Foundations of Classical Electrodynamics
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Lecture-63
Wave Equation, Maxwell's Equation

Hello student to the course of foundation of classical electrodynamics. So, today we have module 4 and in this module we have under this module in lecture 63. Today we will be going to learn the wave equation and then the Maxwell's equation.

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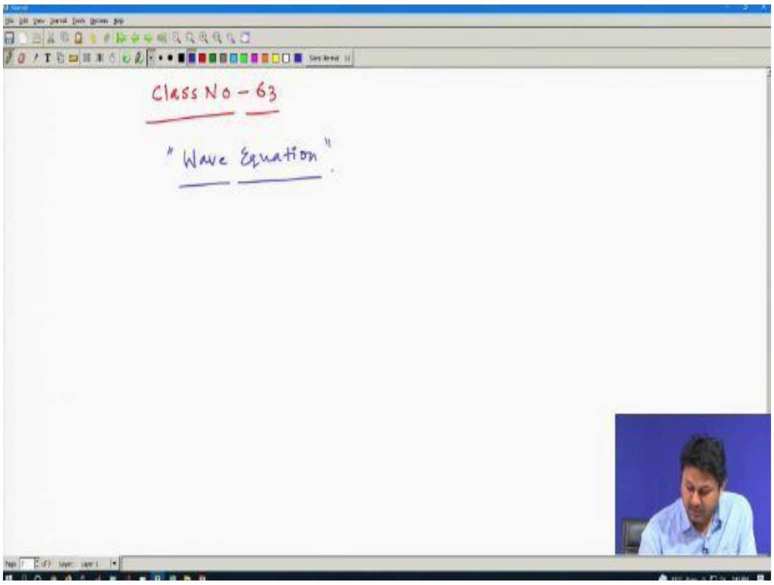

So, we have class number 63 today. So, before going to the Maxwell's equation, it is important that we should know about the wave equation. So, let us first discuss the wave equation. Just most of you may aware of the wave equation in different branches of physics. So, we will be going to make a simple general discussion about the wave equation here.

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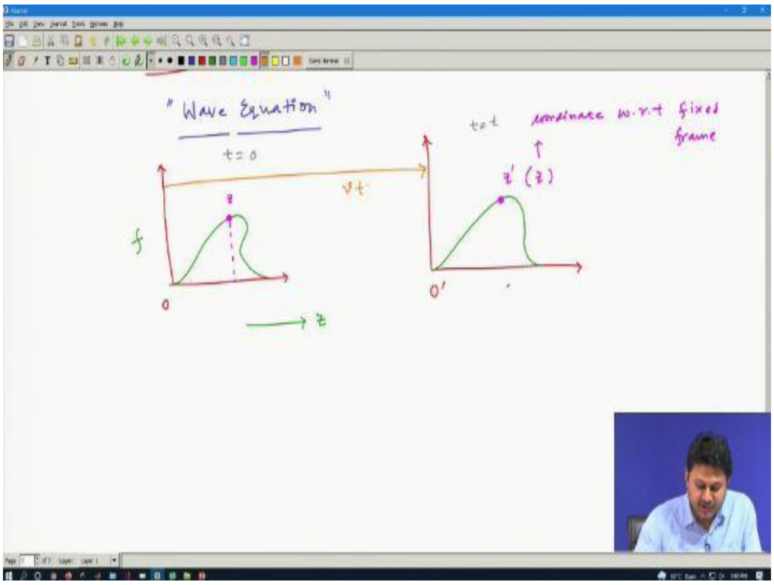

So, suppose I have coordinate systems like this, which is fixed and I called it say O. And in this fixed coordinate system, let me have a disturbance or a wave like this. And say this is z distance, this is z coordinate along this direction is z and the function that is distributed here is f. This point let us define a point over this wave say this one and this point is having a coordinate here and this coordinate with respect to this is z.

And it is at time t = 0. So, whatever the function we are having here is f(z) at t = 0. Now this wave is moving and at some different time t it is located at this place. And this is moving, so, this is the way we are having the same wave and it is moving from this point to this point. And I am just concentrating on the point we marked here. And suppose this is a frame that is also moving in the same velocity of the wave.

