Foundation of Classical Electronics
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Lecture-54
Magnetic Vector Potential

Hello student to the foundation of classical electrodynamics course. Under module 3, today we have lecture number 54. And today we will be going to discuss more about the magnetic vector potential that we started in the last class.

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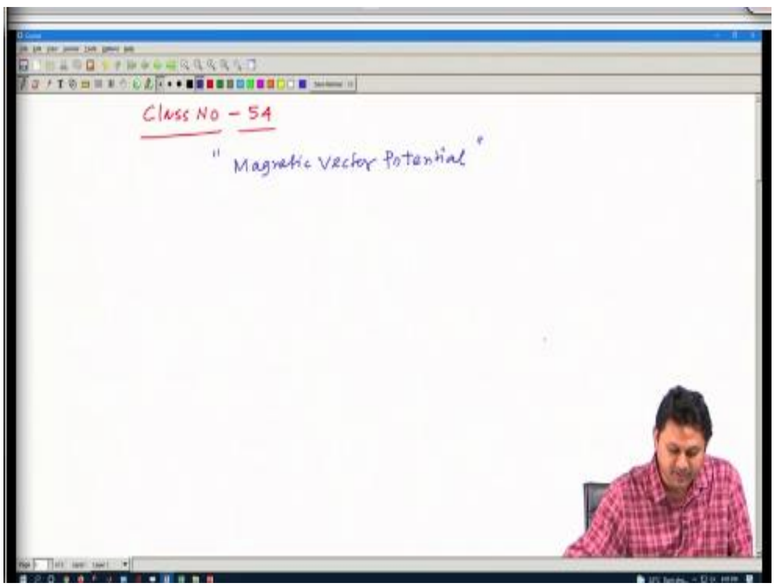

So, today we have class number 54 and our topic today is magnetic vector potential. We will continue this discussion that we started last class.

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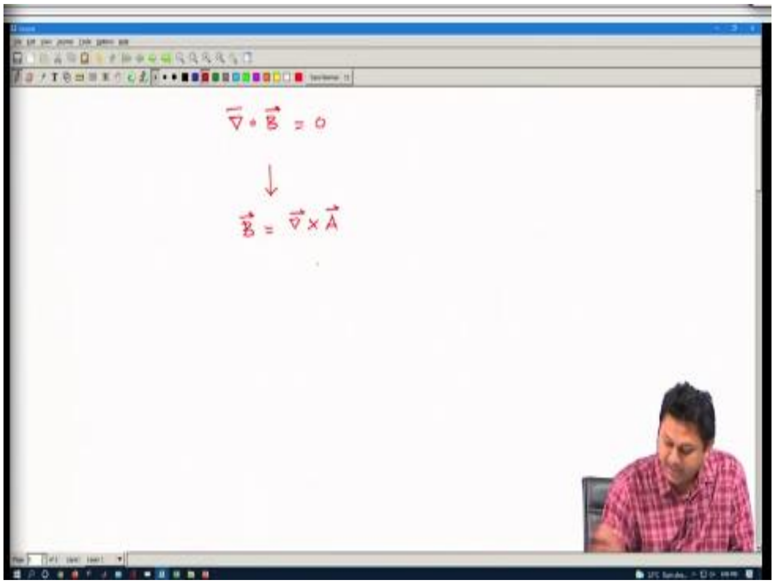

So, we know that the $\vec{\nabla} \cdot \vec{B} = 0$. Since $\vec{\nabla} \cdot \vec{B} = 0$ from that we can write $\vec{B}$ as a curl of a vector function say $\vec{A}$.

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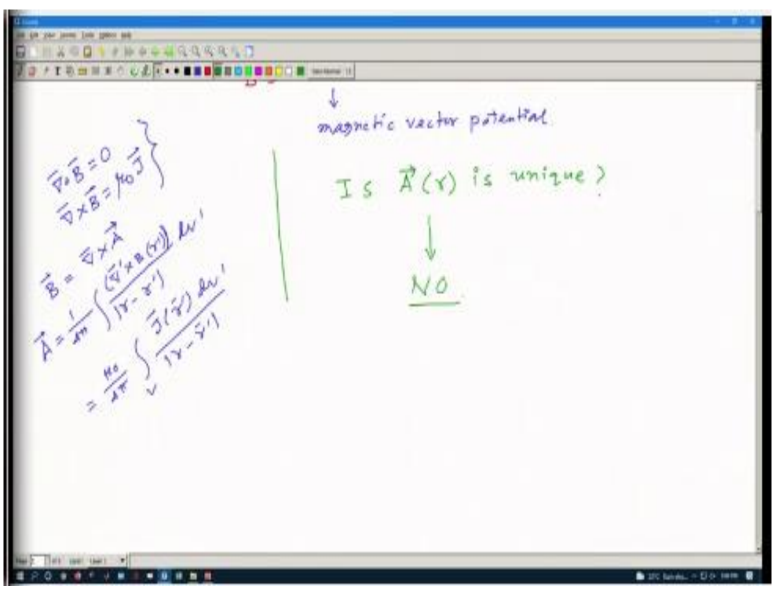

And this vector function $\vec{A}$ is termed as magnetic vector potential. Now this magnetic vector potential because if you go back to again the Helmholtz theorem, I have done this several time that $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B}$ is $\mu_0 \vec{J}$. If these 2 information are known then I can able to write $\vec{B}$ as a function of $\vec{A}$. In this way where $\vec{\nabla} \times \vec{A}$ gives me the $\vec{B}$, where $\vec{A}$ can be determined as according to the Helmholtz theorem as $\frac{1}{4\pi}$ integration and divided by $\vec{r} - \vec{r}\,'$.

