Foundations of Classical Electrodynamics
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology-Kharagpur

Lecture-56
Magnetic Dipole Moment

Hello student to the foundation of classical electrodynamics course. So, under module 3, today we have lecture 56. And today we will be going to discuss about the magnetic dipole moment.

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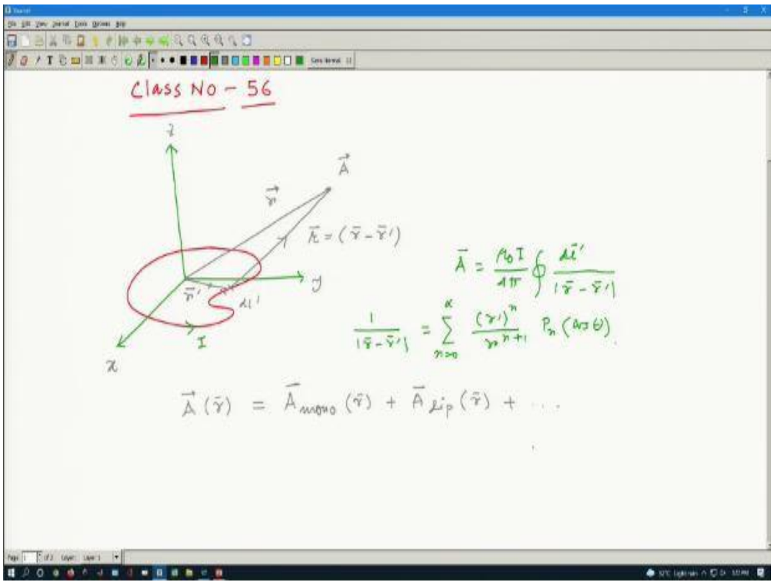

So, we have today class number 56. So, let me quickly remind what we did in the last class. That we had try to find out the contribution of so this is the coordinate system and we had a current carrying loop. And try to find out what is the vector potential $\vec{A}$ due to the current carrying loop and we figure out then that was the structure. So, let us put it as x, y and z. This is at some distance $\vec{r}$ from the origin.

And then actually we expanded. So, if I have a small section here say d$\vec{l}$' and where the d$\vec{l}$', this is $\vec{r}$'. So, from here to here, it was our $\vec{\mathcal{R}}$, which is $\vec{r}$ - $\vec{r}$'. And when we figure out the value of $\vec{A}$ there at point $\vec{r}$ it was the contribution of $\vec{A}_{\text{monopole}}$ contribution and then we have a contribution of dipole and also the higher order terms are there.

So, the monopole contribution so, $\vec{A}$ we started with the expression of $\vec{A}$ with say $\frac{\mu_0 I}{4\pi}$ using the line current and then d$\vec{l}$' and this is $\vec{r}$ - $\vec{r}$' and that was for close line. So, there was a close line integral, because this is a closed loop where we are having the amount of current I. And that 1 by this term actually we expanded. So, $\frac{1}{|\vec{r}-\vec{r}'|}$ that term we expanded.

And that is say n equal to 0 to infinity $\frac{(r')^n}{r^{n+1}}$ and then we had the Legendre polynomials. So, we used it, we derived it during the multipole expansion problems in electrostatics the same thing here. Then the monopole contribution, if I write down.

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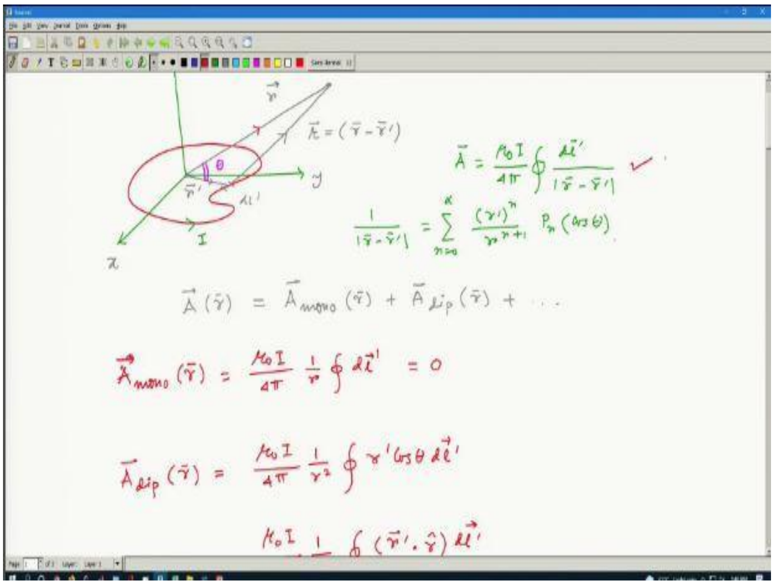

The monopole contribution $\vec{A}_{\text{monopole}}$ that was $\frac{\mu_0 I}{4\pi}$ and then I had $\frac{1}{r} \oint d\vec{l}'$ and that term was 0, because if you have a close line integral over this $d\vec{l}'$ it has to be 0. So, what about the dipole contribution? Dipole contribution the expression was $\frac{\mu_0 I}{4\pi}$ and then we had $\frac{1}{r^2}$ and then this is r' and then cos θ $d\vec{l}'$; that was the expression. And then we mentioned that this term actually this r cos θ so if I consider this angle to be θ.

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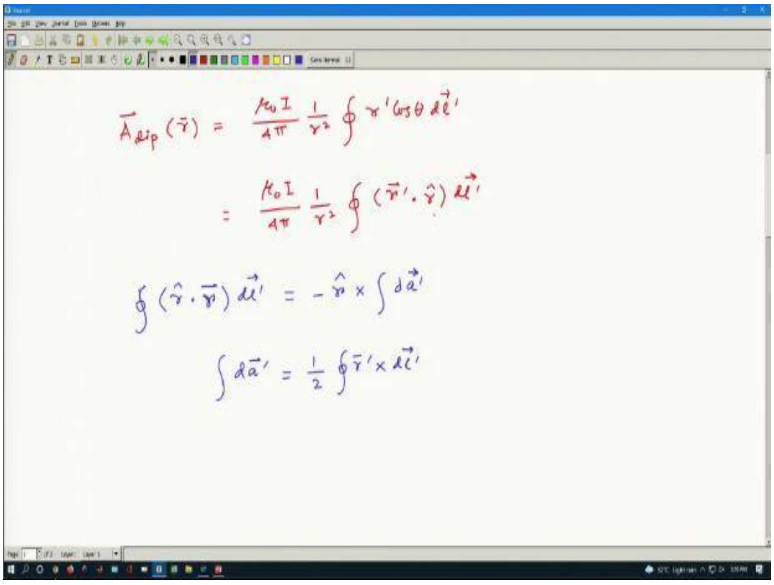

Then what we can get from here is this $\frac{\mu_0 I}{4\pi}\frac{1}{r^2}$ and then I can write it as $\vec{r}' \bullet \hat{r}$ and then $d\vec{l}'$. So, I just absorb this cos θ by putting the dot product between $\vec{r}'$ and $\hat{r}$, which one can explain from here this figure. But after that we did one thing and that is this quantity dot $\hat{r}$ we can write it as $\hat{r} \times \int d\vec{a}'$. Where this integration of $d\vec{a}'$ was $\frac{1}{2}\oint \vec{r}' \times d\vec{l}'$. Last day we just use that and eventually figure out. We used it what is written in blue line.