So, somebody who are in this frame O' will be going to see that there will be no change at all in the wave. So, we can also measure this point and this coordinate, this is a prime frame. So, this will be simply z'. Now this is the same coordinate, so, I should write bracket z and I should say that this is the coordinate with respect to the fixed frame. The same coordinate, so, it is measured from here to here, we measured this is a z coordinate. So, this should be the z same coordinate with respect to the fixed frame.

So, this z is a coordinate I should not write all this I simply write coordinate with respect to the fixed frame is here. So, now we just try to understand what is the coordinate, how the coordinates are related? So, if the velocity is v and if it is measured at time t, so this is say time = 0. So, this is at t = 0, this is at t = t. So, if the velocity is v, so I can have from here to here this distance I can have it is simply vt, v is the velocity and at time t this move to this place. Now what is z’ if I want to find out in terms of z?

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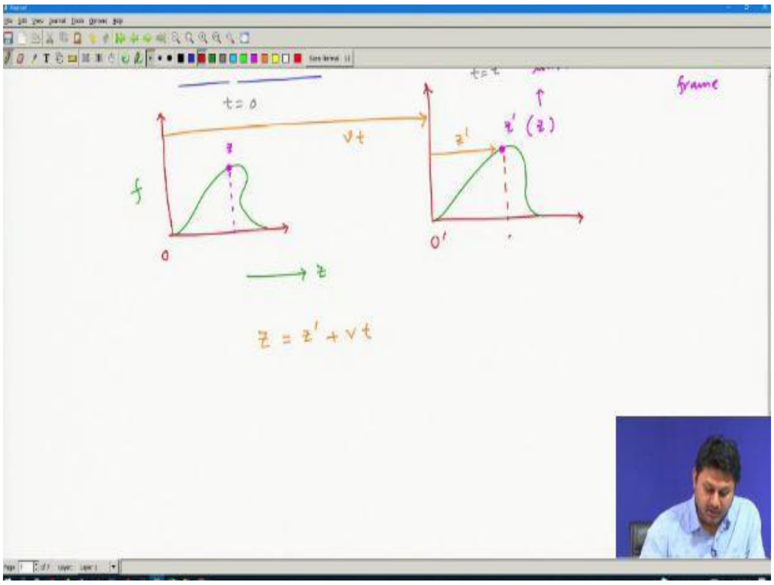

So, this point is z and from here to here it is z’. So, eventually z = z’ + vt. So, z I am measuring the same coordinate with respect to this and this is measured with respect to the frame that is fixed. So, it should be z. So, z should be z’ + vt. The v is a velocity of the frame. So, the frame is moving or v is the velocity of the wave itself. So, wave is moving with this velocity v, which is fixed this is.

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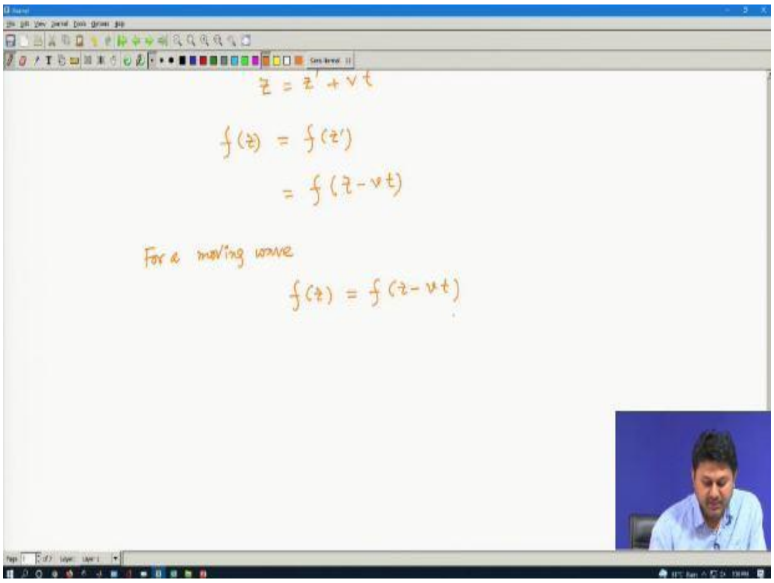

Now I can have the function here. So, the function of z, this is the same point that we are having at z’. So, these two things are eventually same. Now z’ if I write it should be simply z - vt. So, for a moving wave we have an interesting relation that for a moving wave we simply have f(z)

= f(z – vt). So, the z coordinate and t coordinate can be written together like z - vt form and that tells me that the wave is moving that is a some sort of indication.

