Foundations of Classical Electrodynamics
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Lecture-62
Self and Mutual Inductance

Hello student to the course for foundation of classical electrodynamics. Under module 4 today we have lecture number 62, where we discuss self and mutual inductance.

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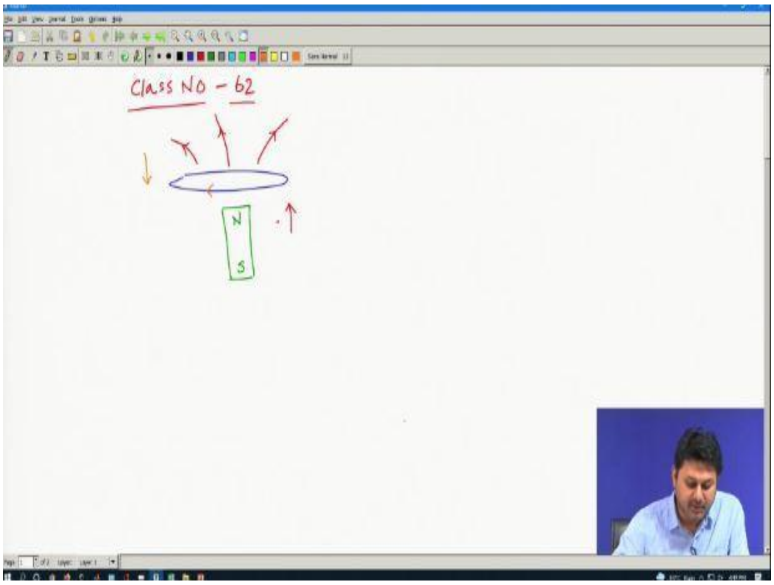

So, we have class number 62 today and in today's class we will be going to discuss the self and mutual inductance. But before that let us quickly discuss what we had so far. So, if we have a current loop like this and if we throw a bar magnet, then there will be the change of flux and this change of flux give rise to a current here. And this current will be such that it will go to oppose the motion of this bar magnet.

So, in this case the current should be such that, so, the magnetic field here in this direction due to the magnet. And the magnetic field due to the current should be in the opposite direction. So, that means I should have a current flow in this direction. So, they will be opposite, so, that you need to do some work. This is the Faraday’s law last day we discussed. So, then the EMF will be going to generate and this EMF is called the electromotive force.

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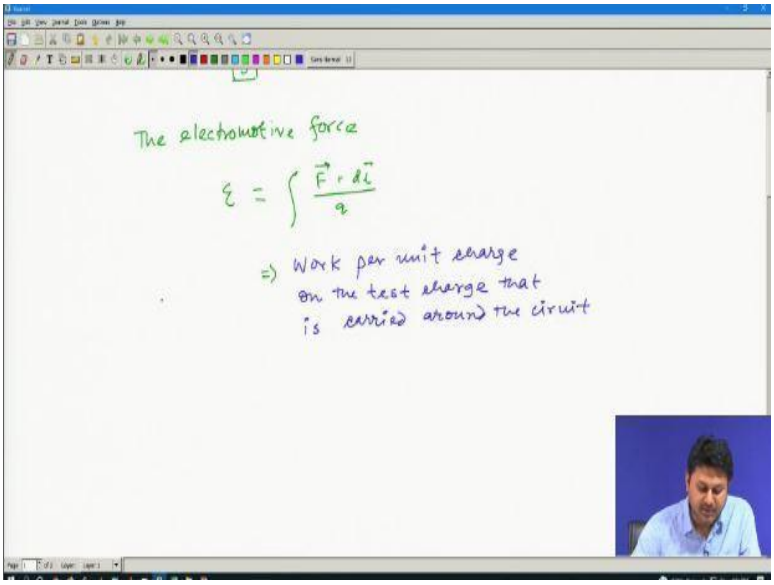

So, this electromotive force if I now calculate that this, the general description of this force is this it basically generates something called EMF in short. So, $\frac{\vec{F} \cdot d\vec{l}}{q}$. So, that means this is the amount of work per unit charge on the test charge that is carried around the circuit. So, around the circuit the force that basically the work done to move the test charge around the circuit is eventually and for per unit charge of that is eventually, the amount of the electromagnetic force and that is the way one can write.

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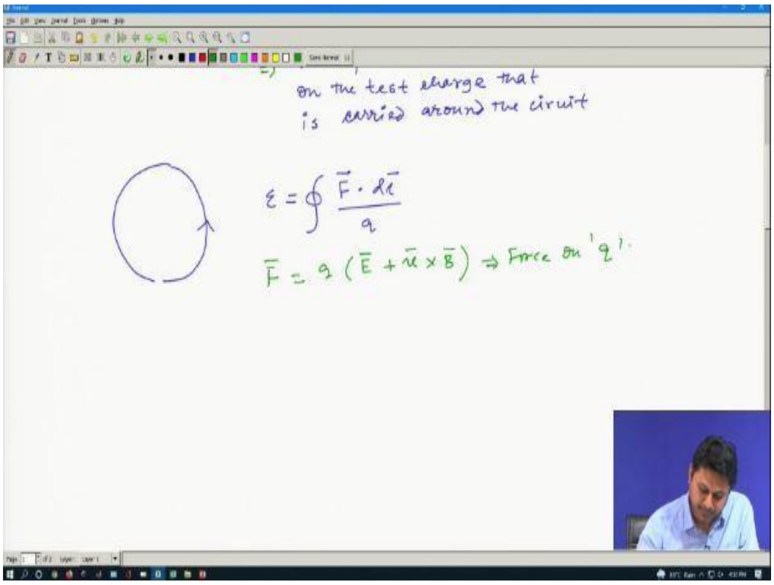

Now if the circuit is closed. So, the expression one can write is this, there should be a closed integral and I should have the same expression $\frac{\vec{F} \cdot d\vec{l}}{q}$, by the way, here $\vec{F}$ is the force experienced by the charge. So, this q multiplied by $(\vec{E} + \vec{v} \times \vec{B})$ that is the force on the charge q. Now if I want to find out what should be the EMF for the circuit, which is moving in?

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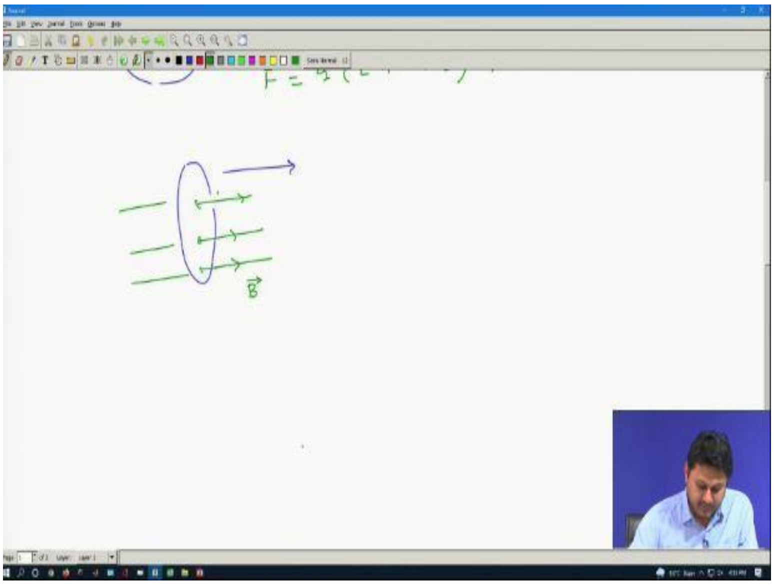

So, suppose I am having a circuit. So, that circuit is moving in an external electric field. So, these are the external electric fields. So, let me so in this external electric field it is moving.