And here I should use the expression of $\vec{\nabla} \times \vec{B}$ in prime frame beta to write $\vec{B}$ as a function of $\vec{r}\,'$ with dv'. And from that I can simply write that this quantity is $\frac{1}{4\pi}$ μ_0 I can take outside because curl cross these things is nothing but μ_0 $\vec{J}$. So, it should be like $\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$ dv', so this we derive last day. Next the question was the uniqueness of $\vec{A}$; next the thing that we discuss that is $\vec{A}$, the vector function that we are talking about is unique? And the answer is No.

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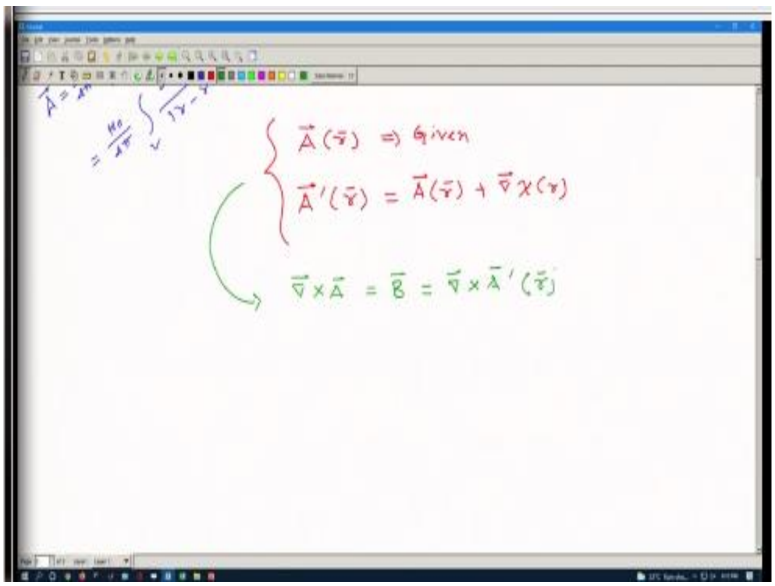

Because you can form a vector function $\vec{A}\,'$ suppose $\vec{A}$ is given, $\vec{A}$ as a function of is given and you can construct another function $\vec{A}\,'(\vec{r})$ using the given function $\vec{A}$ in this way that gradient of a scalar field $\vec{r}$. Then $\vec{A}$ and $\vec{A}\,'$ both leads to the same $\vec{B}$, so from here we can write that $\vec{\nabla} \times \vec{A}$ will give the $\vec{B}$ and that is the same quantity that $\vec{\nabla} \times \vec{A}\,'$. So, this will give rise to 1 $\vec{B}$ and this will again give rise to the same $\vec{B}$, where $\vec{A}$ and $\vec{A}\,'$ are different quantity. That was the discussion we had in the last class, I quickly recap. Now today we will be going to impose certain constraint, so that is the thing.

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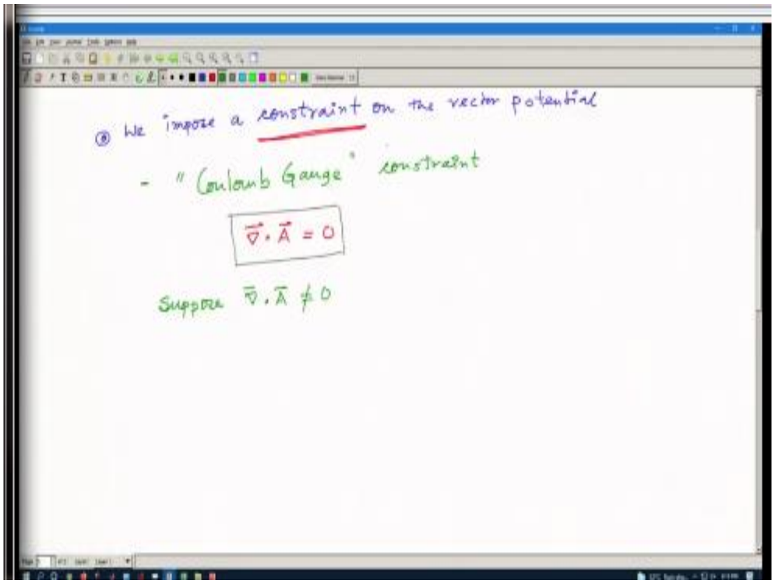

So, now we impose a constraint on the vector potential and that constraint is called the Coulomb Gauge, so what is the meaning of constraint? I will discuss. So, that constraint is eventually the Coulomb Gauge this is called Coulomb Gauge constant. So, what is that constraint? The constant is the $\vec{\nabla} \cdot \vec{A}$ has to be 0 that is the constraint that I impose; it is not necessarily that the vector potential that is given for which I am going to calculate the magnetic field should have this property.

That if I make a divergence of over that vector potential it gives to 0 but we put a constraint over that, that we want that $\vec{A}$ in such a way that if I make a divergence over $\vec{A}$, it should be 0. Now as I mentioned it is not always true that this is not always a valid constant. So, suppose the $\vec{\nabla} \cdot \vec{A}$ is not equal to 0 because the Coulomb gauge constraint we need to put but $\vec{A}$ is given for which the divergence is not equal to 0.

That means simply the $\vec{A}$ is not satisfying this equation, whatever the equation we are having it is not satisfying. But we know that there is no such uniqueness over $\vec{A}$, I can construct my $\vec{A}$ in a different way that I discussed in the last day, today also I mention it. So, what we do that since it is not equal to 0.