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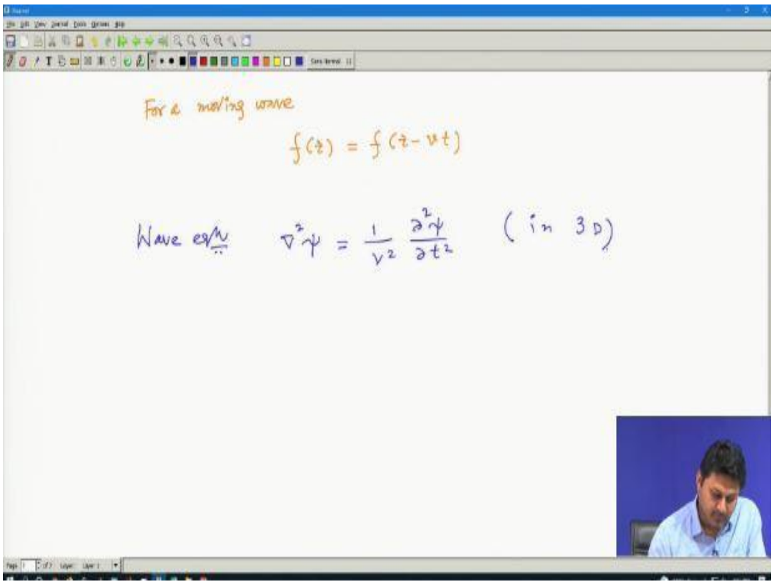

Now we know the wave equation several places we find it. So, the wave equation simply I can write in 3D it should be something like this. This is a differential equation or 2 variables differential equation and this is nothing but the wave equation, in 3D. In 1D the wave equation simplifies for example, this is the 1D problem the figure I just mention here.

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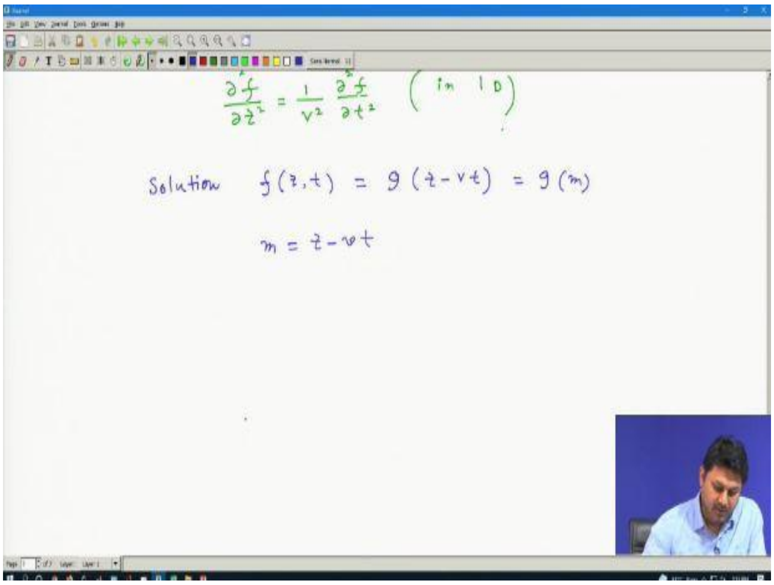

In 1 dimensional problem we can simply have the expression like, $\frac{d^2\Psi}{dz^2}$ that should equal to $\frac{1}{v^2}\frac{d^2\Psi}{dt^2}$, this is in 1D. Now the solution of this equation that we are looking for the solution is of the form. Solution of such differential equation should have an interesting form and the solution should have a form like f I used f so better I just replace this to f not Ψ because, I used a function f.

So, it should be like f. So, f(z, t) any point z and t that I can write a compact way like a function g(z – vt), instead of writing zt separately I can write in a compact way z - vt and that tells me that this is a solution I said I can write it this variable I simply write g(m) say, where my m is z - vt.