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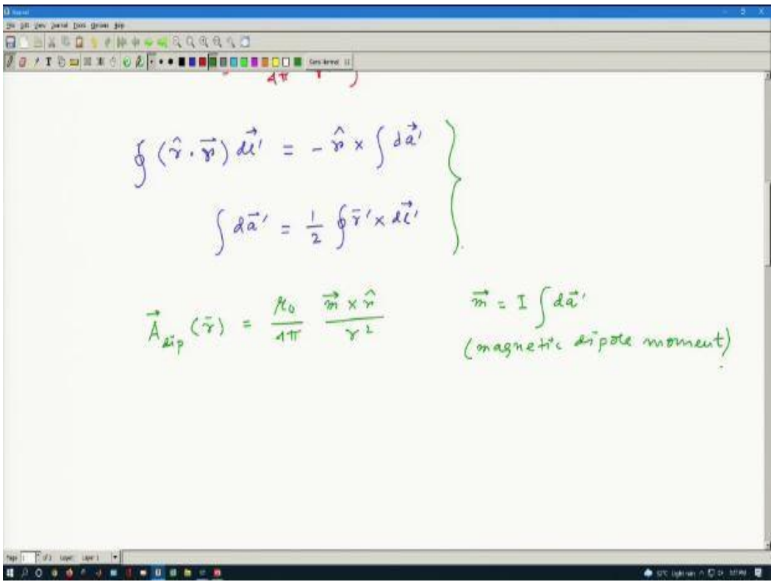

And then finally we got the expression of the dipole the vector potential due to the dipole contribution is something like $\frac{\mu_0}{4\pi}$ and then a term called $\vec{m}$, which is our magnetic dipole moment and r^2, where $\vec{m}$ was defined as current into area. This term is our magnetic dipole moment**.** But this thing we did not prove actually, we just stated that. So, let us try to prove it.

In Griffith's book, I guess there is a problem and if you solve this problem then eventually you prove that. So, let me try to do this.

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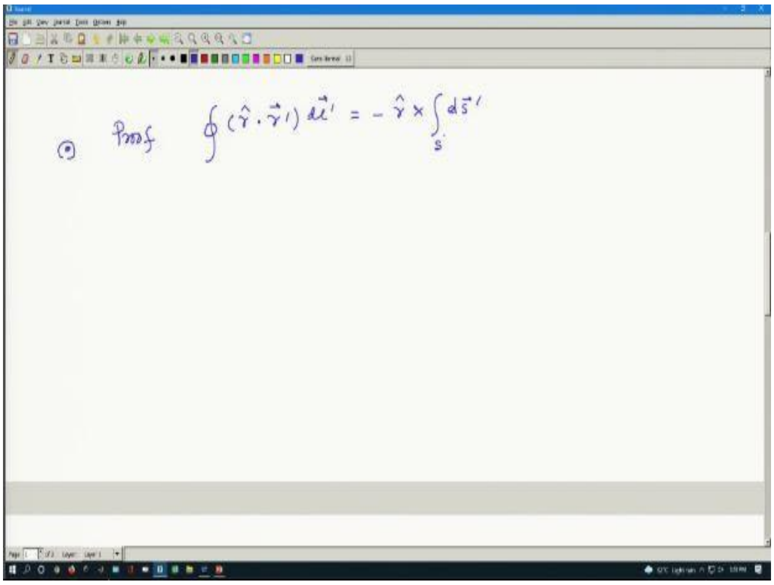

So, our goal here is to the prove that close line integral of $\hat{r} \bullet \vec{r}'$ and then $\mathrm{d}\vec{l}'$ that is equal to -$\hat{r} \times \mathrm{d}\vec{a}$ or $\mathrm{d}\vec{s}'$ with s surface integral. So, that I need to prove and let us start with the Stoke's theorem.

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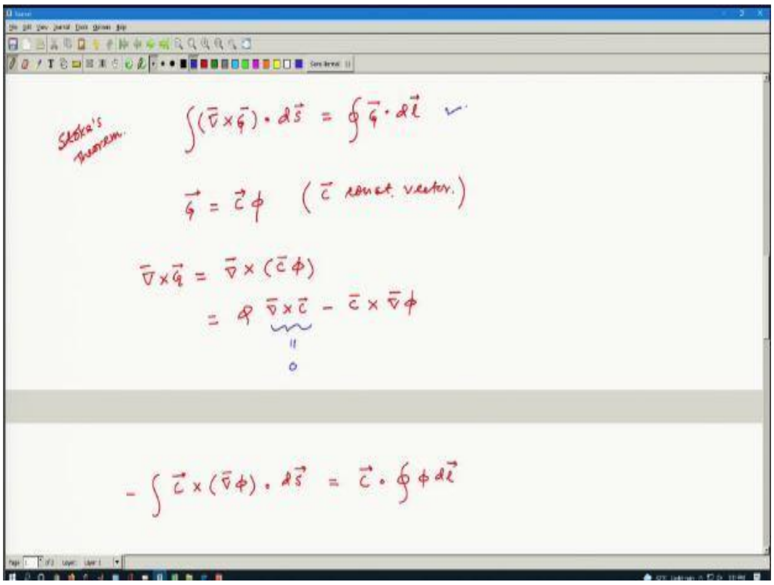

So, Stoke's theorem is saying that the curl of a vector field $\vec{A}$ over the surface is simply the closed line integral of $\vec{A}$ over this. This is the Stoke's theorem. Now let us take this vector field $\vec{A}$ this is any arbitrary vector field is not $\vec{A}$ here does not mean that it is the vector potential.

This is any arbitrary vector field. So, better I just put a different name to you know the confusion.

So, let us put this as something any vector field says $\vec{G}$. So, $\vec{G}$ is here say some constant vector and a scalar field ϕ, C is a constant vector. Now if I make $\vec{\nabla} \times \vec{G}$ what we get. We get $\vec{\nabla} \times \vec{C}$ and then ϕ. That is simply ϕ and then $\vec{\nabla} \times \vec{C}$ - $\vec{C} \times \vec{\nabla}\phi$. But this quantity should be 0, because C is constant. So, this will be 0. So, eventually we have this. So, so $\vec{\nabla} \times \vec{G}$ whatever is here I will put this side.

So, in the Stoke's theorem. So, what we get is this $-\int \vec{C} \times (\vec{\nabla}\phi) \bullet d\vec{s}$ is $\vec{C} \bullet \oint \phi \, d\vec{l}$. So, I just put this $\vec{\nabla} \times \vec{G}$ here in this equation. And $\vec{G}$ is also $\vec{C}\,\phi$. So, $\vec{C}$ then I take outside. So, in this Stoke's equation here I just put the value whatever I get.