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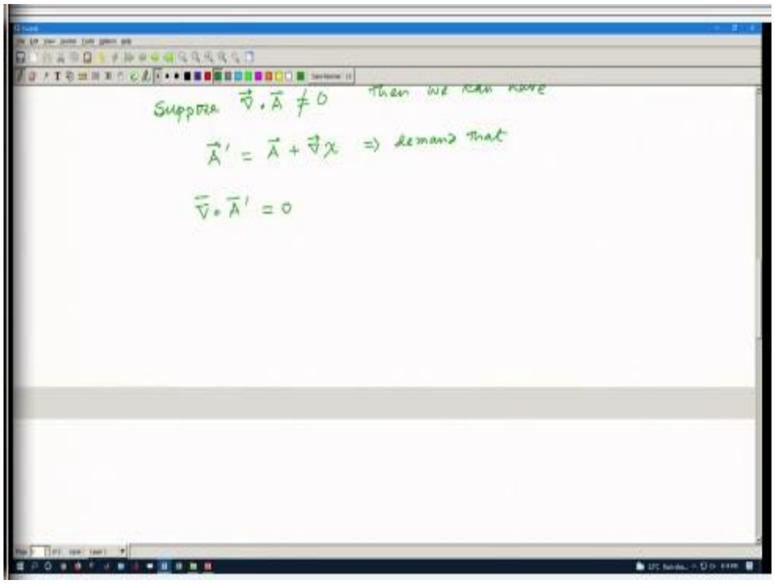

Then we can make it 0, we can have another $\vec{A}'$ obviously that $\vec{A}'$ can also produce the same $\vec{B}$, this is my new vector potential and that is this we know, by mathematical form this is the form. And demand that I construct $\vec{A}'$ in such a way that we demand that the $\vec{\nabla} \cdot \vec{A}$ is 0. So, if $\vec{\nabla} \cdot \vec{A}$ is 0 from this equation what I can see that.

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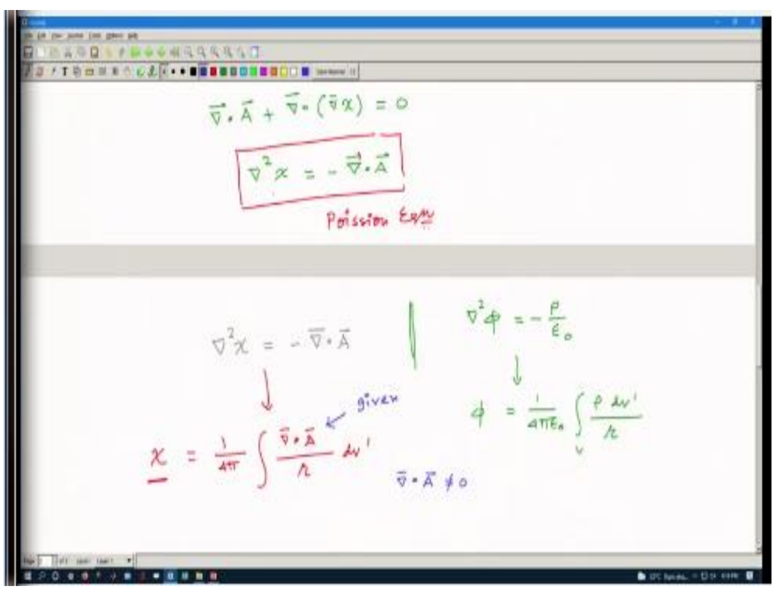

If I make the $\vec{\nabla} \cdot \vec{A}$ both the side and keep this equal to 0 then eventually we have $\vec{\nabla} \cdot \vec{A}$ plus divergence of right-hand side I am just writing $\chi = 0$ or $\nabla^2\chi$ is a vector field is equal to minus of this. So, that in order to make $\vec{\nabla} \cdot \vec{A}'$ 0 what we need to do is that the χ should satisfy this equation. And this is nothing but the Poisson equation this is whose solution is known.

So, let me write it down once again, so that means the χ, which we have a freedom to choose should satisfy this Poisson equation in order to hold the Coulomb gauge. So, it is like it should satisfy this equation and this is equivalent to. Now if I want to write it an equivalent equation here, so this is equivalent to an electrostatic, which is a known equation, so let me write it down and then we will go to compare this.

So, in electrostatic we already had an equation like this. Let me write it properly square this is equal to $-\frac{\rho}{\epsilon_0}$, that is our usual Poisson equation in that for the potential that we had for electrostatic. And whose solution is known and the solution ϕ was $\frac{1}{4\pi\epsilon_0}$ then integration of $\frac{\rho\, dv'}{Л}$. Here also since the solution is this, this is a volume integral, so these 2 equations are identical, so I can readily have the solution.

And the solution for χ is simply if I want to find out the solution here I can write this, this is nothing but $\frac{1}{4\pi}$ and then integration of this quantity over Л dv'. So, under Coulomb gauge these I can satisfy, I can prepare my, I can construct my $\vec{A}'$ in such a way that it satisfies this equation, which is the condition for Coulomb gauge. And in order to satisfy this equation I need to choose my my scalar field χ in such a way that this is the solution, this is the way I need to choose.

Because this $\vec{\nabla} \cdot \vec{A}$, which is non-zero is given, so these things is given, this is given. So, exploiting that information I can construct the thing. So, this is given and this is not 0 that is the condition we start with, that suppose these things is not 0, so this is not 0 some value is there, so that thing I will put here, plug it and then find out my χ. And then I construct $\vec{A}'$ that should satisfy the Coulomb gauge. Now if the Coulomb gauge is there what we get? Let us check what should be the expression of $\vec{A}$ then.