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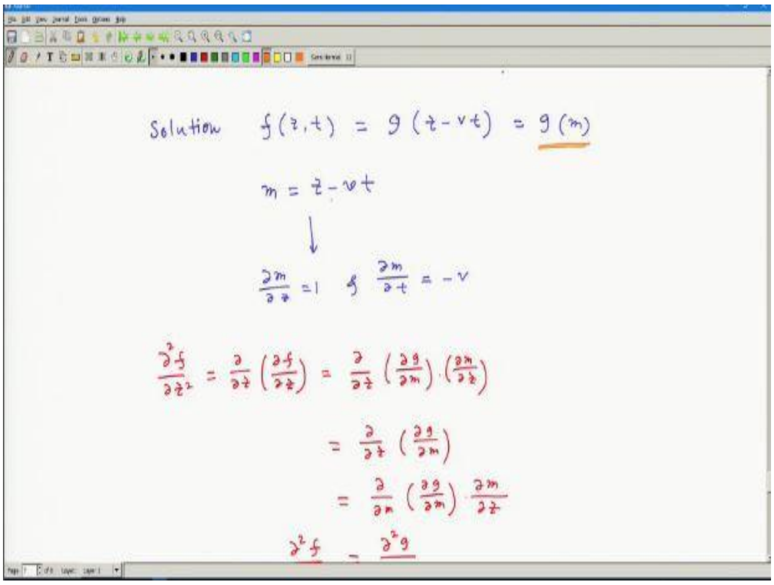

So, I can have from here, I can have a relation that $\frac{\partial m}{\partial z}$ = 1 and $\frac{\partial m}{\partial t}$ = - v, that two information we are having that we are going to exploit later. Now if I demand that this should be a solution then I just simply check it because, we know f is a solution. So, I just replace like $\frac{d^2 f}{dz^2}$ that quantity, I simply have $\frac{d}{dz}$ ($\frac{df}{dz}$) and $\frac{df}{dz}$ I use the chain rule $\frac{d}{dz}$ because, if I am going to use these things g(m).

So, I should have like $\frac{dg}{dm}$ and $\frac{dm}{dz}$. So, this simply I have $\frac{d}{dz}$ this quantity is one. So, I should have $\frac{dg}{dm}$. This again I can use like $\frac{d}{dm}$ ($\frac{dg}{dm}$) and then $\frac{dm}{dz}$.

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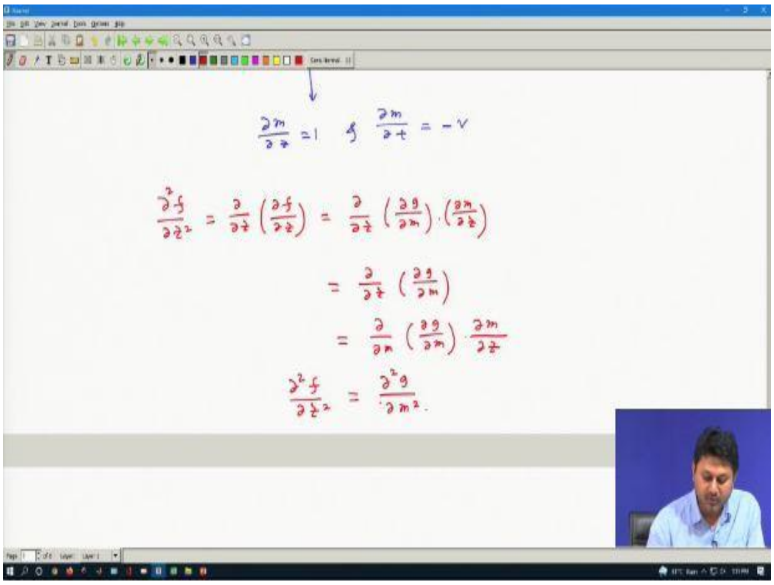

So, this quantity eventually is $\frac{d^2 g}{dm^2}$. So, I have $\frac{d^2 f}{dz^2}$ is equivalent to this quantity. In the similar way now we will check that what we have for the time derivative. So, let us do that.