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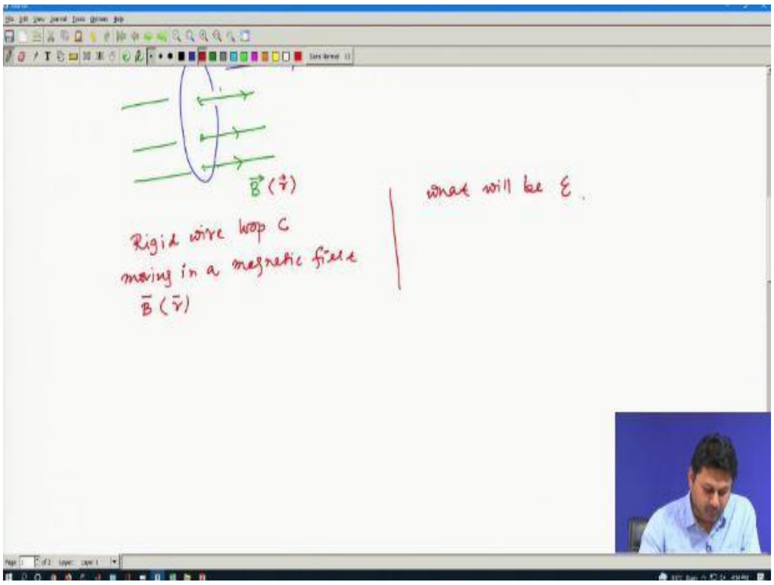

Then we should write here, that the rigid wire loop c moving in a magnetic field, which is not uniform, which is steady, but not uniform, which is not depend on time but maybe a function of $\vec{r}$. So, it is moving. So, the question is what will be the EMF? So, one expression that I just derive, so, that I will be going to exploit here.

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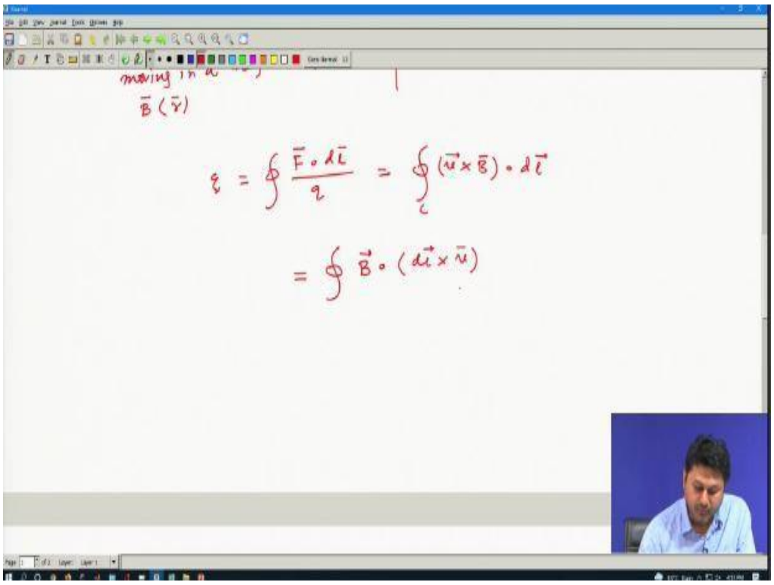

So, EMF should be $\oint \frac{\vec{F} \bullet \mathrm{d}\vec{l}}{q}$, $\vec{E}$ is not here. So, $\frac{\vec{F}}{q}$ I should write as $(\vec{v} \times \vec{B}) \bullet \mathrm{d}\vec{l}$. And then close line integral, so, that I can write in another way that $\oint \vec{B} \bullet (\mathrm{d}\vec{l} \times \vec{v})$. This is another expression to find out what is the EMF of this system. So, one application of this expression whatever, the expression I wrote is called the Faraday disk generator.

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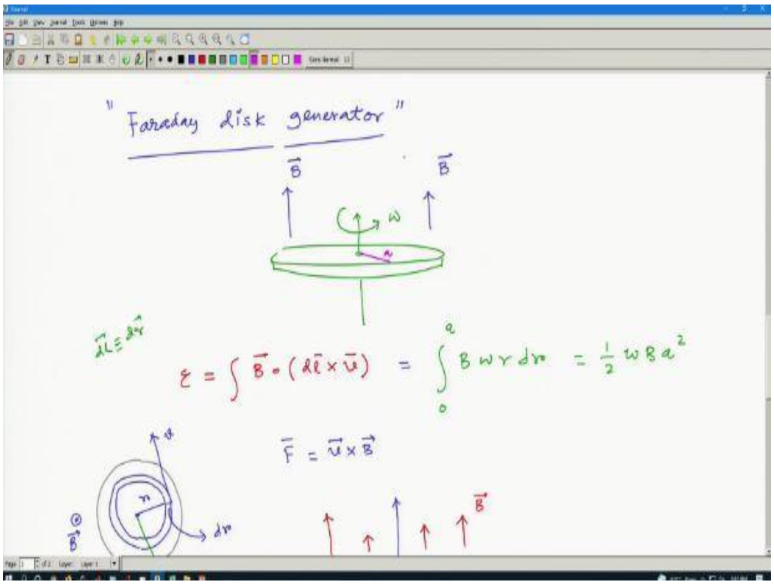

So, let me quickly discuss the principle of that. So, it is called Faraday disk generator. And here what we have? We have a disk like this. This is made of a conductor. And suppose it is rotating with a certain angular velocity ω okay. So, this is that, So, EMF that will going to generate here by the way I derived last day. In the last page that $\vec{E}$ is $\int \vec{B} \bullet (d\vec{l} \times \vec{v})$. So, now $\mathrm{d}\vec{l} \times \vec{v}$, so, if I make a top view of the system, then what we have is this is top view.

By the way, so it is rotating but I forgot to mention that it is in uniform it is rotating in some external magnetic field. So, magnetic field is already there. Obviously, the magnetic field should be there. And in the environment of this magnetic field this stuff is rotating. Now since, the magnetic field is there and it is a conducting material. So, it will go to the free electrons there will be going to experience some kind of force.

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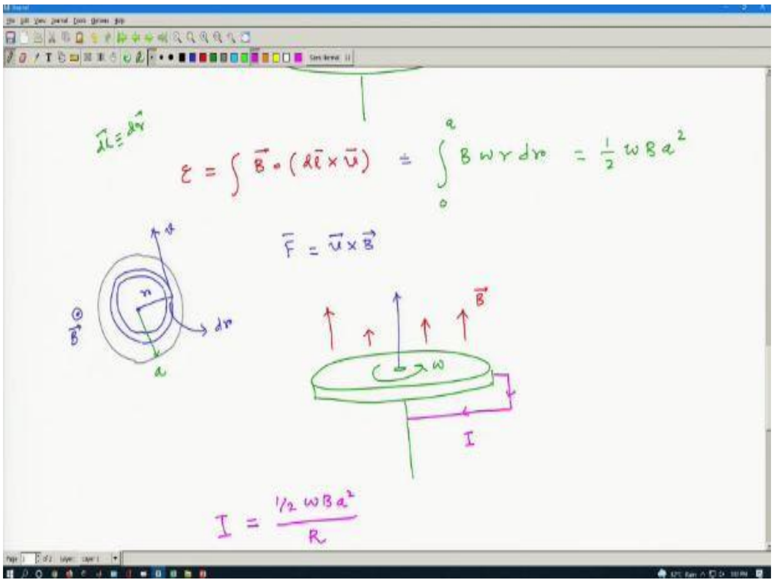

And that force makes this free electron the separation of this charge. So, if I make a top view you can see that so suppose, I have a small section here of the disk. And from here to here say it is r and this portion is dr. The velocity will be along this direction and magnetic field is perpendicular to the plane $\vec{B}$**.** So, now you can see that the $\vec{v} \times \vec{B}$. So, what is the force? The force here is $\vec{v} \times \vec{B}$.