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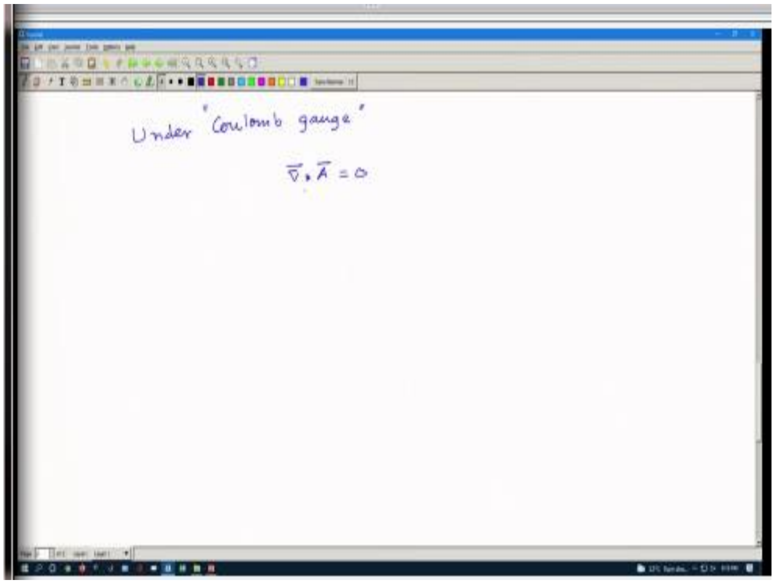

So, under Coulomb gauge what we have? We have $\vec{\nabla} \cdot \vec{A}$ is 0.

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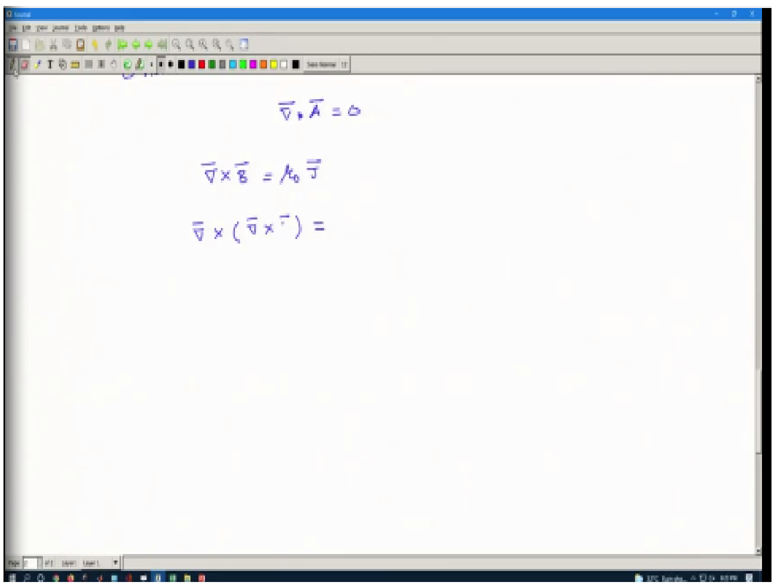

From this information I can have $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ and then $\vec{\nabla} \times \vec{B}$ I am making curl of both the side that I should write this is $\vec{A}$, I do not want to make curl of both the side now.

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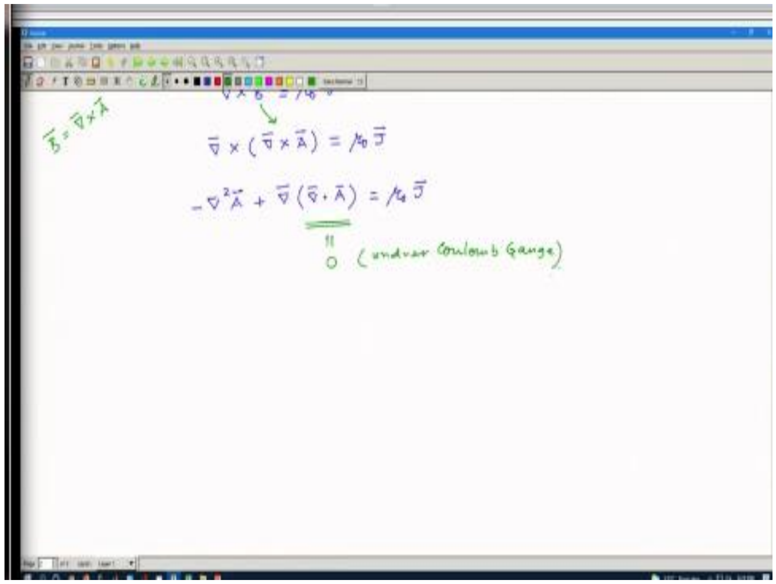

So, from this equation I write because $\vec{B}$ is $\vec{\nabla} \times \vec{A}$ and I just replace $\vec{B}$ here and that thing should be $\mu_0 \vec{J}$. Now $\vec{\nabla} \times \vec{A}$ we know this is a very famous identity, we did it during the vector calculus and it is simply the $-\nabla^2\vec{A} + \vec{\nabla} (\vec{\nabla} \cdot \vec{A})$ and that thing should be equal to $\mu_0 \vec{J}$. Now here we had this condition $\vec{\nabla} \cdot \vec{A}$, which should be 0 under Coulomb gauge. So, this should be 0, so then we simply have a vector Poisson equation.

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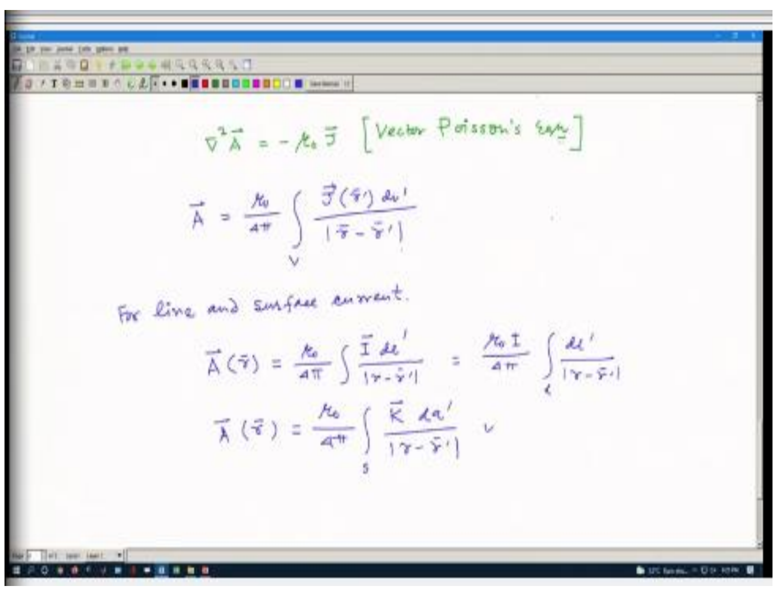

Which is $\nabla^2\vec{A}$, which is a vector quantity = - $\mu_0 \vec{J}$, this is nothing but a Poisson equation but vector vectorial form, so this is a vector Poisson equation. So, now from that if I want to extract the value of the $\vec{A}$, the solution is simply $\frac{\mu_0}{4\pi}$ because this solution is already there written, the solution of the

Poisson equation. And then $\frac{\vec{J}(\vec{r}\,')dv'}{|\vec{r}-\vec{r}\,'|}$, this is some volume integral over the entire volume v and that thing for volume current density.