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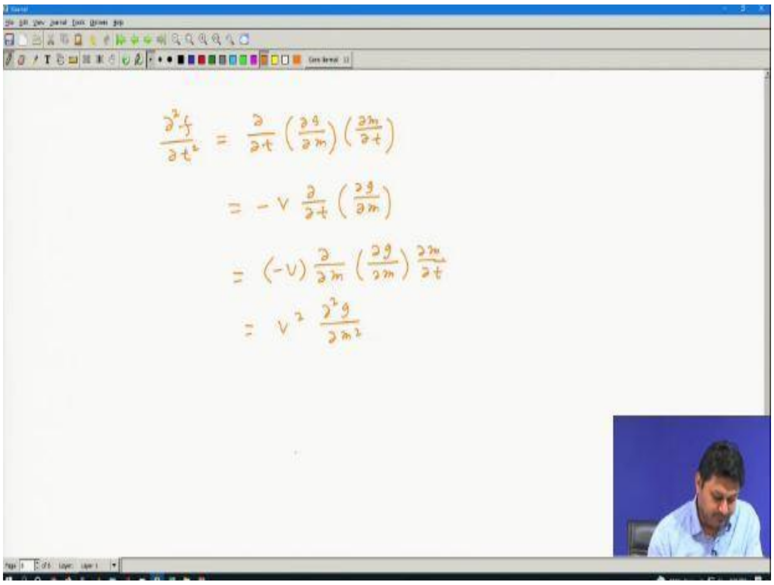

So, I have $\frac{d^2 f}{dt^2}$, so that quantity if I want to calculate because, that is in the right-hand side we are having here in the wave equation. So, that quantity again in the previous way I have $\frac{d}{dt}(\frac{dg}{dm})$ and $\frac{dm}{dt}$, g is a function of m. So, I can have a partial derivative. So, this quantity $\frac{dm}{dt}$ we know it is -v. So, I have minus of v $\frac{d}{dt}(\frac{dg}{dm})$, further I can have - v $\frac{d}{dm}$ $(\frac{dg}{dm})$ and $\frac{dm}{dt}$ chain rule dt. So, this again gives a – v. So, eventually I have v^2 $\frac{d^2 g}{dm^2}$.

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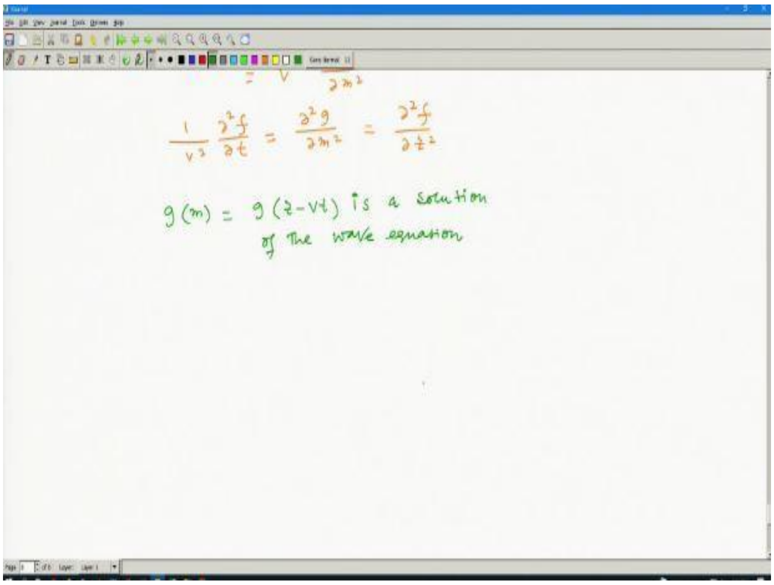

So, now I can see that divided by $\frac{1}{v^2}\frac{d^2 f}{dt^2}$ is something like $\frac{d^2 g}{dm^2}$, which is I $\frac{d^2 f}{dz^2}$. That means, that g(m) is a solution because, when I put g(m) here in this equation it basically satisfy this equation $\frac{1}{v^2}\frac{d^2 f}{dt^2}$ is the same value of 1 $\frac{d^2 f}{dg^2}$ when I put f as a f the solution as g(m), where m the argument is written in this form z - vt.

So, we can conclude that g(m), which is equal to g(z – vt) this form is a solution of the wave equation**.** And the most general solution because, in the similar way if the wave is moving not in the right-hand side but the left-hand side with the negative velocity. Then I should have another solution.