And due to this force there will be separation of the charge. In the ring area and in the central area, so, there will be separation and that leads to the EMF. So, we need to calculate the EMF anyway we already have the expression of this EMF. So, from here we can readily calculate here d$\vec{l}$ is nothing but this d$\vec{r}$. So, I can simply have and they are perpendicular to each other. So, I can simply have and $\vec{B}$ is also perpendicular to the disk.

So, this expression simply gives us $\int B\omega r\, dr$. ω is an angular speed and I can replace this v to ω r and since this dl is equivalent to dr. So, I can simply replace this expression in this way. And if I integrate the value is $\frac{1}{2}$ ω B and then a^2 because, I need to integrate this stuff from 0 to

a. If a is a full radius. So, let me draw the system once again I have this conducting disk and it is rotating along its axis and the external magnetic field is there.

So, it is rotating in the environment of external magnetic field $\vec{B}$ and this is rotating like this way with angular frequency ω. Now what happened that it will already there is a separation of the charge and that leads to EMF. So, if I just have the circuit here if I complete the circuit if I join this part and this part, if I complete the circuit then there should be a flow of current.

And this current if I say I, then I will be simply whatever the EMF we have that is $\frac{1}{2}$ ω B a^2 and divided by the resistance R. So, this is the amount of current so if I increase the amount of current. Then what happened that I can increase the radius of this disk, which is a, say this is the area or I can increase the amount of the rotation. And under that environment, what we do? That we can able to generate some current.

So, this kind of system is called the Faraday disk generator. So, I just exploit the expression of EMF that I discuss and then exploiting that I can have the expression of the current. So, now let us move to today's topic, which is the self and mutual inductance. And we are going to calculate few things out of that. So, let us try to understand fact, what is the meaning of that.

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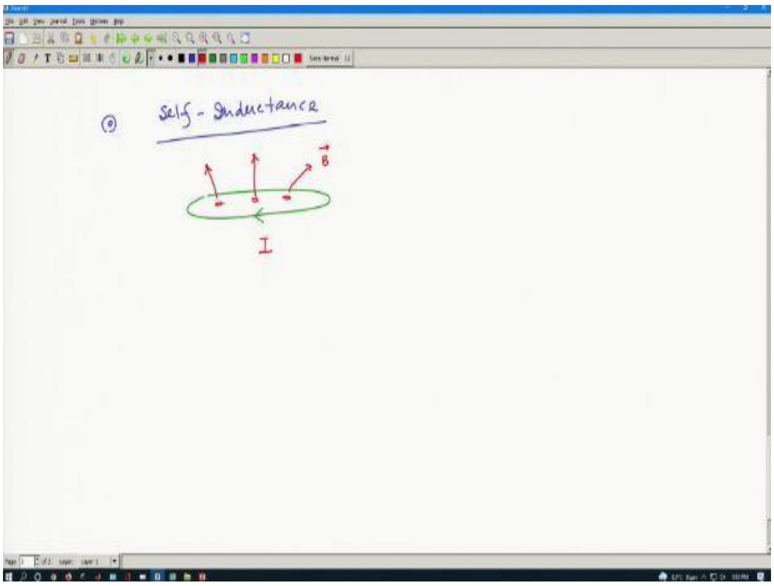

So, next we will discuss about the self inductance. What is self inductance? So, if I have a wire, where the current is flowing because, of the flow of the current, what happened it will produce some kind of magnetic field and that magnetic field will be going to cross it is this area. Because of the current that is flowing in this loop itself. So, this is the B we are having. Because of the flow of the current of the loop it will be going to generate some magnetic field and that magnetic field itself will be going to cross this. So, that means this current will going to link some kind of flux.

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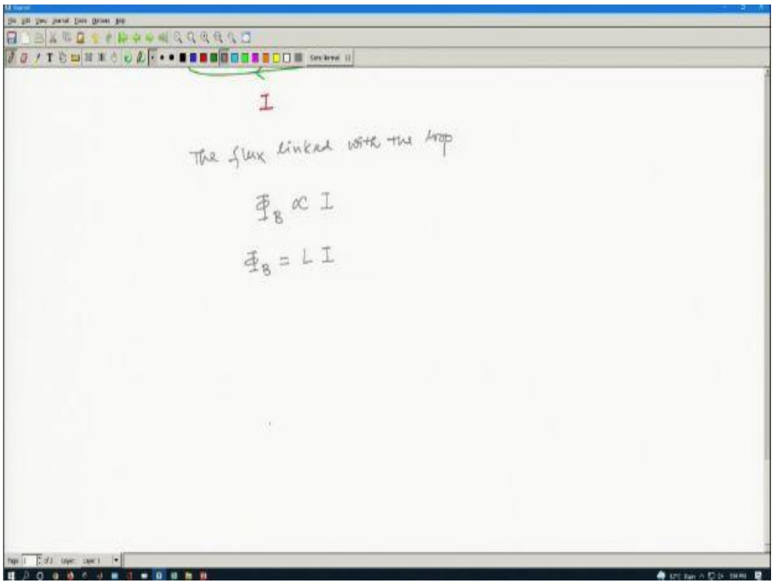

So, that the flux linked with the loop, the amount of flux that is linked with the loop is simply proportional to the amount of current. So, if I write this flux is ϕ_B. So, ϕ_B will be simply proportional to the amount of current that is flowing in the wire. And if I put the proportionality constant then ϕ_B is equal to the proportionality constant multiplied by I. This proportionality constant L is called the self inductance. This is called the self inductance.

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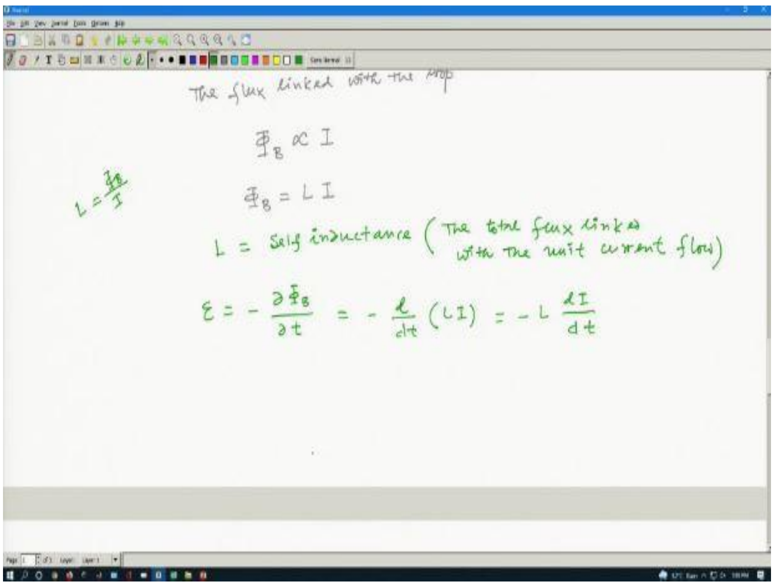

So, here the quantity L is called the self inductance. So, what is self inductance? The total flux linked with the unit current flow. The total flux ϕ_B linked with the unit current flow. So, L is eventually $\frac{\phi_B}{I}$. So, the total flux linked by unit current flow is defined as L, the self inductance. Now if I calculate this. So, we know that this EMF is $\frac{\partial \phi_B}{\partial t}$.

So, that quantity for self current is $-\frac{d}{dt}$ (LI) and we simply have $-L\frac{dI}{dt}$. Now this is the way we define now we can calculate what is the self inductance for a solenoid? This is the standard problem. So, what is then maybe we can have some idea.