So, for surface and line current density we can also have an expression in some places we may require that. So, for line and surface current we can have $\vec{A}$ is $\frac{\mu_0}{4\pi}$ then $\int \frac{\vec{I}\,dl'}{|\vec{r}-\vec{r}\,'|}$. And I can if for a steady current we know that then I can take I outside, so this is simply $\frac{\mu_0 I}{4\pi}$ and then $\int \frac{dl'}{|\vec{r}-\vec{r}\,'|}$, so this is for line current.

And for surface current the vector potential can be written like $\frac{\mu_0}{4\pi}$ integration surface current density $\frac{\vec{K}\,da'}{|\vec{r}-\vec{r}\,'|}$, so that will be over surface that will be over line. And previously the expression is over volume, so these are the 3 forms for different current density or different current line, surface and volume current and that is the expression for the vector potential $\vec{A}$. Now after having that let us now calculate few cases, where, how to calculate the current density for different cases, how to calculate the vector potential?

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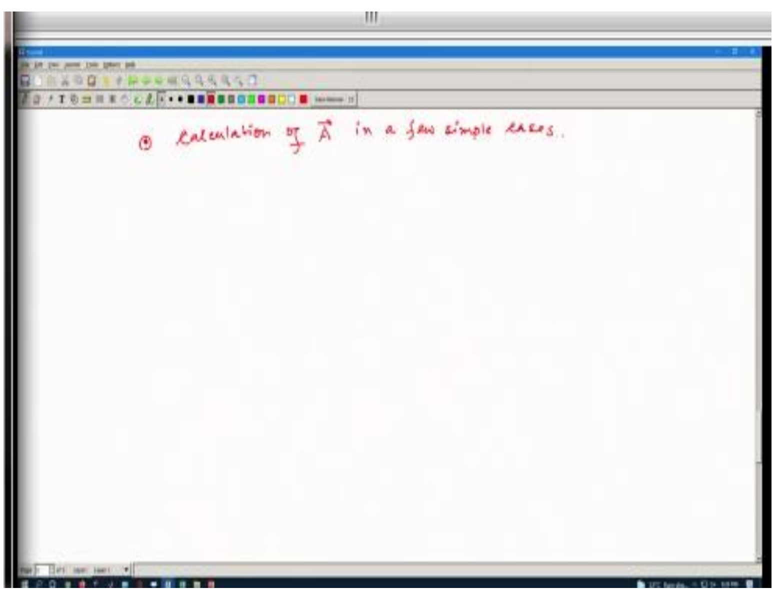

So, calculation of vector potential $\vec{A}$ in a few simple cases we are going to calculate.

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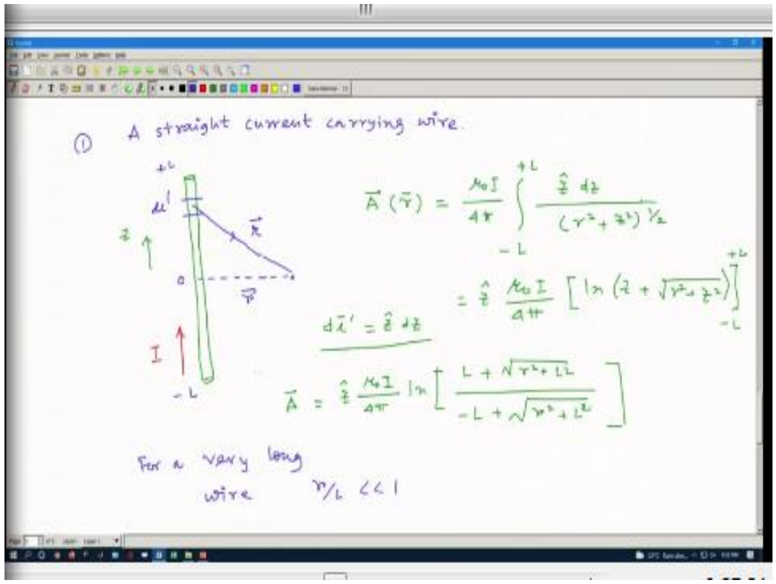

So, the first thing, for the first case we calculate is a straight current carrying wire. So, let me draw the current carrying wire first. Suppose I have a straight wire like this and the current that is flowing through is I and I want to say find out the vector potential at some point here as a distance $\vec{r}$. Because some magnetic field should be here because of the flowing of the current through this wire and as a result we should have some vector potential $\vec{A}$, so that we are going to calculate.

So, from here to here this is say $\vec{r}$ and from here to here, so this is d$\vec{l}$ section and the length is say -L to +L and this is our origin. So, I can use this expression my $\vec{A}$ should be at $\vec{r}$ should be $\frac{\mu_0 I}{4\pi}$ integration -L to +L and then the current is flowing along this direction and that is say if this is z direction, so $\frac{\hat{z}\, dz}{(r^2+z^2)^{1/2}}$.

