ is now $-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$. So, what we get so far is this.

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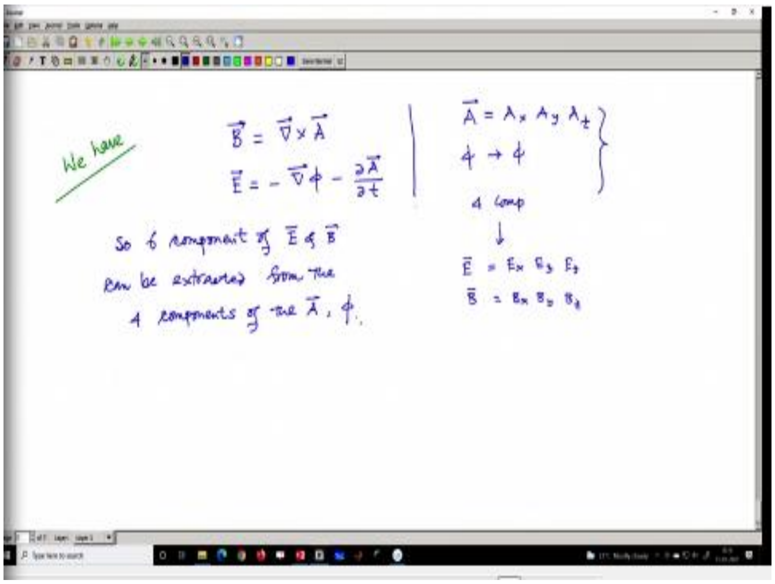

We have done this earlier but now we are try to understand in using the 4 Maxwell's equation few things. So, $\vec{B}$ is now $\vec{\nabla} \times \vec{A}$ and $\vec{E}$ is now $-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$, both the equation we derived earlier but now I am writing again. And this is the way one can write the electric and magnetic field in terms of the vector and scalar potential, this is right. So, you can see here 6 component of $\vec{B}$ and $\vec{E}$ can be exhausted from the 4 components of $\vec{A}$ and ϕ.

So, whatever, so how many components are there? So, at the end of the day I need to find $\vec{E}$ and $\vec{B}$, so $\vec{B}$ has 3 components B_x, B_y, B_z, E as 3 component E_x, E_y, E_z. But from these 2 equations I can see that this 6 component can be compressed or can be extracted from the 4 component of $\vec{A}$ and ϕ. Because $\vec{A}$ is a scalar quantity, $\vec{A}$ is a vector quantity having 3 component but ϕ is a scalar quantity.

So, $\vec{A}$ is having A_x, A_y, A_z, ϕ is a scalar, it should only have one component ϕ, these 4 component are capable enough to describe $\vec{E}$ and $\vec{B}$, where $\vec{E}$ and $\vec{B}$ are having 6 component E_x, E_y, E_z and B_x, B_y, B_z. Now, so what should I get what should I write? So, 6 component of $\vec{E}$ and $\vec{B}$ that is 3 component of $\vec{E}$ and 3 component of $\vec{B}$ can be extracted from the 4 components of the vector field $\vec{A}$ and the scalar field ϕ.

Now I would like to stop here because I do not have much time to discuss further but the next day what we do that will continue this treatment because we are just started to understand few things. So, we write the 4 Maxwell's equation in a form where we had 2 homogeneous and 2 non-homogeneous equations that is the starting point here, in this part. Now we exhausted the information of the 2 homogeneous equation $\vec{\nabla} \bullet \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$.

And after taking the information out of these 2 homogeneous equation we find that $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = \vec{\nabla}\phi$ minus the time derivative of the vector potential $\vec{A}$. So, next day what we do in the next class? We will put this value of $\vec{B}$ and $\vec{E}$ in this non-homogeneous equation. So, we had 2 very important equations here with having source that is this 1 and 4. So, whatever the information we extracted from this we will be going to put here in this equation and in this equation. What we put?

We put this information that the $\vec{B}$ is curl of this information, which is $\vec{B}$ is curl of a vector field $\vec{A}$, so that information and $\vec{E}$ is $-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$. So, we put this information in equation 1 and 4 and then we are going to check that what we get from equation 1 and 4. So, then we will see that the entire expression all the equation will now come into the form of $\vec{A}$ and ϕ and then we have a different 2 important differential equation for $\vec{A}$ and ϕ, I am going to try to solve this.

And from that one can extract information of electric field and magnetic field in terms of the scalar and vector potential. So, with that note I like to conclude here, thank you very much for attention. And see you in the next class where we continue this treatment, thank you and see you in the next class.