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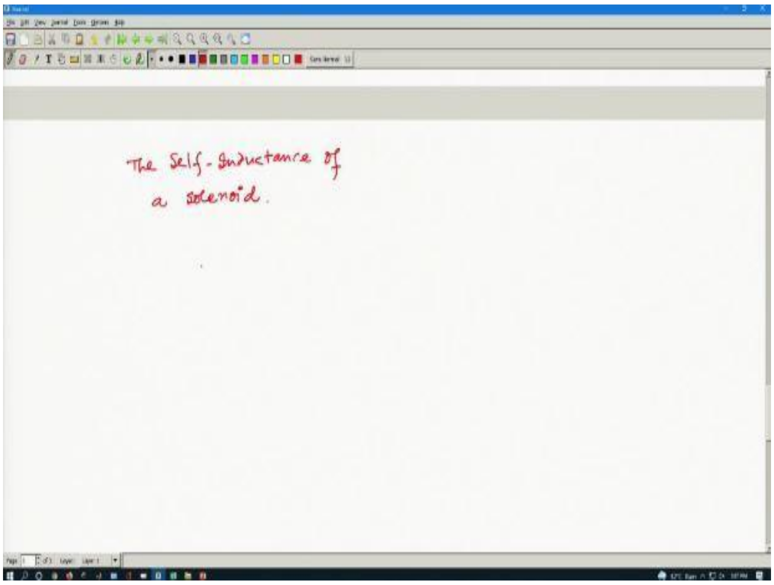

So, now we calculate the self inductance of a solenoid. So, for solenoid, so we know that what is quickly if I draw the figure here.

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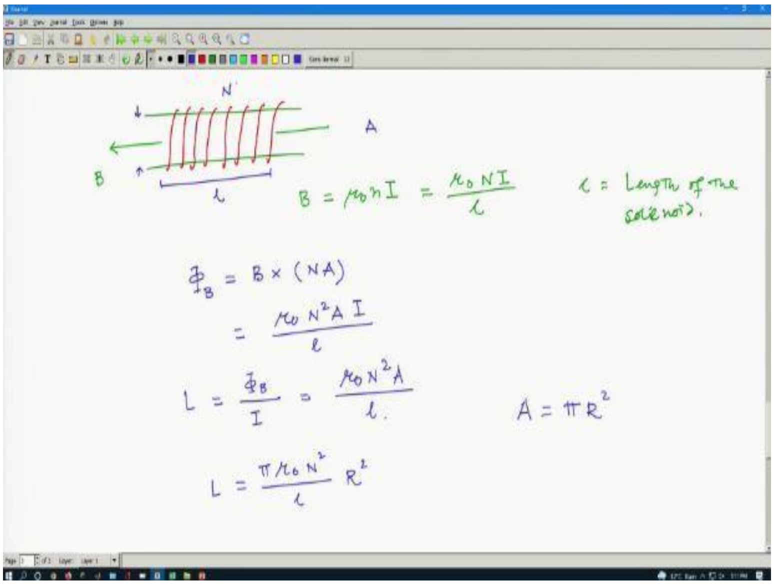

So we have the current loop, the current loop like this. So, the magnetic field that is produced in this core region this magnetic field that is produced the magnitude of that is μ_0 n I or if I write in the with the total number of turns then it should be μ_0 total number of turn multiplied by I divided by the total distance l. l is a total length of the so this is the length of the solenoid.

So, what is the amount of flux linked with this magnetic field? So, the magnetic flux that is linked is simply the B multiplied by the area. And the area is NA. A is the area of the single loop and N is a total number of loop. So, it should be N multiplied by A. So, that gives rise to μ_0 then N^2 A and then $\frac{I}{l}$. Now the mutual inductance if I want to calculate for this, then l will be simply $\frac{\Phi_B}{I}$.

So, this quantity is simply $\frac{\mu_0 N^2 A}{l}$. So, A is again I can write in π say R^2, if R is the radius of this solenoid in some cases. So, I can write it is μ_0 N^2 I have a π here divided by l and R^2. So, in many labs these experiments are there. So, the student need to calculate what is the dependence, what is the dependence of the self inductance with the parameter of the given solenoid that like the l is given. Suppose the value of the l is given and the area they can calculate.

This is the area, area is say A and total number of turn is N. So, this is the way it can depend. So, it inversely proportional to the length that means, if the length is short then the self inductance should be high. It is proportional to the value of the square of the number of turns in number term increase then it will going to increase and it is also square the radius. Another thing we have silts we have self inductance it is quite common that we should also have something called mutual inductance.

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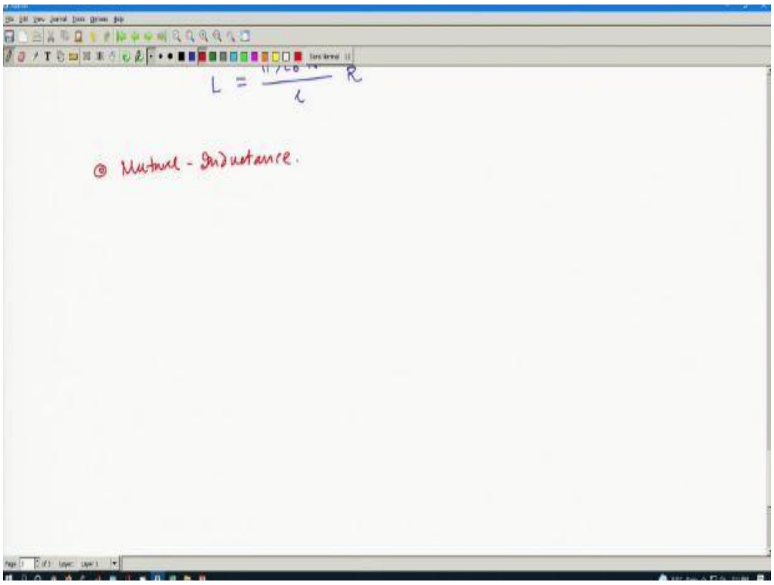

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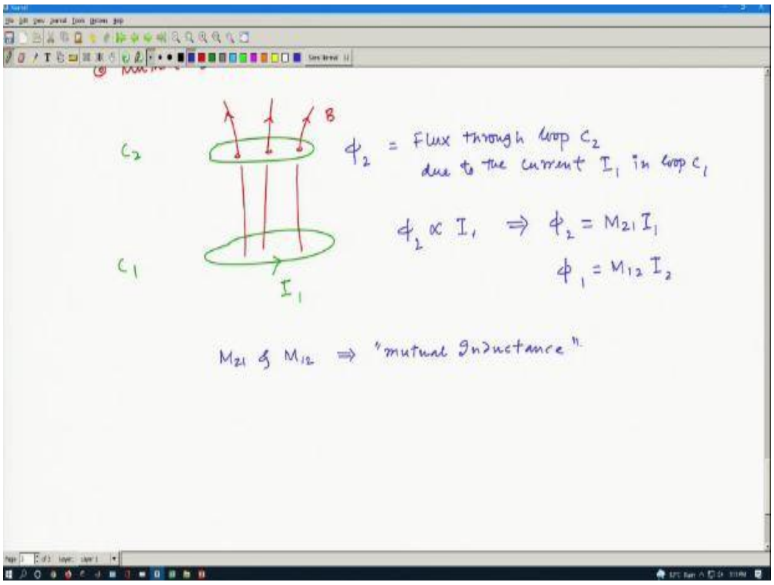

So, unlike one circuit here we have 2 closed loop one is say here, another is say here. So, now what happened that for this loop I can have a current flow I_1. So, this is loop c_1 say and another loop says c_2. So, I am having a current flow. So, I need to draw this loop properly and because of the flow current flow I_1. Because of the flow of the current we have a magnetic field that is produced. And this magnetic field will be going to cut the other wire or the closed loop like this.

So, it will going to cut, but the magnetic field that is produced by the other loop, so, ϕ_2 is a flux. So, if I now define, the amount of flux that is here is ϕ_2. This is the flux through loop c_2. Due to the current I_1 in loop c_1 that is. So, ϕ_2 again will be proportional to I_1 and that gives rise to proportionality constant. And I can write that $\phi_2 = M_{21} I_1$**.** In the similar way, I can have $\phi_1 = M_{12} I_2$, that is the amount of suppose I am now the current is flowing in c_2 and because of that some flux will be there in c_1.