So, r^2 z^2 is simply this one and since it is z direction, so simply we have d$\vec{l}$ vector is $\hat{z}$ dz considering the wire is over this z axis. So, I simply now evaluate this integral, that is all and if I do this then this is $\hat{z}$. Then $\frac{\mu_0 I}{4\pi}$ and it should be ln because this is we know (z + $\sqrt{r^2 + z^2}$) and evaluated at -L to +L. So, if I evaluate that.

So, my $\vec{A}$ should be $\vec{A}$ is $\hat{z}\ \frac{\mu_0 I}{4\pi}$ and this is simply ln I just put the value of this boundary value, so it should be $\frac{L+\sqrt{r^2+L^2}}{-L+\sqrt{r^2+L^2}}$L + $\sqrt{r^2 + z^2}$, that is the value. Now if the wire is very, very long, so for a very long wire the condition simply when $\frac{r}{L}$ this ratio is very, very less than 1.

That means I try to find out the vector potential as a distance $\vec{r}$ but that distance should be much, much smaller than the distance of the length of the wire itself, so that is the condition here. And if I put this condition under the limit that this is very, very small we can simplify the expression a bit, we can approximate the expression a bit.

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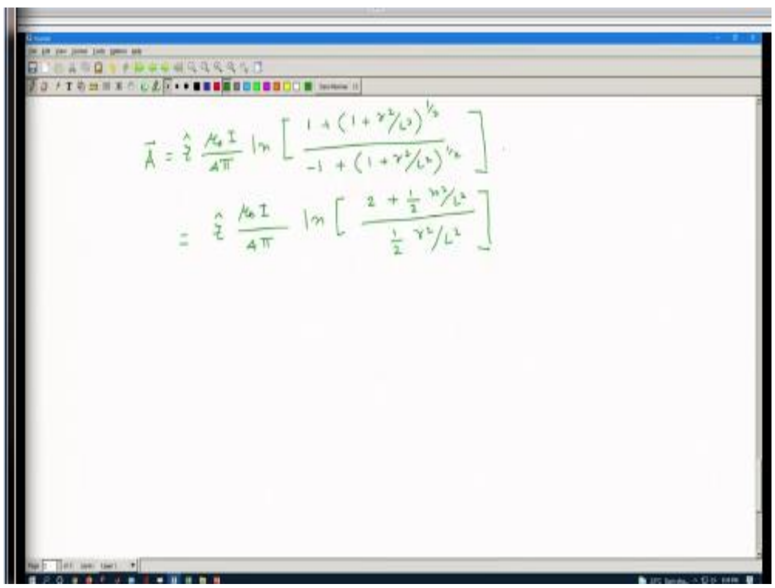

So, $\vec{A}$ is simply $\hat{z}$, then $\frac{\mu_0 I}{4\pi}$ and then I have ln and take L common so it should be (1 + (1 + $\frac{r^2}{L^2}$))$^{1/2}$ and in a denominator we have (-1 + (1 + $\frac{r^2}{L^2}$))$^{1/2}$. And then we make a binomial series kind of expansion of this quantity that simply leads to $\hat{z}\ \frac{\mu_0 I}{4\pi}$ and then ln $[\frac{2+\frac{1}{2}\frac{m^2}{L^2}}{\frac{1}{2}\frac{r^2}{L^2}}]$.

Please note I cannot neglect $\frac{r^2}{L^2}$ then we cannot do this. Because then we have an undetermined quantity, so we need to just make a binomial expansion here.

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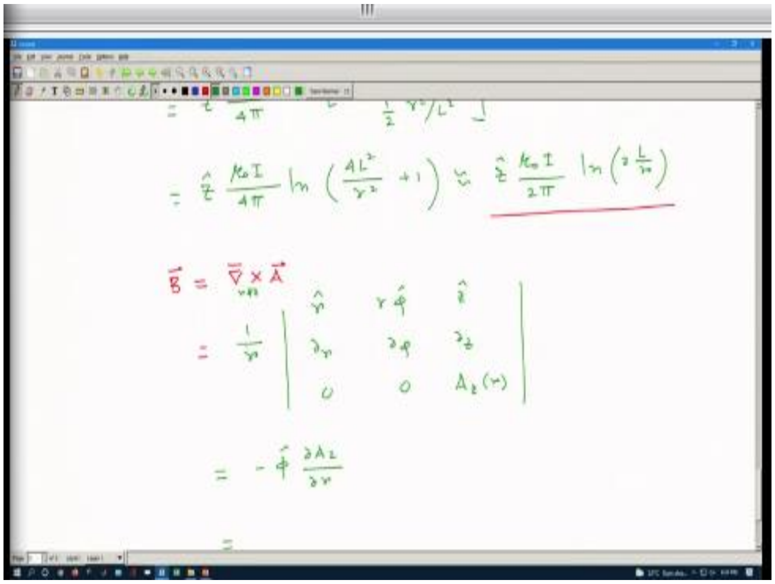

And then further simplification leads to $\hat{z}\,\frac{\mu_0 I}{4\pi}$ and then ln $(\frac{4L^2}{r^2} + 1)$, just divide everything. And that is again simplify as $\hat{z}\,\frac{\mu_0 I}{2\pi}$ and ln $(2\,\frac{L}{r})$ because $\frac{L}{r}$ is much, much greater than 1. So, I can neglect this 1 and I can simply write this as A^2 and this 2 can come out and we are going to get this. Now this is the expression we are getting, finally my $\vec{A}$ should be this one.

Now if I want to extract my $\vec{B}$ out of this $\vec{A}$, so what should be my $\vec{B}$? My $\vec{B}$ is $\vec{\nabla} \times \vec{A}$ and $\vec{A}$ is there, so I can simply have if I execute it $\frac{1}{r}$ because mind it $\vec{A}$ is in polar coordinates I need to write everything in polar coordinate or in cylindrical coordinate at least. So, it should be $\hat{r}$ then r $\hat{\varphi}$ and $\hat{z}$ in cylindrical coordinates. So, I need to exploit this at r φ z because $\vec{A}$ is in this way, written in this way, so this is ∂_r, this is ∂_φ and this is ∂_z and we have 0 0 and A_z.