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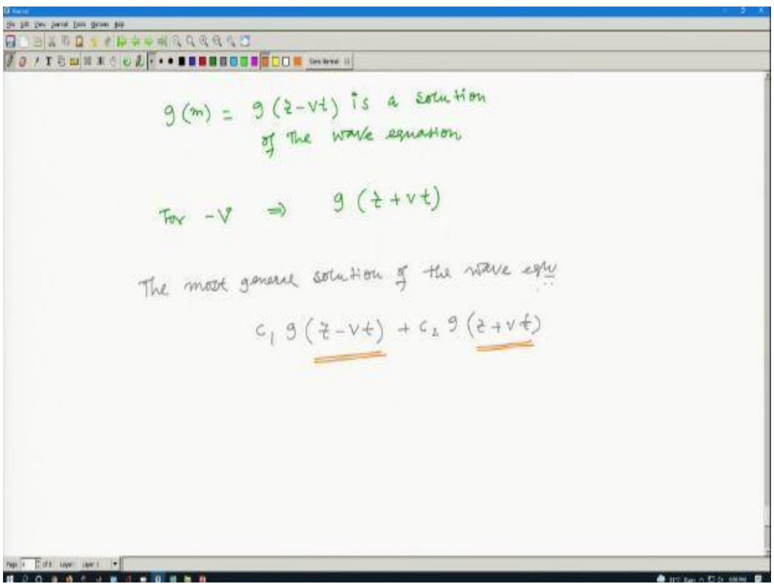

So, for negative v that is moving in the opposite direction, I should have another solution, which is g(z + vt) just replace the sign of the v because, now the wave is moving in opposite direction. So, in general I can have a general superposition of these 2 solutions can be the most general solution. So, the most general solution can be written as the most general solution of the wave equation one can write as a constant c_1 g(z – vt) + c_2 g(z + vt).

This should be the most general solution of the wave equation, but the interesting fact is that the solution can always be written like z - vt or z + vt, z and t can be written in a same footing like z - vt or z + vt. So, this is the signature of a solution that should satisfy this wave equation. So, whenever you have this kind of solution, then readily you can understand that this is a solution for some wave equation.

So, we have a very rough idea about the wave equation and solutions and we will be going to use this concept in Maxwell's equation when we derive the Maxwell's solution. So, let us now directly jump to the Maxwell's equation a very, very important topic in electrodynamics. So, we will be going to start this topic today.

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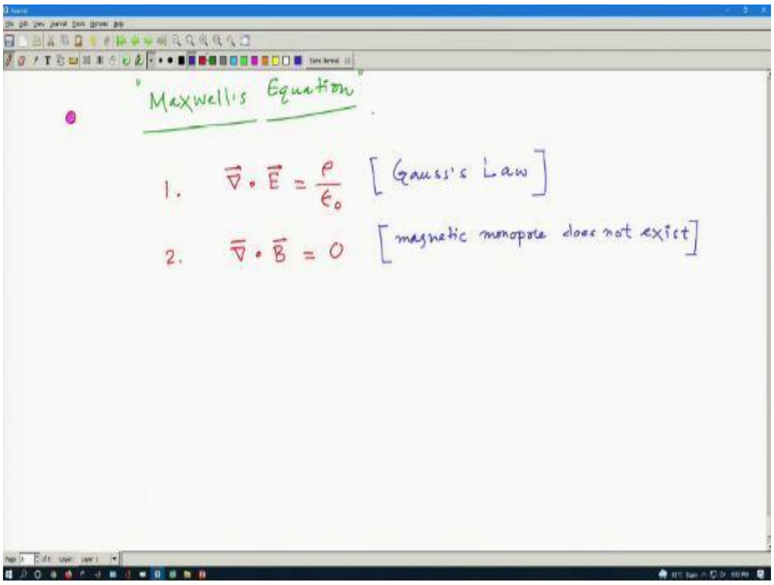

So, the topic is Maxwell's equation. So, one by one I should write all the equations are well known. The first equation that we find in electrostatic that $\vec{\nabla} \cdot \vec{E}$ is $\frac{\rho}{\epsilon_0}$, this is Gauss's law. This is our first Maxwell's equation. What is next? Next is $\vec{\nabla} \cdot \vec{B} = 0$ that we find in magnetostatic one of the fundamental equations and that tells us that the magnetic monopole does not exist, that is the physical understanding of this.