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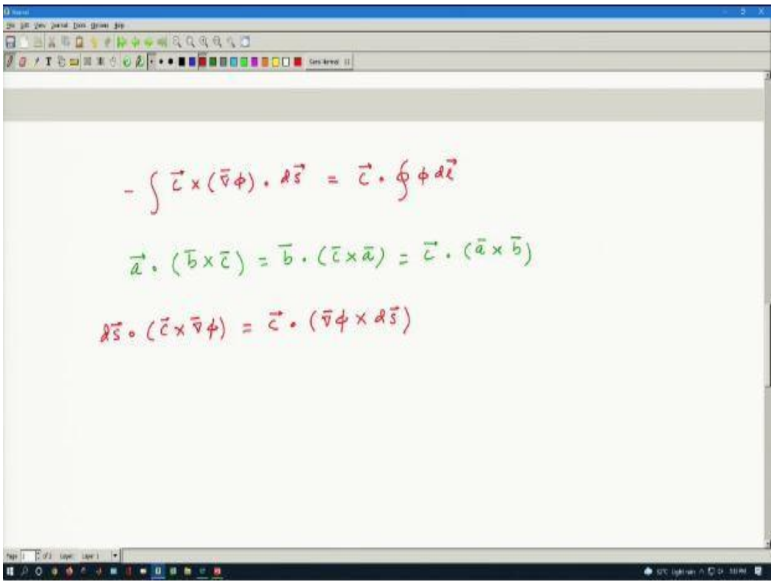

So, then after that we will be going to exploit the expression like $\vec{a} \bullet \vec{b} \times \vec{c} = \vec{b} \bullet \vec{c} \times \vec{a}$ and that is equal to $\vec{c}$ this is a vector identity $\vec{a} \times \vec{b}$. Exploiting this vector identity, I can write as $d\vec{s} \bullet$ ($\vec{c}$ $\times$ this quantity) is eventually $\vec{c} \bullet \times d\vec{s}$. Just playing with the vector identity and then I am going to put it here. Because in one term we have $d\vec{s} \bullet$ this $\vec{c} \times \vec{\phi}$.

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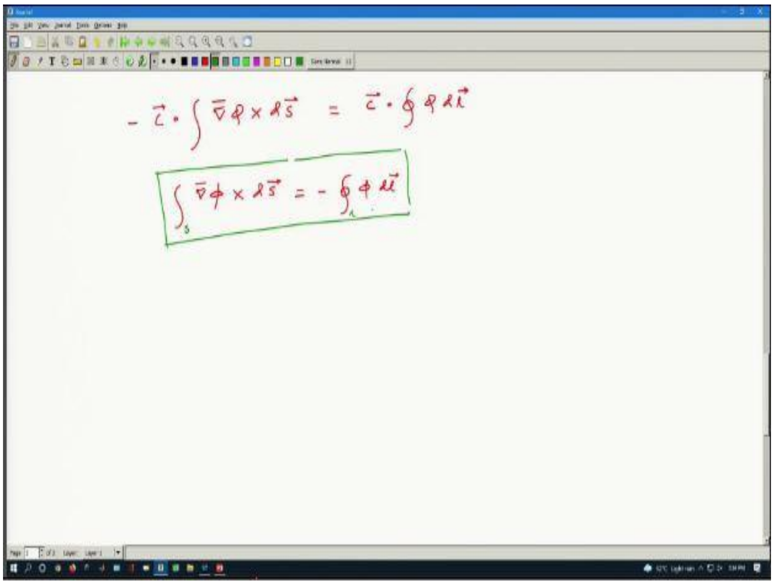

So, this term I am going to replace this $\vec{c}$ dot. So, I have $-\vec{c}$ dot then integration of this quantity cross d$\vec{s}$. Right-hand side we already have $\vec{c}\ \bullet\ \phi$ d$\vec{l}$. So, you can see that this is true for any given $\vec{c}$ and ϕ. So, that means I can safely write that cross d$\vec{s}$ is equal to $-\oint \phi\ d\vec{l}$. So, this is one very interesting vector identity like Stoke's law, like Gauss's integral theorem.

Here we have a divergent a gradient of a scalar field cross over a surface. That should be simply the integration of the scalar potential over this close line. So, this is interesting this is over surface and this is over close line. So, this interesting vector relation that we figure out.

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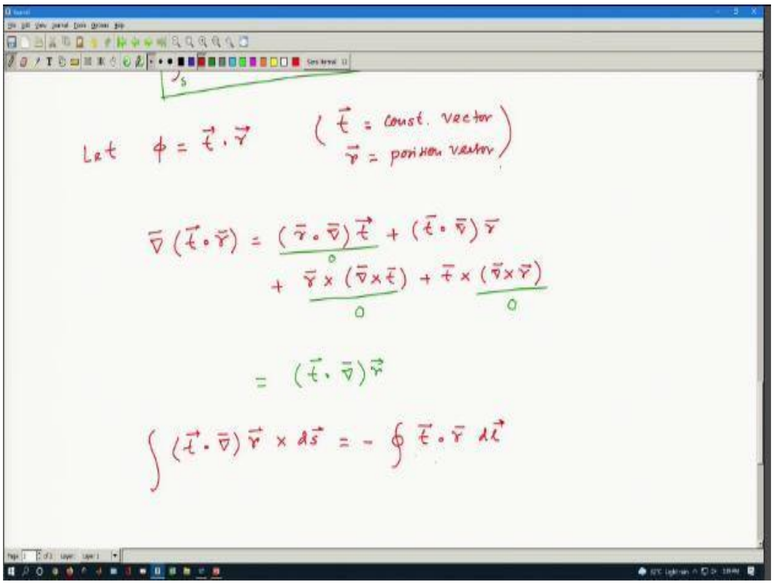

Now after that what we do let ϕ our scalar field to be some vector say $\vec{t}$ dot the position vector $\vec{r}$, $\vec{t}$ is a constant vector and $\vec{r}$ is a position vector. So, $\vec{t}$ this is a constant vector and $\vec{r}$ is our position vector. So, that thing if I try to first I need to find out the gradient of these two quantity and if I do if I try to find out the $\vec{\nabla}(\vec{t} \bullet \vec{r})$. Then so this is the gradient of these 2 vector $\vec{t}$ and $\vec{r}$ I can expand with the usual.

So, this is not very simple to be very honest. So, I just write the expansion. So, it should be $\vec{r}$ dot this over $\vec{t}$ but mind if $\vec{t}$ is constant. I am just writing the full expansion. Then $\vec{t}$ dot this over $\vec{r}$ then we have plus $\vec{r}$ cross this cross $\vec{t}$ and then $\vec{t} \times \vec{r}$ that is a vector identity. But you can see that there should be many term that is 0, $\vec{t}$ is a constant and this will go to operate. So, this term is 0, $\vec{c}$ is a constant it will operate. So, this term is also 0 and this term $\vec{\nabla} \times \vec{r}$ is also 0.

So, this I have 0, this I have 0 and this I have 0. So, eventually this left the term that we have here is simply $\vec{t} \bullet \vec{r}$. So, this is equal to this for constant $\vec{t}$. So, then I can have because I put this here. So, $\vec{\nabla}\phi$ now I put this $\vec{\nabla}(\vec{t} \bullet \vec{r})$ and $\vec{\nabla}(\vec{t} \bullet \vec{r})$ is now $\vec{t}$ dot this quantity. So, I just put it here and I should have $(\vec{t} \bullet \vec{\nabla})\, \vec{r} \times d\vec{s}$ that is the right-hand side with the negative sign.

I should have the integration and this quantity over $d\vec{l}$. So, I should have $\vec{t} \bullet \vec{r}$ and $d\vec{l}$. So, here I am just writing this whatever the ϕ I am having. And that ϕ is here $\vec{t} \bullet \vec{r}$. So, I just write $\vec{t} \bullet \vec{r}$ as a ϕ, just exploiting this expression whatever we had.