Because only z component is there, which is a function of $\vec{r}$, that is all because this is only the direction of the $\vec{A}$ is along z. Now this value most of the terms are 0. So, only we have if I execute $-\hat{\varphi}\,\frac{\partial A_z}{\partial r}$ and that if I execute because A_z I know this is the quantity and if I make a derivative with respect to r.

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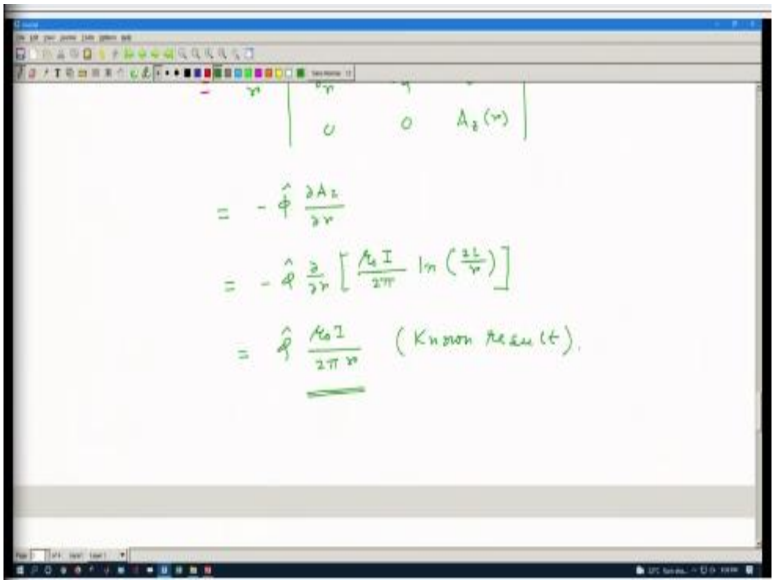

Then simply I have $-\hat{\varphi}\,\frac{\partial}{\partial r}$ and whatever the A_z I am having I am writing this $\frac{\mu_0 I}{2\pi}$ then ln $\frac{2L}{r}$ and this value is simply $\hat{\varphi}\,\frac{\mu_0 I}{2\pi r}$. So, this is the value I have because if I write so this I can write it as ln 2L - ln r, so ln 2L is a constant, so this will going to cancel out and ln r should be 1 by r and this negative sign is going to absorb here, so that is why I will be going to get this result.

But if you look carefully this result we already have, so this is a known result because we already derived in earlier class that what should be the magnetic field for a wire infinitely extended wire or very long wire. And the result was $\frac{\mu_0 I}{2\pi}$, the length, the distance that is r, so this result again we derive but this time in a different way, this time we are not using the Biot-Savart law, we are not using the Ampere's law.

We just use this expression to figure out $\vec{B}$ and here $\vec{A}$ is the value that we basically calculated here. So, this is the way we calculate $\vec{A}$ and from that we calculate $\vec{B}$, so let us do another problem.

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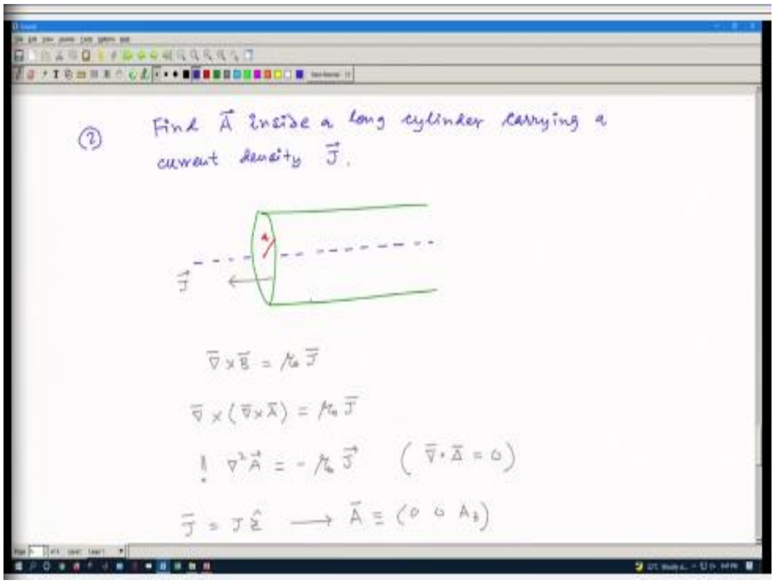

So, problem 2 we calculate find $\vec{A}$ inside a long cylinder carrying a current density $\vec{J}$. So, long cylinder is there and it is carrying a current density $\vec{J}$ and I want to find out. So, that is the cylinder, suppose this is the axis and the radius is say $\vec{A}$ and the current density that is given. So, suppose this is $\vec{J}$, so I need to find out the value of $\vec{A}$ that is the vector potential. So, $\vec{\nabla} \times \vec{B}$ is $\mu_0 \vec{J}$ and $\vec{\nabla} \times \vec{B}$ is called of is I just put this is simply $\mu_0 \vec{J}$, I am writing the Poisson equation once again.

So, that leads to $\vec{A} = -\mu_0 \vec{J}$ under this Coulomb gauge where $\vec{\nabla} \cdot \vec{A}$ is 0 under Coulomb gauge. So, now $\vec{J}$ is J $\hat{z}$ and that basically leads to because $\vec{J}$ and $\vec{A}$ are should be same direction here from this equation that we understand.