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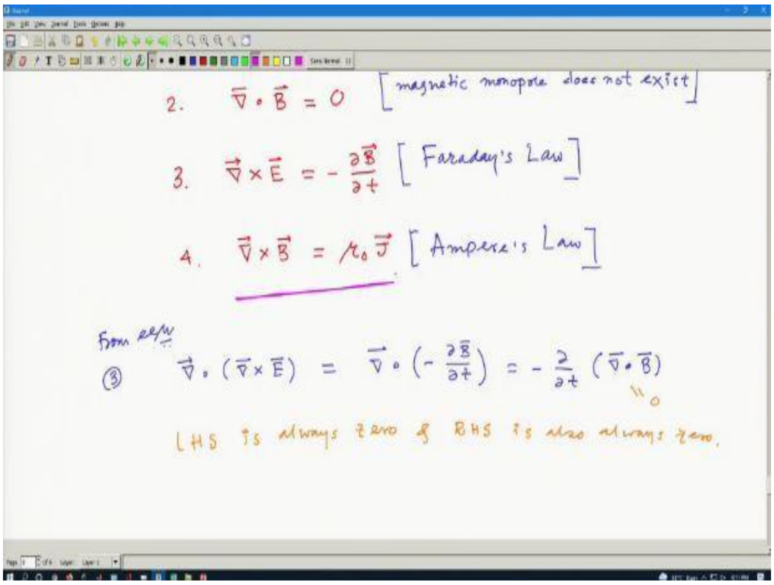

Then we have third equation $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, that is our Faraday's law. This is the first time we have a relationship with $\vec{E}$ and $\vec{B}$ first two equation. The first equation from electrostatic, second equation is from magnetostatic, third equation we find the first time there is a relationship between $\vec{E}$ and $\vec{B}$. And now we are entering in the domain of electromagnetism. So, here this is our Faradays law.

And finally the fourth equation that we got in magnetostatic that is $\vec{\nabla} \times \vec{B}$ is $\mu_0 \vec{J}$ up to this. And that we mention that this is Ampere's law. So, these are the 3 laws are there and one some sort of identity we had. So, 2 equations related to the divergence and 2 equations related to the curl of electric field and magnetic field respectively. So, now what we do that will check few things.

So, after having this equation now we are in a position the 4 equations are there in my hand. So, we can check few things. So, from equation 3, so first thing we check is from equation 3 what we get? I can have the $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{E})$ if I make the $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{E})$, then this side should be zero and this condition should be reflected in the right-hand side as well.

So, if I do I will get like it is divergence of in the right-hand side from equation 3 it is $-\frac{\partial B}{\partial t}$ and that I can write at $-\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{B})$. I can exchange this operator because, they are independent. But this quantity is 0. So, I can have the left-hand side. So, I should write, so this quantity is 0. So, the left-hand side is always 0 because, I am taking a divergence of curl of something. So, it has to be zero.

So, left-hand side is always zero and right-hand side so far reciprocate well is also always zero. So, there is no problem with these 2 equations. Now let us do the same thing for equation 4.

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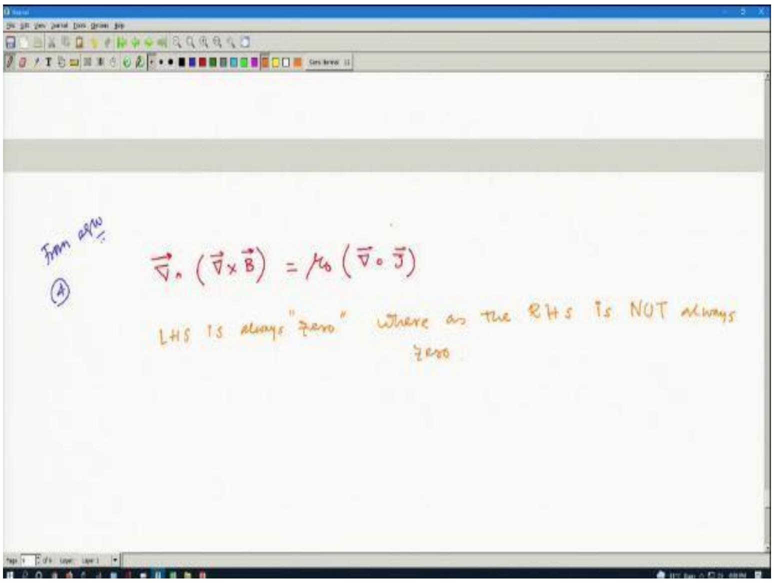