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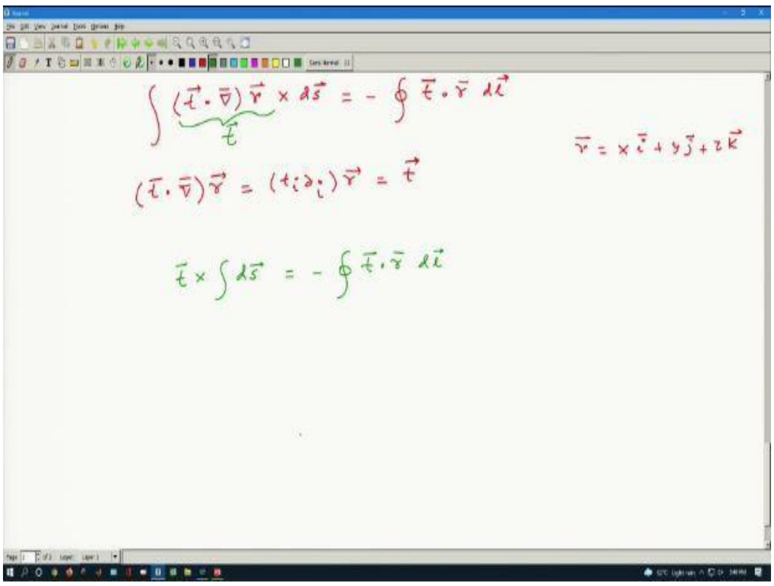

Now let us check what is this quantity? So, $\vec{t}$ dot this over $\vec{r}$ is eventually. So, here I should write simply $t_i\ \partial_i$ that is operating on the $\vec{r}$ and it simply gives me $\vec{t}$ please check it. So, it should be t_x dx, t_y dy + t_z dz delta and the operator z partial derivative with respect to x, y and z. And then I will be going to operate over the $\vec{r}$. So, $\vec{r}$ in Cartesian coordinate system is simply x i +

y j + z k. So, it simply gives me $\vec{t}$. So, that thing gives me $\vec{t}$ cross. So, from here I can have this quantity is simply $\vec{t}$ and $\vec{t}$ is a constant. So, I can write it as $\vec{t} \times d\vec{s}$ is $-\oint \vec{t} \bullet \vec{r}\, d\vec{l}$.

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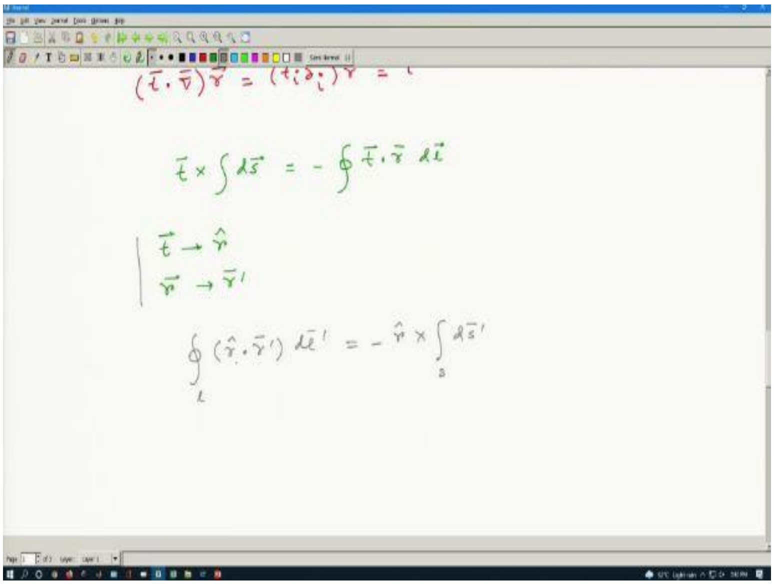

Now we are there actually so if I just replace $\vec{t}$ to $\hat{r}$ and $\vec{r}$ to $\vec{r}\,'$. Then I simply have our desired expression that we wanted to prove. So, let me write it here. So, then I simply have closed integral over line. Then $\vec{r} \bullet \vec{r}\,'$ and then $\vec{l}\,'$ is equal to $-\hat{r} \times \int d\vec{s}\,'$ or $d\vec{a}\,'$. This is surface integral and this is close line integral. So, this is essentially.

So, from here I just replace these 2 things, these 2 vectors and I finally figure out the desired result, that you wanted to prove. So, this is a lengthy proof but very tricky. So, you just have a look, this is just for once you just need to check that you can understand this or not. But the thing that you need to remember is the expression of this how the magnetic vector potential for the dipole contribution is coming.

And the magnetic dipole is defined like current into area in this way. However I can also have an alternative way to show but in terms of say vector potential the current density J. And that I quickly, again there is some trick to prove certain identity. But I think I should show at least I should mention this.

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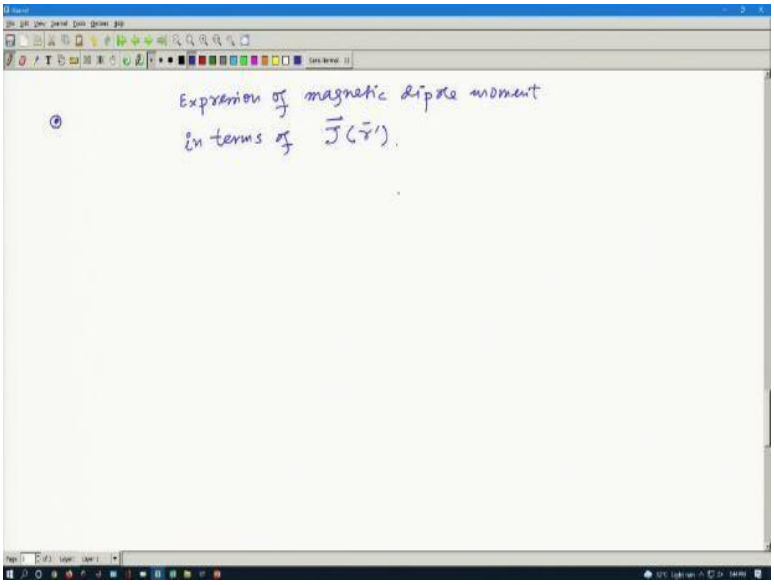

So, we know so this is the magnetic dipole moment. The expression of magnetic dipole moment in terms of the current density J, which is normal in $\vec{r}'$, the function of $\vec{r}'$. So, how to do that? Because here the magnetic dipole moment $\vec{m}$ is defined if I go back and check it is defined here in terms of the line current I. But I can have another expression where the magnetic dipole moment can be expressed in terms of the volume current density. Because normally volume current density is something through which it is better to represent the magnetic dipole moment and other?