And that flux if I write ϕ_1 that should be proportional to I_2 and I the proportionality constant should be M_{12} in this case. Now this M_{21} or M_{12} this is the constant quantity and this is called the mutual inductance.

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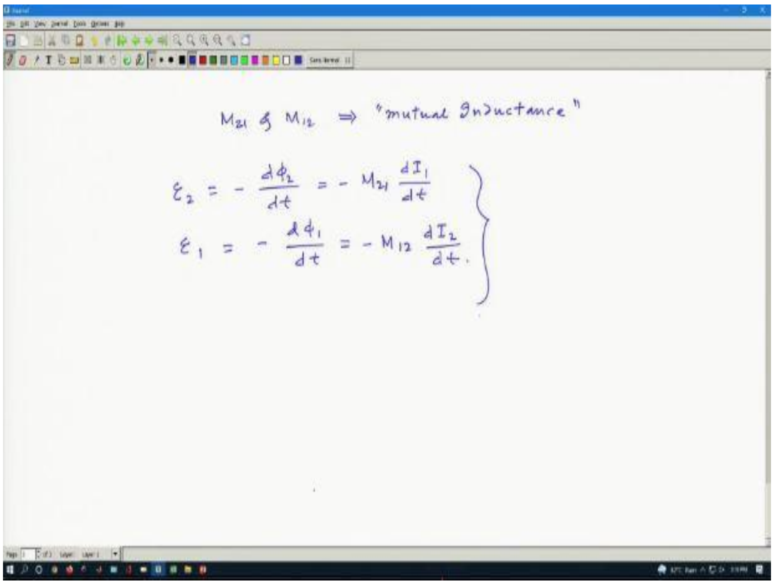

So, the E_2 that is generated now in loop 2 is $-\frac{d\phi_2}{dt}$ that is $-M_{21}\frac{dI_1}{dt}$. And E_1, that is the EMF generated in loop 1 that is due to the current I_2. So, I can have this is equal to $-M_{12}\frac{dI_2}{dt}$**.** So, after having the rough idea of mutual inductance now we are going to calculate something, which is called the Neumann's formula. That how they are related and under what condition and we show some interesting thing there.

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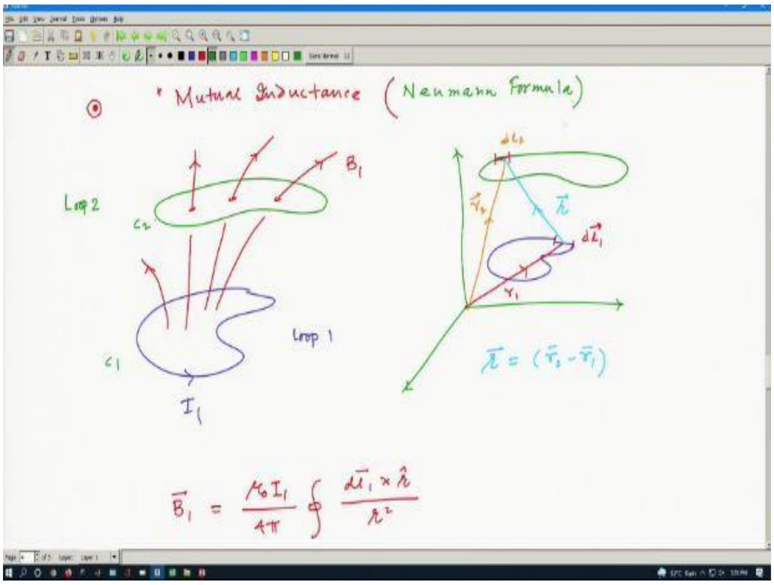

So, next what we do is calculate the mutual inductance again, but in a different way. And the formula is called the Neumann's formula. What was that? Suppose we have two arbitrary loop one is like this and another is maybe I can do some different colour another. So, these are two arbitrary loops. And the current that is flowing in this is I_1 and this is called this is loop 1 and this is loop 2. Current is flowing here in one.

And it produces some magnetic field and that magnetic field the field line may be like this. It will be going to cross here this. I am going to cut the loop 1. The amount of magnetic field that is producing is say B_1 magnitude**.** So, now we know what is B_1 because, the current flow is this loop and at some distance what is the B_1 that we know. So, let me first the systematically draw these things. So, I had a loop here.

So, this is my coordinate system. And suppose I had the loop here, this is my loop 1 and my loop 2 something is like this is sitting here. So, these are the 2 loops. So, this is the coordinate system. So, the small section here if I draw say from here to here and that length from here to here is say r_1 and this is this length I called dl_1. And this is another section, which we call say different. So, this section is say dl_2. And the coordinate of dl_2 say r_2.

And the distance between d1 and dl this length is our notation this Л. Where $\vec{Л}$ is $\vec{r}_2$ - $\vec{r}_1$ that is the system we are having. Now I know what is my $\vec{B}_1$? Let us write straight way $\vec{B}_1$ is $\frac{\mu_0 I_1}{4\pi}$ integration close loop. So, I have $\oint \frac{d\vec{l}_1 \times \hat{Л}}{Л^2}$. This is the magnetic field due to loop 1 at some point and this some point is here**.**

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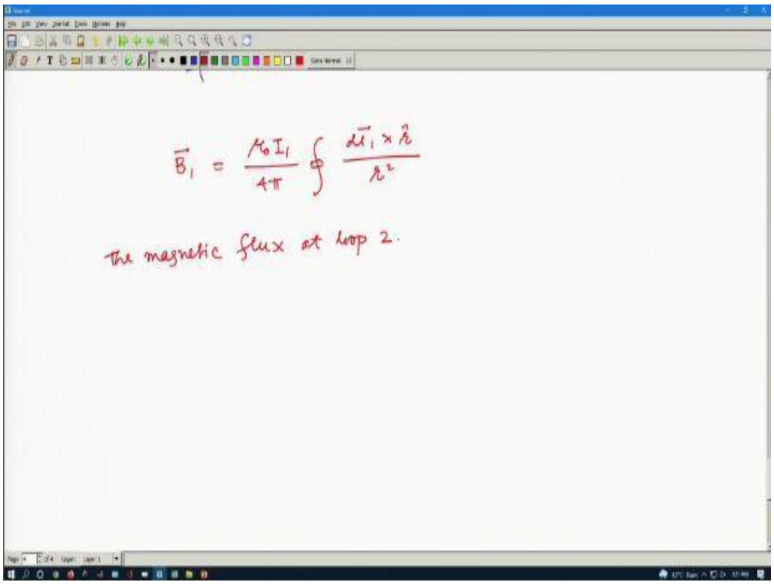

So, magnetic flux at loop 2 is how much?

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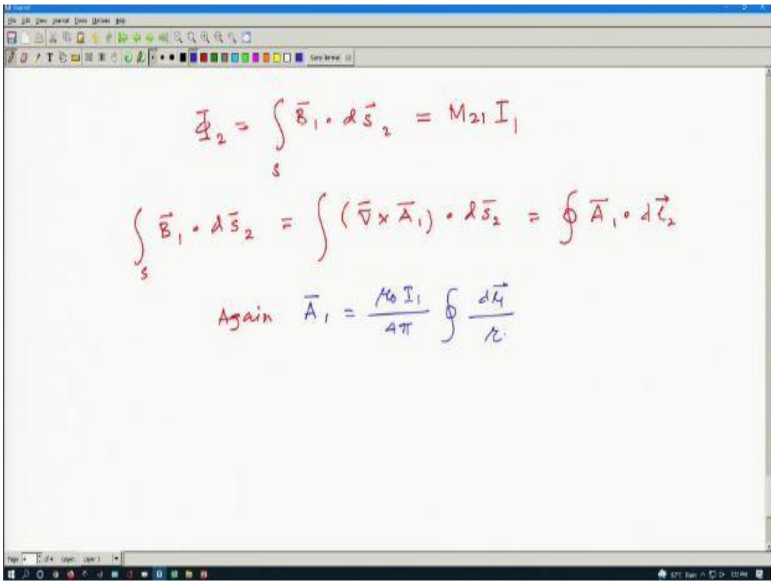

It is ϕ_2 equal to $\int \vec{B}_1 \bullet d\vec{s}_2$ and that quantity we know is $M_{21}\ I_1$. So, what is the surface $\int \vec{B}_1 \bullet d\vec{s}_2$ that I can write in terms of the vector potential $\vec{A}$. And let us write in this way dot $d\vec{s}_2$ that quantity is simply close line integral, because I am using this close line integral $\vec{A}_1$ Stoke's theorem $d\vec{l}_2$. Again I have $\vec{A}_1$ is $\frac{\mu_0 I_1}{4\pi}$ that we also calculated. This is a close line, so, $\oint \frac{d\vec{l}_1}{r}$.