So, this is from equation 3. So, from equation 4, what we get? From equation 4 if I do the same treatment like let us take a $\vec{\nabla} \bullet (\vec{\nabla} \times \vec{B})$. So, the left-hand side again should be always 0 unconditionally. What is the right-hand side? Right-hand side we have μ_0 and then $\vec{\nabla} \bullet \vec{J}$. Now here we are having some issues because, left-hand side is always zero; this is unconditional because, if I make a divergence of a curl of something then it has to be zero.

But what about the right-hand side. Whereas, the right-hand side is not always zero. For steady condition may be it is zero. But this is not always zero. But left-hand side is unconditionally zero. So, what is the conclusion here, the conclusion is we need to make certain modification of equation 4 that is the conclusion we can make at this stage.

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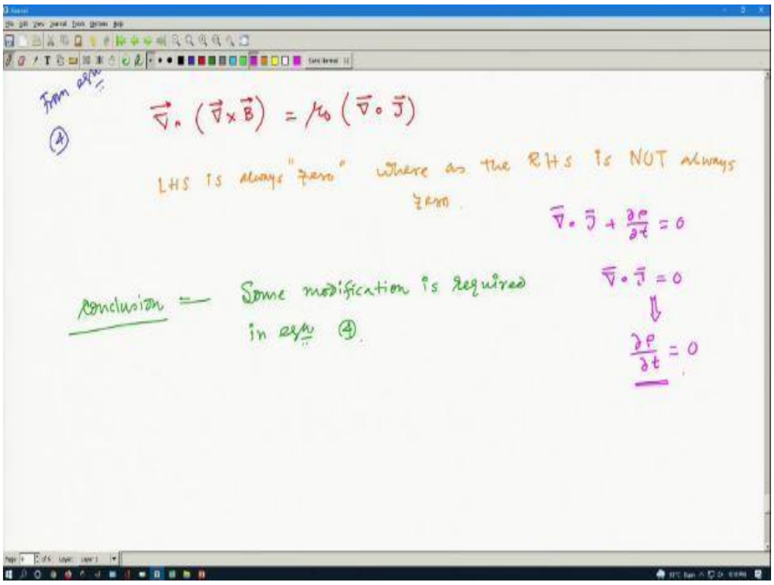

So, the conclusion from this treatment we can write here, some serious modification is required in equation 4. The fourth Maxwell's equation whatever, the fourth Maxwell's equation we are having some modification is required. Mind it, here we mentioned that the $\vec{\nabla} \cdot \vec{J}$ is 0. So, I should write a small note here before concluding. So, $\vec{\nabla} \cdot \vec{J}$ and then $+\frac{\partial \rho}{\partial t}$ this is a continuity equation the one of the fundamental equations is zero. So, $\vec{\nabla} \cdot \vec{J}$ is 0 means, $\frac{\partial \rho}{\partial t} = 0$.

So, for steady current so $\vec{\nabla} \cdot \vec{J} = 0$ basically, leads to a condition that $\frac{\partial \rho}{\partial t}$ has to be 0 and that is for steady current. When we have the steady current this condition is satisfied but not always. So, I like to conclude here in today's class. And in the next class we will discuss what kind of modification really required in equation 4. So, we should consider this equation is under scrutiny that we find that some problems are having in this equation, in equation 4.

And we need to put this modification in equation 4, such a way that left-hand side and right-hand side are compensating each other. They are complementing each other. If the left-hand side is always zero, we should make the modification in such a way that the right-hand side should also gives the value zero. So, with this note I like to conclude today's class. So, in the next class we will discuss more about Maxwell's equation. We find that there is something very interesting called the displacement current.

And we will be going to include this measurement current. And by including this displacement current we can solve this issue. So, that part we will do in the next class. Thank you for your attention and see you in the next class.