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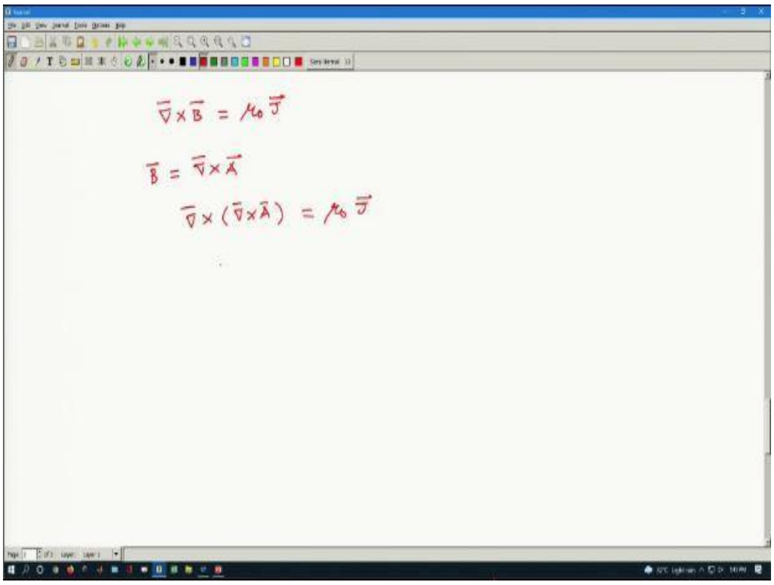

So, let me do that. We know $\vec{\nabla} \times \vec{B}$ quickly is equal to $\mu_0\,\vec{J}$. This is the source term $\vec{J}$. Now $\vec{B}$ is $\vec{\nabla} \times \vec{A}$ and that gives me $\vec{\nabla} \times (\vec{\nabla} \times \vec{A})$ as $\mu_0\,\vec{J}$.

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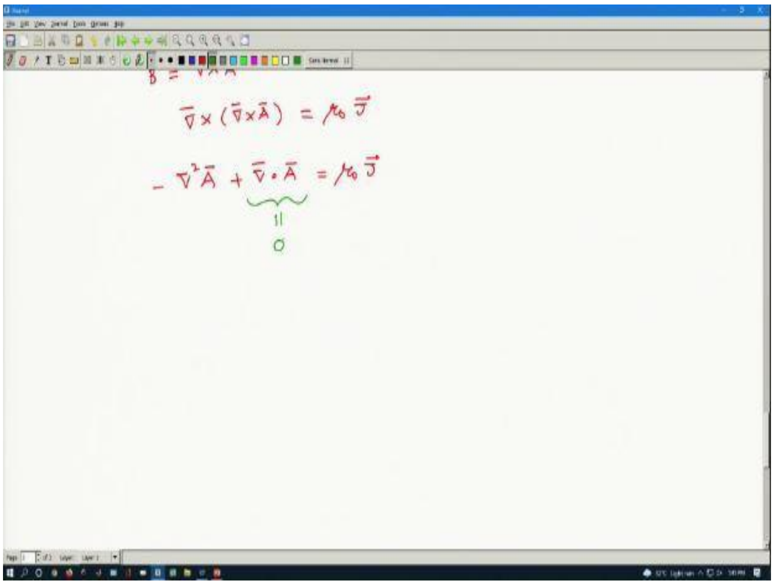

Now $\vec{\nabla} \times (\vec{\nabla} \times \vec{A})$ that is a very famous identity and I can write that $-\nabla^2\vec{A} + \vec{\nabla} \bullet \vec{A} = \mu_0 \vec{J}$. Now exploiting the coulomb gauge you can have this equal to zero that we discussed last day.

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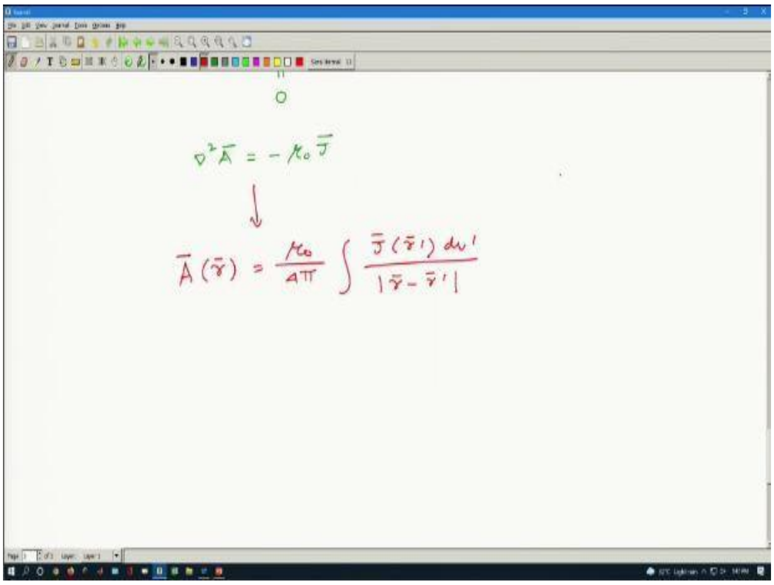

So, eventually the expression becomes a vector Poisson equation. This part also we did in the last class is say let us put the plus sign. So, let us put this and this $-\mu_0 \vec{J}$ and the solution for this Poisson equation we wrote in this way. The solution is $\vec{A}$ that is a function of $\vec{r}$ and that value is $\frac{\mu_0}{4\pi}$ and then $\int \frac{\vec{J}(\vec{r}\,')dv'}{|\vec{r}-\vec{r}\,'|}$. So, please check it.

When I expanded this today, the value of $\vec{A}$ I wrote the value of $\vec{A}$ the very beginning. It was in line current. Exploiting the expression of the line current I started my calculation. But here the same expression I am writing in terms of the current density $\vec{J}$. Now if I expand $\frac{1}{r}$.

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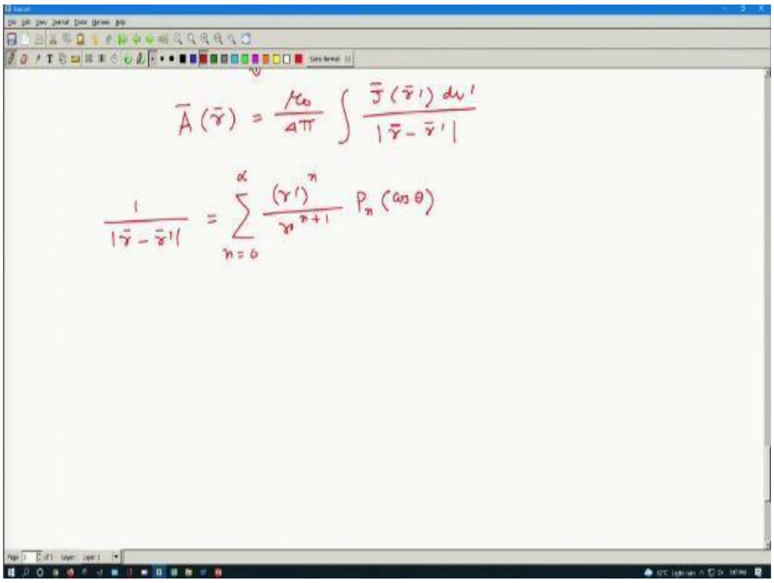

So, again I have a term $\frac{1}{|\vec{r}-\vec{r}'|}$ here. And that is simply n 0 to infinity I have $\frac{(r')^n}{r^{n+1}}$ and then I have P_n (cos θ).