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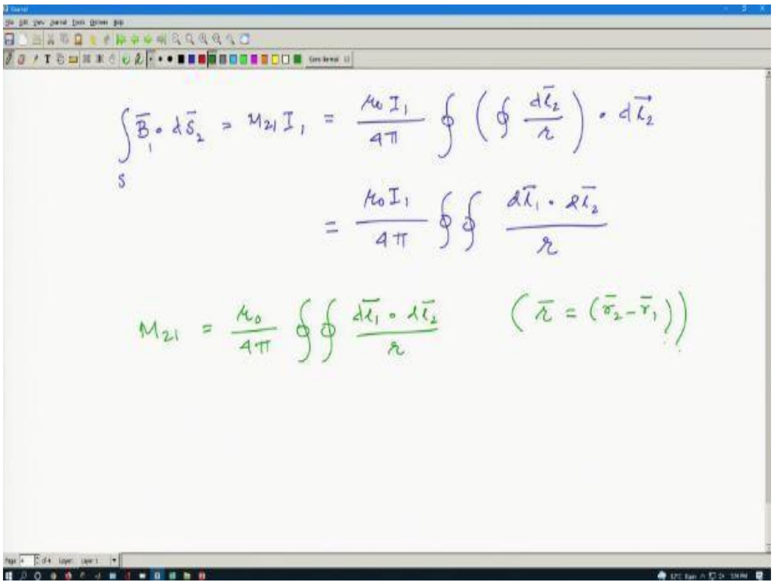

So, I can have like s $\vec{B}_1 \bullet d\vec{s}_2$, which is M_{21} I_1 is $\frac{\mu_0 I_1}{4\pi}$ and then we have a close integral and then a I just simply put, which is like $\oint \frac{d\vec{l}_1}{Л} \bullet \ d\vec{l}_2$. So, this quantity I simply write $\frac{\mu_0 I_1}{4\pi}$ and then I have this $\oint\oint \frac{d\vec{l}_1 \bullet d\vec{l}_2}{Л}$. So, from here if I compare left-hand side and right-hand side we have an interesting expression $M_{21} = \frac{\mu_0}{4\pi}$ and $\oint\oint \frac{d\vec{l}_1 \bullet d\vec{l}_2}{Л}$. Mind it, $\vec{Л}$ is $\vec{r}_2$ - $\vec{r}_1$.

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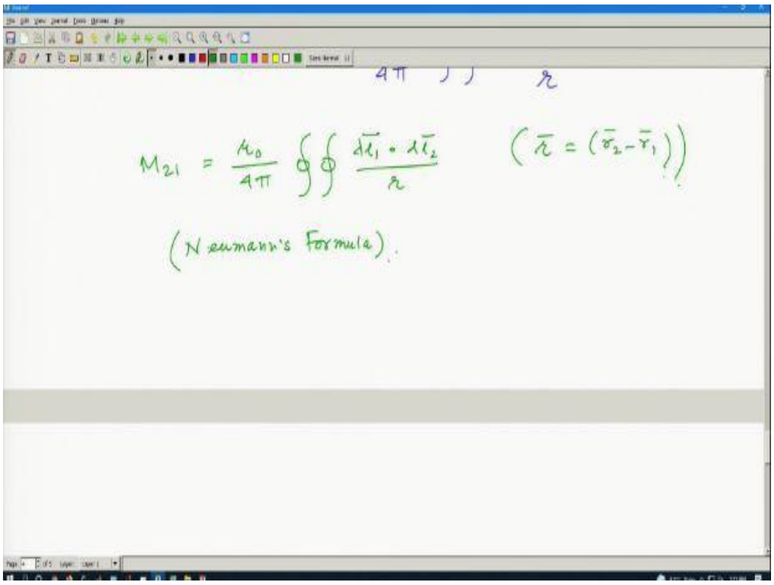

Now if I calculate M, so this is the Neumann's formula. How to calculate this M_{21}, so, this expression is eventually called Neumann's formula or Neumann formula.

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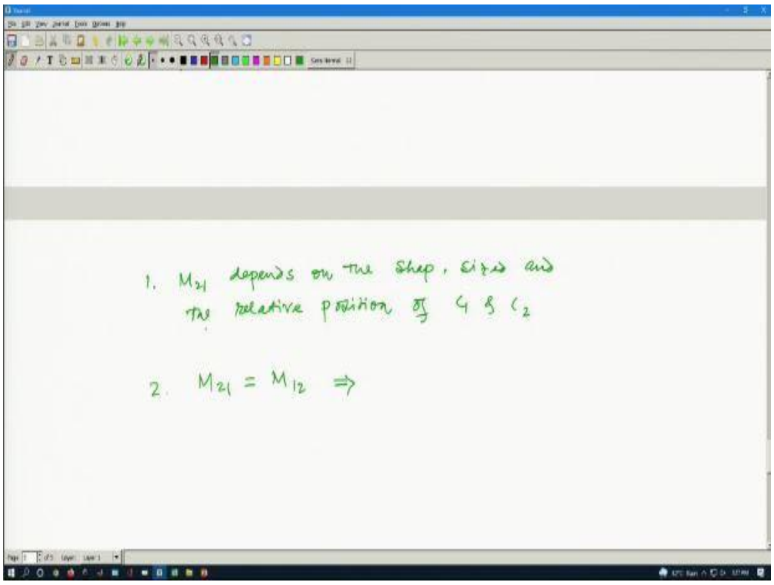

Now few interpretation I can make 1 M_{21} depends on the shape, size and the relative position because, you can see the relative position r is sitting. So, how the 2 loops are distributed and how what is the relative difference relative distance between these 2 loops are important to calculate M_{21}. So, M_{21} and the relative position of the 2 loops depends on shape, size and the relative position of c_1 and c_2. c_1 and c_2 are 2 loops.

Say this is say c_1 and this is c_2**.** And another thing also that M_{21} should be equal to M_{12} because, if you just replace these in this expression if you just replace 1 2 to 2 1 right-hand side will not going to change. So, $M_{21} = M_{12}$, that eventually tells that whatever the shape and position of the loop. The flux through loop 2 due to the current I in loop 1 is same as the flux through the loop 1 due to the same current passing through loop 2.

So, what happened that if I have a current flowing here. So, whatever it is producing whatever the field $\vec{B}$ in the similar way if you have the current the same current that is flowing here I, whatever the field produce both the cases. The value of the corresponding mutual inductance should be same. So, that we find from this expression**.** Now after having these we will try to find another important thing that we will do today itself.