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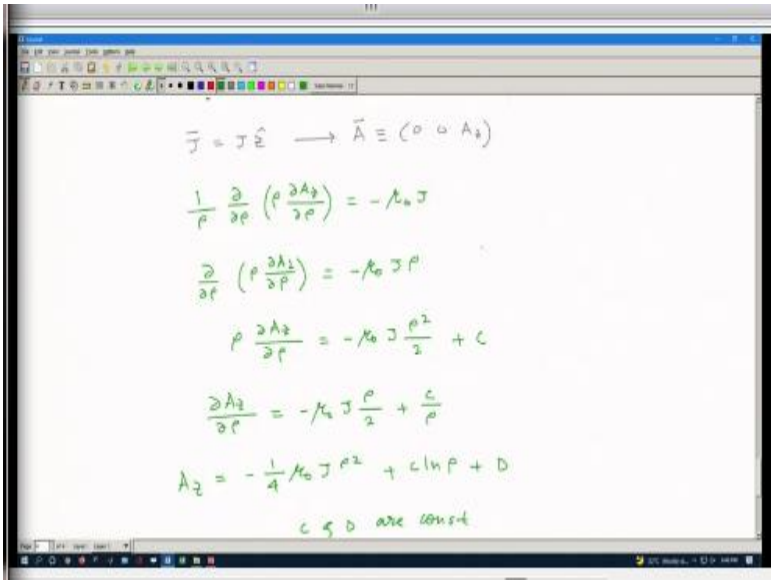

So, the $\vec{A}$ should be having no x component, no y component only the z component and that is A_z. Because from this equation we can say that $\vec{A}$ and $\vec{J}$ should be in same direction. Now using the cylindrical coordinate, so I am exploiting this equation in cylindrical coordinate what we can get is this. So, $\frac{1}{\rho}$ and then $\frac{\partial}{\partial \rho}$ and then A_z $\partial \rho$ should be $-\mu_0 \vec{J}$.

I am just writing only the ρ part because other components are not there. So, I have then $\frac{\partial}{\partial \rho}$ and then $\rho \frac{\partial A_z}{\partial \rho} = -\mu_0 J \rho$ or $\rho \, \partial A_z$, a partial differential equation we are solving now it should be $-\mu_0 J \frac{\rho^2}{2}$ and then + C, C is integration constant. Further, ∂A_z our aim is to find out $\frac{\partial A_z}{\partial \rho} = -\mu_0 J$ 1 ρ will going to cancel out, so $\frac{\rho}{2} + \frac{C}{\rho}$.

Again if I integrate then A_z simply comes out to be minus of 1 by say 4 and μ_0 then J, then ρ^2 then + C ln ρ + D. Now C and D are constant, which can be evaluated by exploiting the boundary condition. And the boundary conditions suggest that we have a ln ρ term that means if I want to find out something in the axis then ρ tends to 0. So, under ρ tends to 0, A_z should be finite but here we can see that this term can have a undefined quantity.

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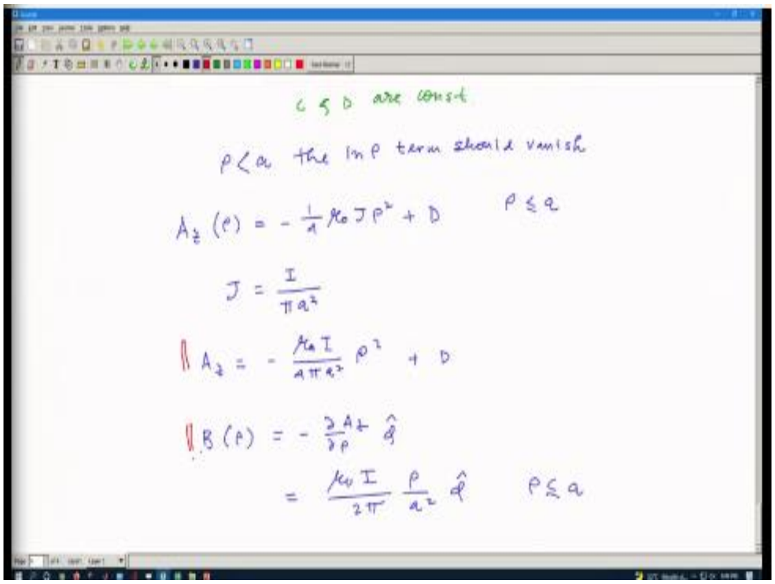

So, that means the C is 0, so when ρ is less than a the ln ρ term should vanish or ρ tends to 0, this terms will vanish. Under that condition I can write that A_z, which is a function of ρ should be - of $\frac{1}{4}$ then μ_0 J ρ^2 + D, ρ is less than equal to A inside that I want to find. Now what is J? J is the amount of current flow per area here, so I should write $\frac{I}{\pi a^2}$, a is given in the problem because the radius is a that is known.

So, in terms of that I can simply have A_z to be - μ_0 $\frac{I}{4\pi a^2}$, I just replace J here and then ρ^2 plus the constant D. If I write and without, so plus some constant D. From that expression I can also figure out my B, B(ρ) = $-\frac{\partial A_z}{\partial \rho}\hat{\varphi}$. And that quantity simply become μ_0 $\frac{I}{2\pi}$ μ_0 $\frac{\rho}{a^2}$ $\hat{\varphi}$ for ρ less than equal to a.

So, I figure out my a, I also figure out from A, I also figure out the magnetic field for this cylindrical system, which carries a current I. So, this is the way, so I just give 2 different examples to find out A, now I should stop because I do not have that much of time to continue. So, in the next class what we do that we will discuss more about the magnetic vector potential and then try to understand if a current carrying wire is placed, then how it is producing some magnetic vector potential.

And again like scalar potential we can have a multiple expansion and how the dipole momentum, magnetic dipole moments come appear etcetera. So, with that note I like to conclude here, so please try to practice few problems regarding calculating the magnetic field through A. So, first you need to calculate A the way I discussed so far a given system having the symmetry, use the symmetry and find out the value of the magnetic vector potential. And then from that calculate the value of the magnetic field B. So, thank you very much; see you in the next class.