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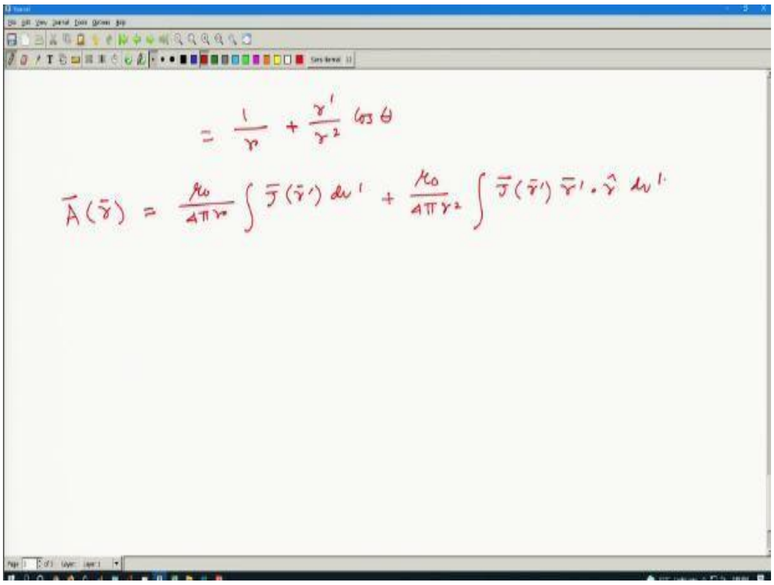

Now, these things if I expand is $\frac{1}{r}$ first term, second term is $\frac{r'}{r^2}$ and then cos θ like before the same thing. Now I put these two here in this expression of $\vec{A}$. So, I can write my $\vec{A}$, which is a function of $\vec{r}$ as $\frac{\mu_0}{4\pi r}$ and then $\int \vec{J}(\vec{r}')dv'$. And then I have the second term $\frac{\mu_0}{4\pi r^2}$.

And then I make integral and then I have $\int \vec{J}(\vec{r}')\, \vec{r}' \bullet \hat{r}\ dv'$ .So, here this is the contribution also the higher order terms are there, because this is going on here, for the time being I am not bothering about the higher order term.

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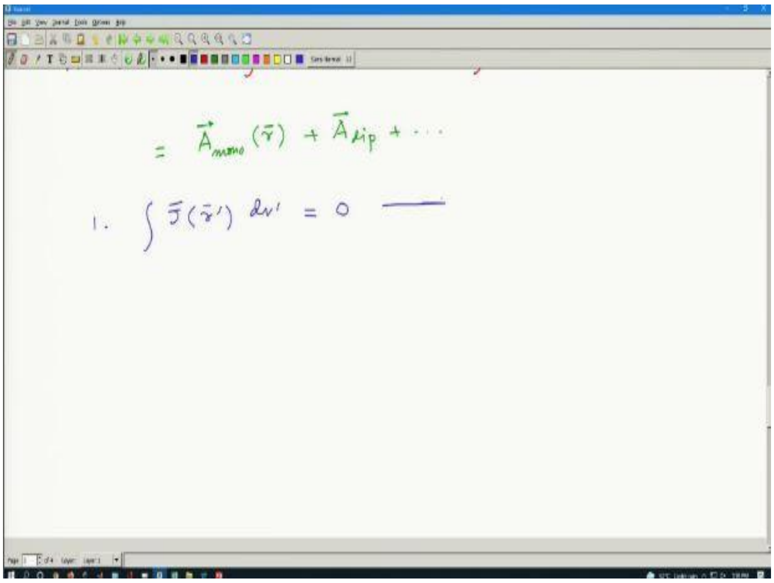

So, this thing is equivalent to the monopole contribution, which is a function of $\vec{r}$ and the next term is the contribution of dipole and so on. Here two very important expressions are there. So, I will be going to prove that but if I have some time in some tutorial class I will prove this as a tutorial problem. So, it can be shown that $\int \vec{J}(\vec{r}\,')dv'$ is 0 and that makes this a tricky proof.

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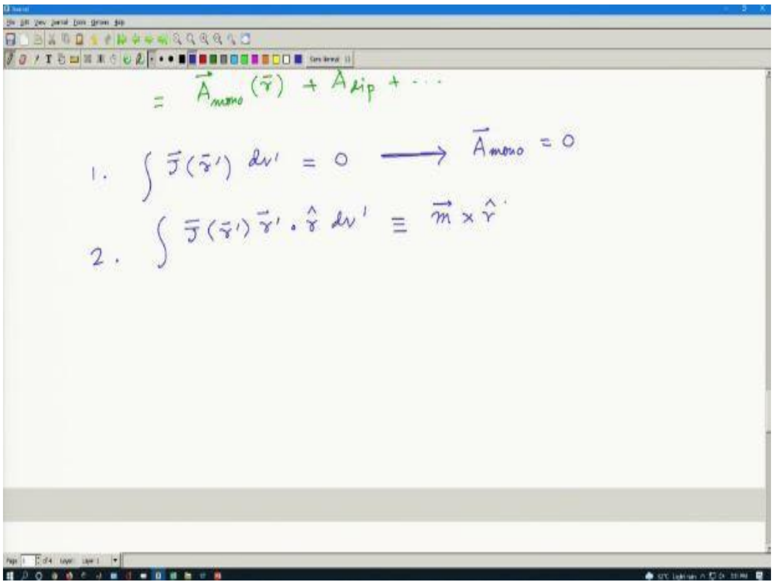

If you want to check this proof, the stamp lock book it is there. But if I have some time as I mentioned I will to do that here in class. So, $\vec{A}$ that makes $\vec{A}_{\text{monopole}}$ term is 0. And the second identity that we are going to use is the integral of $\int \vec{J}(\vec{r}\,')\, \vec{r}\,' \bullet \hat{r}\, dv'$. It can be written as a quantity like $\vec{m} \times \hat{r}$, where $\vec{m}$, which is the dipole moment.

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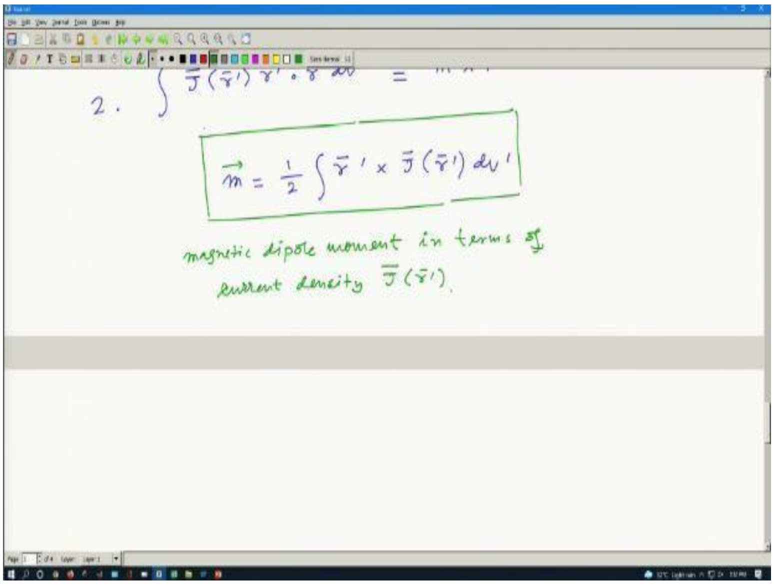

Now is defined in this way. This is now $\frac{1}{2}\int \vec{r}' \times \vec{J}(\vec{r}')dv'$.So, magnetic dipole moment in terms of so this basically gives me this expression, whatever the expression we are now having. So, I should have a vector sign here. So, whatever the expression now we are having is the magnetic dipole moment. This is the magnetic dipole moment in terms of current density $\vec{J}$.