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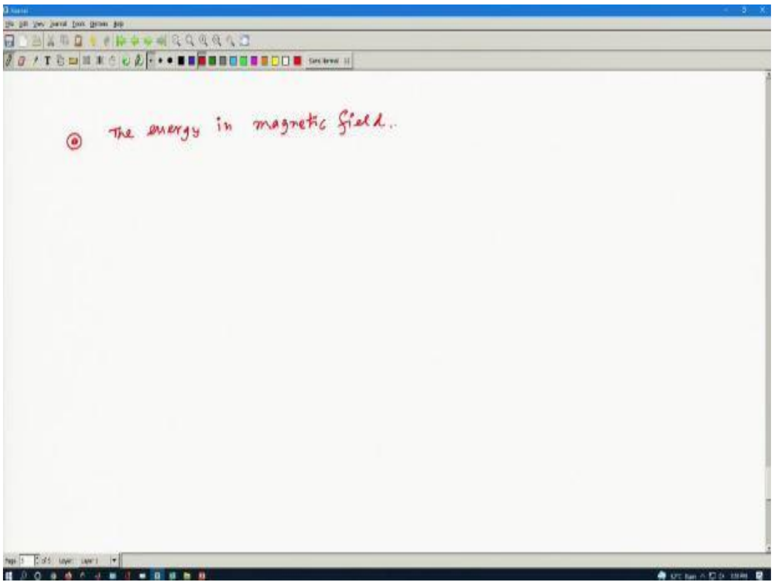

And that is the energy; the next thing that we discussed today is the energy in magnetic field. What is the energy magnetic field, how to calculate the energy in the magnetic field? So, we will do in this way.

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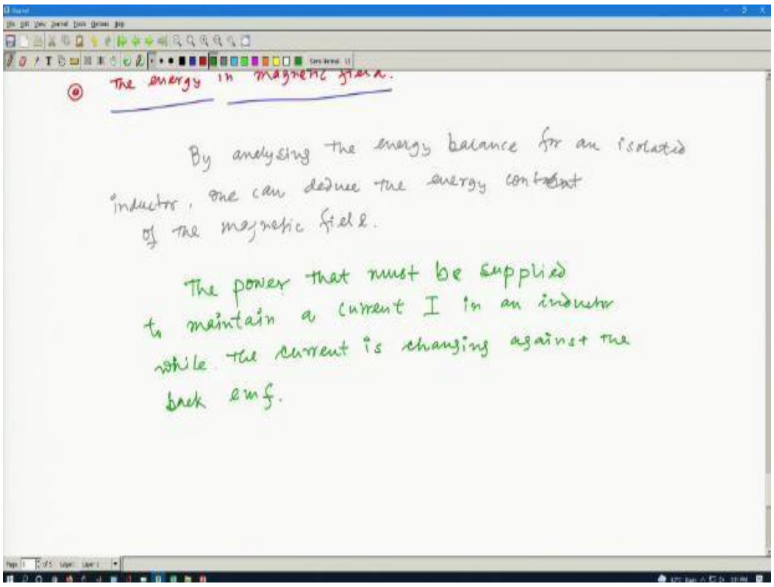

We one can calculate by you know analyzing the energy balance for an isolated inductor; one can deduce the energy content of the magnetic field. What kind of energy balance we will going to discuss. So**,** if you look carefully, the power that must be supplied to maintain a current say I in an inductor while, the current is changing against the back EMF. So, some amount of power must be supplied to maintain the flow of the current because, there will be a back EMF because, we have isolated inductor.

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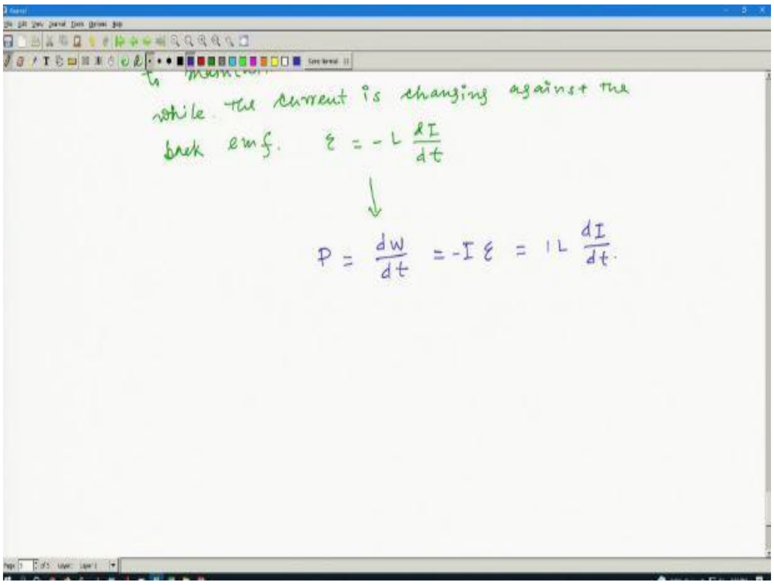

And this back EMF simply E = -L $\frac{dI}{dt}$. And the amount of power that we need to work with is this power the rate of change of work done and that is equal to the current with a negative sign obviously and the EMF, that quantity because, EMF is this is I L $\frac{dI}{dt}$ because, EMF is this quantity. So, this amount of power in order to balance that, so, the flow of the current in an inductor.

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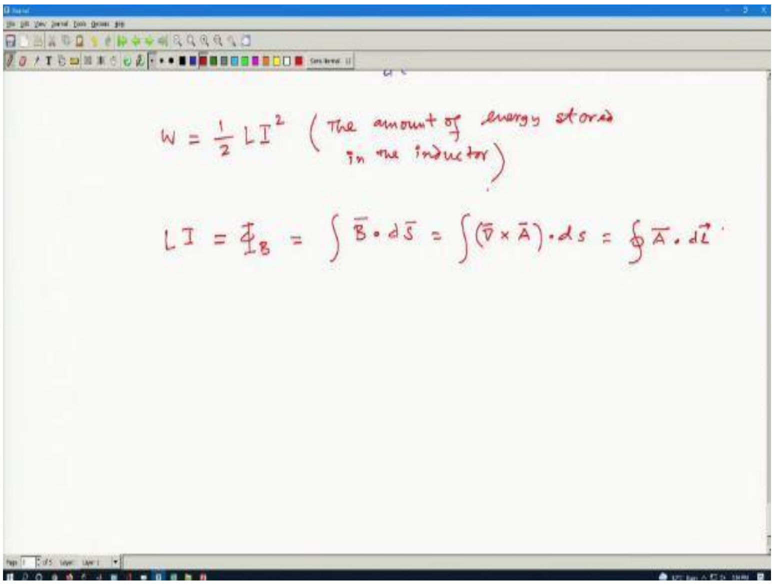

So, you can see that w can come out to be from here w amount of work done is simply $\frac{1}{2}$ of if I integrate this side with respect to t. Then it is $\frac{1}{2}$ LI^2. So, this is the amount of energy that is eventually stored in the inductor. So, I should write, this is the amount of energy stored in the inductor. Now I have LI is equal to the flux ϕ_B and that quantity I write in terms of $\vec{B}$ like $\vec{B} \cdot d\vec{s}$.

And again $\vec{B} \cdot d\vec{s}$ I can write in terms of the vector potential $\vec{A}$ like $(\vec{\nabla} \times \vec{A}) \cdot d\vec{s}$. And this flux eventually I can write in terms of $\vec{A}$ like $\vec{A} \cdot d\vec{l}$. So, this we have done earlier how to calculate the magnetic flux in terms of vector potential. This calculation we did it. So, this steps.