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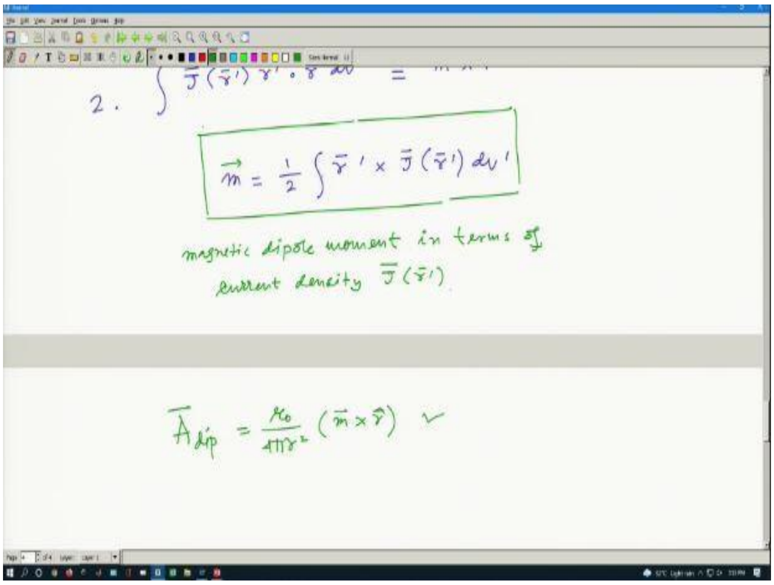

And $\vec{A}$ from that I can write the $\vec{A}_{\text{dipole}}$ contribution to be $\frac{\mu_0}{4\pi r^2}$ then $\vec{m} \times \hat{r}$ that is the dipole. So, you can now see that the magnetic dipole moment here is defined in terms of the current density. So, it is like electrostatics.

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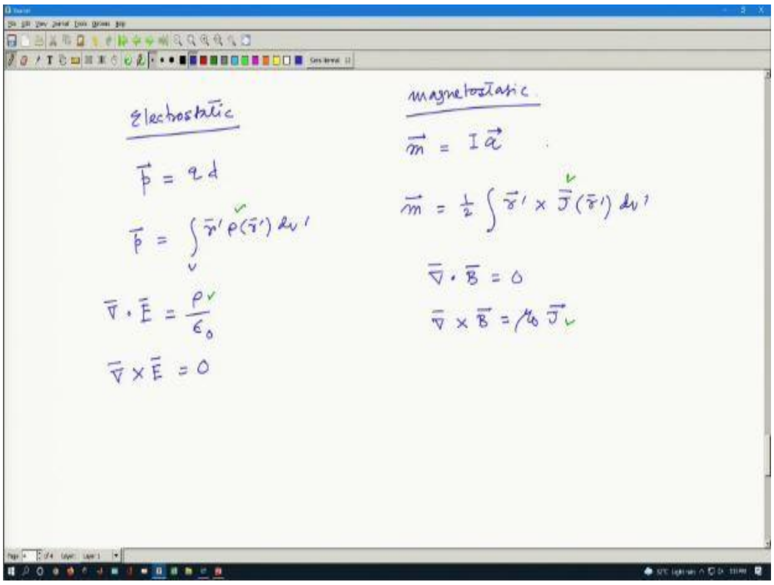

If I go back to the electrostatic problem what we had the electric dipole moment $\vec{p}$ was charge into distance that is the usual way we define. But also it is defined in terms of it is also defined like $\vec{r}'$ and ρ ($\vec{r}'$) dv'. This is the way we define in vectorial form, in integral form, exploiting the source term ρ. ρ is a source term there. In the similar way for magnetostatic, the magnetic dipole moment $\vec{m}$ is defined like current into area in crude way, but also I can write it as some $\frac{1}{2}\int \vec{r}' \times \vec{J}(\vec{r}')dv'$.

Here in electrostatic, we have source term ρ that is why when you calculate $\vec{\nabla} \cdot \vec{E}$ as a source. So, $\frac{\rho}{\epsilon_0}$, this behaves like a source and $\vec{\nabla} \times \vec{E}$ was 0. Here we had on the other hand $\vec{\nabla} \cdot \vec{B}$ is 0, but the source terms is here Ampere's law $\mu_0 \vec{J}$. So, this source term I write here in electrostatic dipole moment, this source term I write here in expanding or in writing a dipole moment in magnetostatic.

So, this is just to show some sort of resemblance. Later we will see that how these magnetic dipole moments are there. But before going to finish, let us quickly calculate the dipole moment for few cases.

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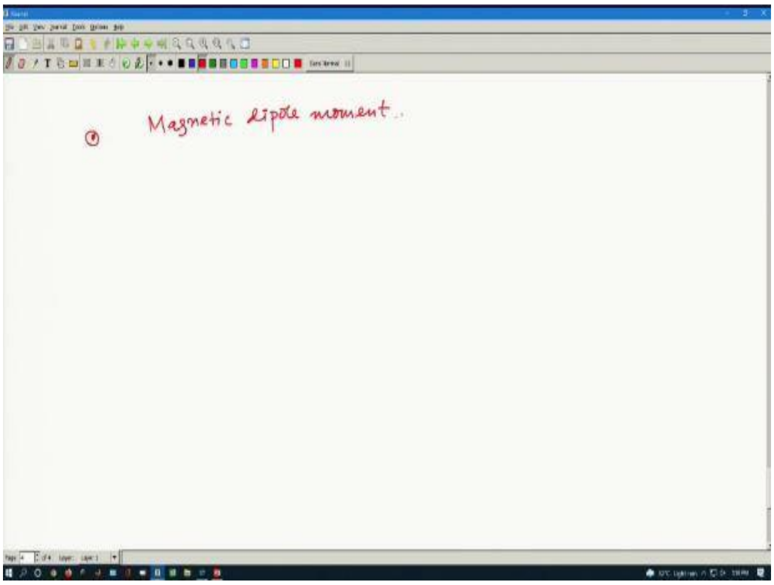

So, the first thing that we do is the magnetic dipole moment. This is simply current into area so let me.

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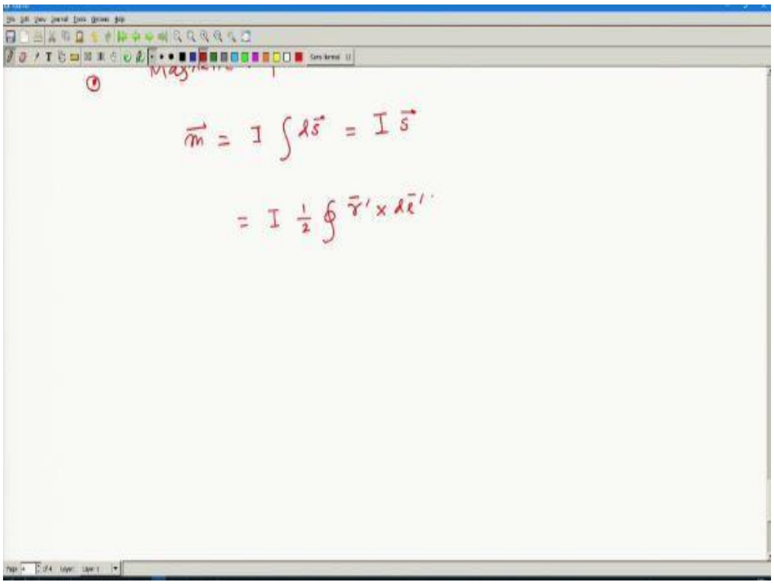

So, this magnetic dipole moment quickly so, $\vec{m}$ I had I $\int d\vec{s}$ or $d\vec{a}$ or simply I into current into area. That I write $\frac{1}{2}$ I and area I can write it as $\frac{1}{2}\oint \vec{r}' \times d\vec{l}'$.

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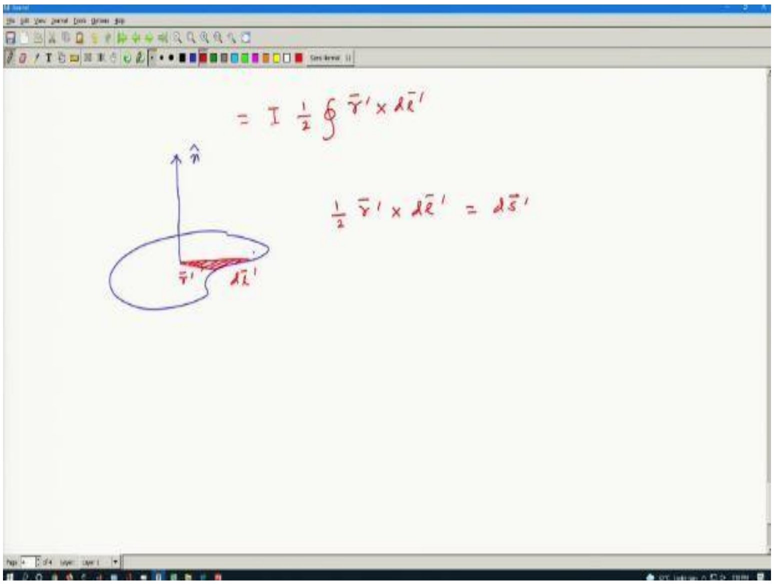

So, that means if I have a current loop. This is a current loop and this is the direction of $\hat{n}$. So, it is forming an area. So, if I have the section from here to here small section. So, this is d$\vec{l}$'. And say this is my $\vec{r}$'. So, $\frac{1}{2}$ $\vec{r}$' × d$\vec{l}$' that is the area we are having right small area d$\vec{s}$'. And that is the shaded region. So, this is the area we are talking about.

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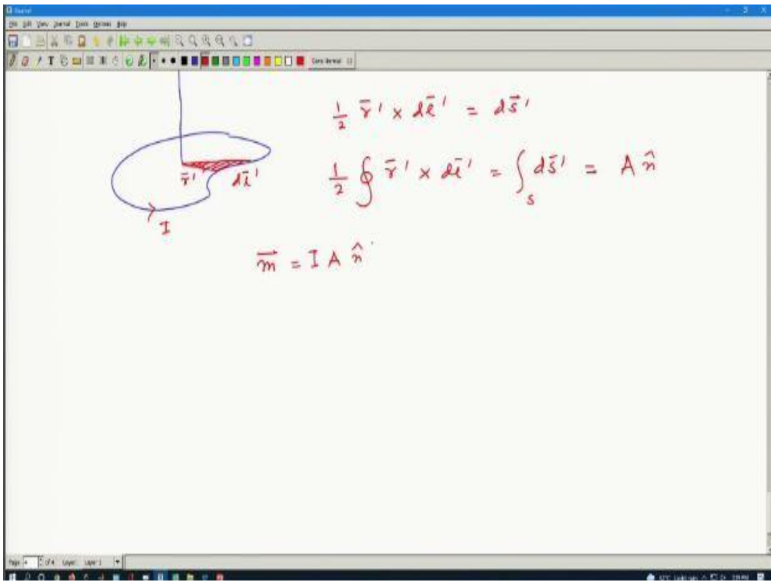

Now if I make a total integration simply over this closed loop when the current is flowing like this. Then I should have $\oint \vec{r}' \times d\vec{l}'$ that is the integration in the right-hand side d$\vec{s}'$ and that gives me simply the total area with the direction, A is area. So, $\vec{m}$ is simply current into area and this is the usual way we define.

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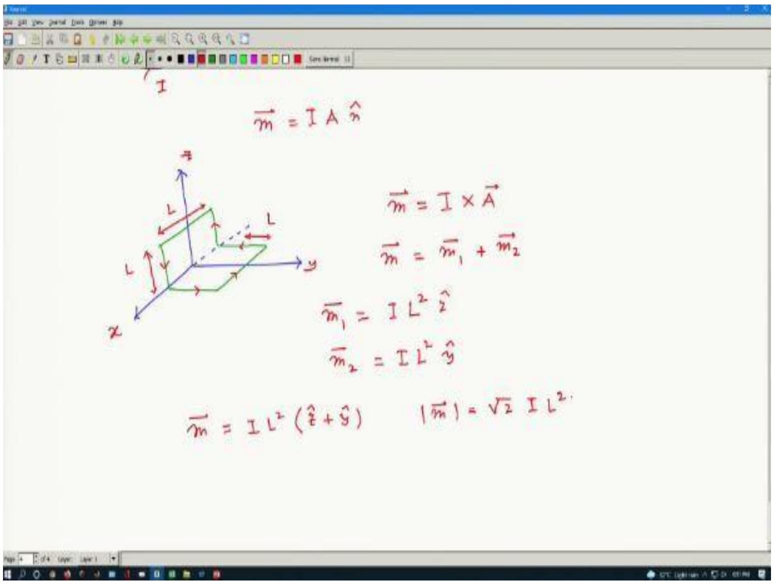

Now what happened if I have a current loop not in a plane, but situated in this way. So, I have this coordinate system and the current loop is like this. So, let us go to the negative side as well. So, the current loop is something like from here to here. Then here to here, it is folded book like current loop and the current that is passing through is say in this direction here, here, here, here.

From here to here say length is L. This length is also L and here to here this length, say L. So, we know the current I mean according to our usual notation $\vec{m}$ is current into area. And here we have 2 areas. So, let us divide by $\vec{m}$ since, we are having 2 areas so $\vec{m}_1 + \vec{m}_2$. So, $\vec{m}_1$ is current into area if this length is L. So, this is L^2 and the direction should be along z**.** Let me put it as x, y and z.

So, for this case it is $\hat{z}$ and for $\vec{m}_2$ similarly, we have I L^2 $\hat{y}$. So, now if I want to calculate the total $\vec{m}$ it should be simply the vector sum that is all. So, it should be I L^2 and the vector sum of $\hat{z}$ and $\hat{y}$. So, if I want to calculate the magnitude of $\vec{m}$ this simply gives us root over of 2 and then I and L^2**.**

So, that is quickly I try to calculate 2 different I mean just to show that how the magnetic dipole moment can be calculated when we have a loop, which is not in the surface. So, next class what we do is we will go further. And try to understand more about the practicality of this magnetic dipole moment. And then maybe we can move to magnetization problem. Or before that we need to do the magnetic dipole movement in an electric external magnetic field if I put a magnetic loop in external magnetic field how the torque will be there etcetera. So, that maybe we can discuss. So, with that note I like to conclude here. Thank you very much for your attention and see you in the next class.