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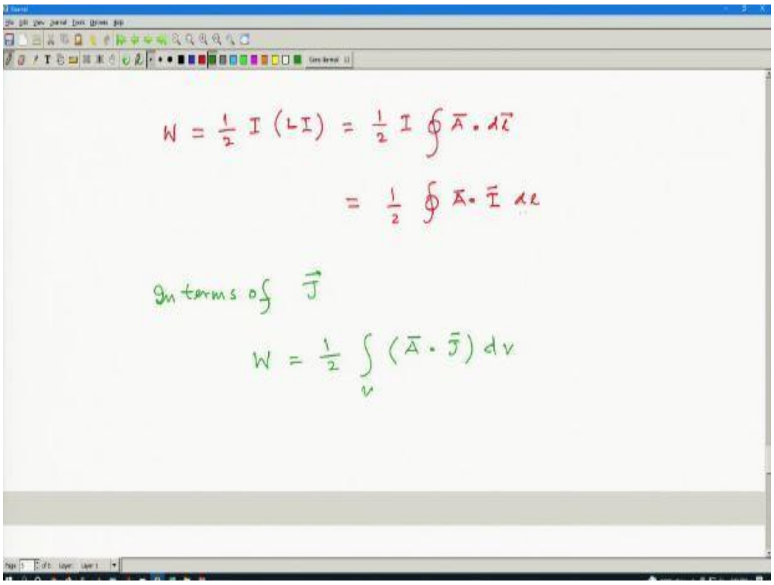

So, w the work done should be $\frac{1}{2}$ I can write it like I into LI because, LI I already calculated. So, it should be $\frac{1}{2}$ I and LI I write $\oint \vec{A} \cdot d\vec{l}$. So, this quantity I can put this I inside and I can write it that $\oint \vec{A} \cdot \vec{I}$ make it vector and $d\vec{l}$ because, the vector in the same direction that of I and L. So, the generation this is the expression of the work.

So, I can make it in terms of the volume current and in terms of volume current I can have this expression simply half of integral. Now the integral will be the volume integral and that is $\vec{A} \cdot \vec{J}$, this I just make it in volume current and dv. It is line current I can just simply make it volume current.

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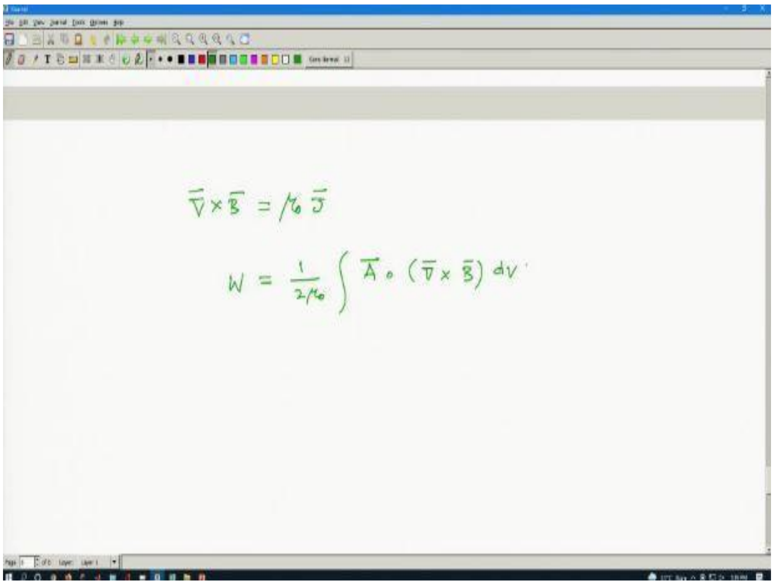

And now I will be going to exploit the expression $\vec{\nabla} \times \vec{B}$ is $\mu_0 \vec{J}$. So, w the work store is $\frac{1}{2\mu_0}$ just replace this there in place of $\vec{J}$ and I should have something like $\vec{A} \cdot (\vec{\nabla} \times \vec{B})$ then dv.

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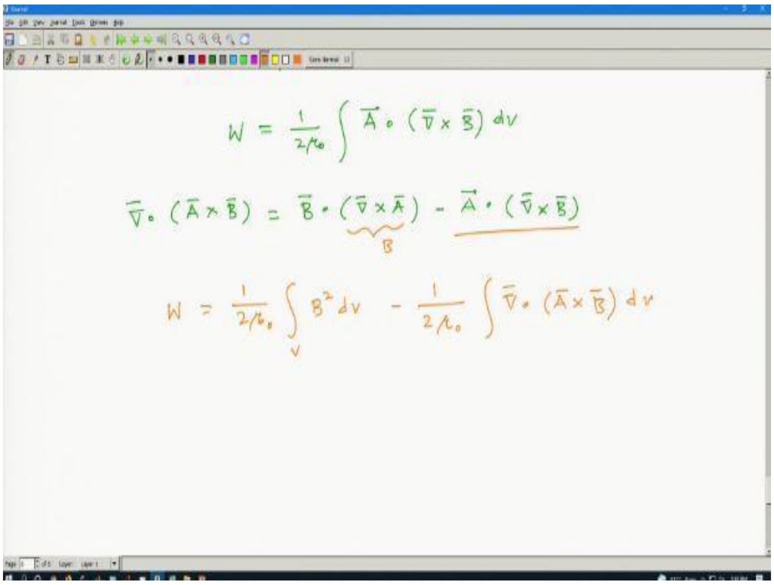

Now I exploit the vector identity this kind of treatment we have done several time in many calculation in electrostatic and magnetostatic. So, this is a usual practice now to exploit, this vector expressions and from that like we can find something extra something, which is having some physical meaning. So, I can use this $\vec{A} \cdot (\vec{\nabla} \times \vec{B})$ here. We have this term here $\vec{A} \cdot (\vec{\nabla} \times \vec{B})$.

So, I replace it here and that basically gives me w = $\frac{1}{2\mu_0}$. One term I am having is $\vec{B} \bullet (\vec{\nabla} \times \vec{A})$. So, $\vec{\nabla} \times \vec{A}$ is itself $\vec{B}$. So, I am having a term like volume $\int B^2$ dv and another term is $-\frac{1}{2\mu_0}$ and then I have this $\vec{\nabla} \bullet (\vec{A} \times \vec{B})$, $\vec{A} \times \vec{B}$ interesting term over dv.

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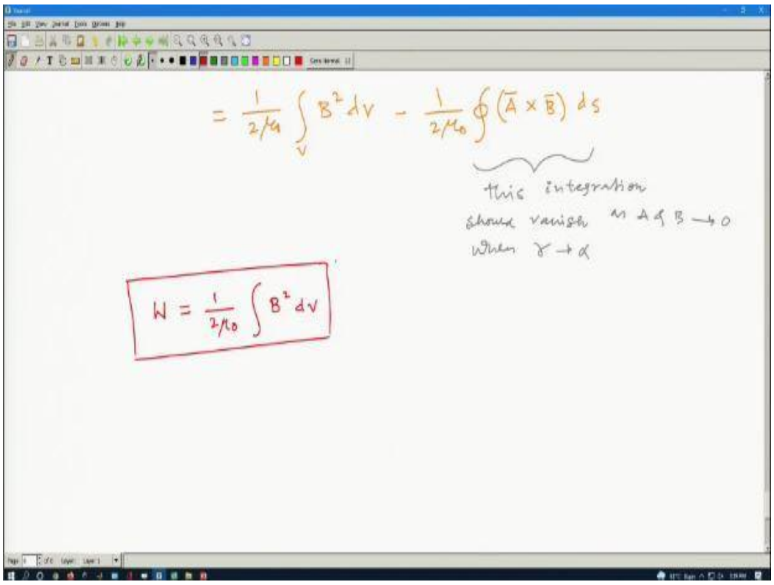

Now this $\vec{\nabla} \bullet (\vec{A} \times \vec{B})$ I can write in terms of so let me write the first part, this is fine. And the next part, I can write $\frac{1}{\mu_0}$ and the closed surface integral. And then I should have $\vec{A} \times \vec{B}$ over the surface. Now here you can see this is the closed surface integral. So, I can take my surface according to my convenience and this quantity $\vec{A} \times \vec{B}$ both are having the dependency r. One is $\frac{1}{r}$ and one is $\frac{1}{r^2}$.

So, that means this quantity is going to vanish if the surface is very large. So, this integration should vanish as A and B both goes to 0, when r tends to infinity. For very large surface this will vanish. So, eventually the amount of energy that is stored in this system should be or in other way the energy of the magnetic field is simply this quantity.

This is a very important expression we will go to exploit this expression again, when we deal with electromagnetic theory and electromagnetic energy density. Well, so, I do not have much time today. So, let me conclude here. Thank you very much for attention and see you in the next class.