Foundations of Classical Electrodynamics
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Lecture-52
Ampere's Law

Hello student to the foundation of classical electrodynamics course. So, under module 3 today we have lecture 52 where we discuss the Ampere's law.

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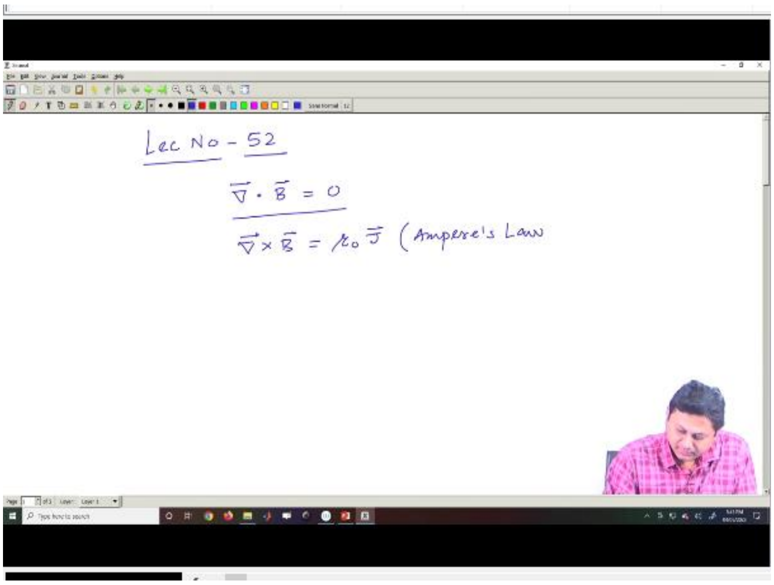

So, we have lecture number 52 today. So, far whatever we have done let me quickly write down. So, the $\vec{\nabla} \cdot \vec{B}$ we found 0 that is for magnetic field always the divergence is 0 showing that the magnetic monopole does not exist. Also if I do the curl of magnetic field $\vec{B}$ it should be $\mu_0 \vec{J}$ that is the Ampere's law and we will be going to derive this today.

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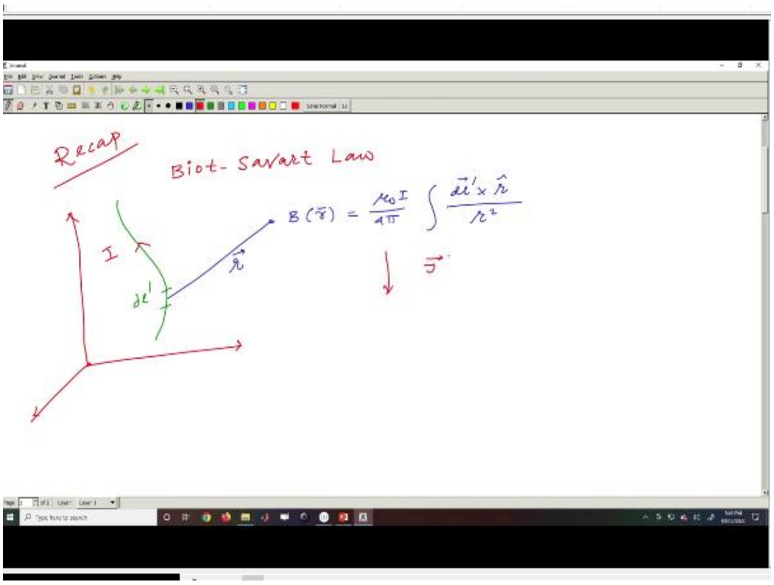

But before that let me recap quickly what we have done, we done the Biot-Savart law and Biot-Savart law it says that if I have a wire having some length say d$\vec{l}$' and the current is carrying the amount of current that is passing through this wire is I. So, it should produce some kind of magnetic field at some distance say at this point the magnetic field that should be a function of $\vec{r}$ and if I have the length here say $\vec{r}$.

So, this quantity is $\frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}$ and this $\vec{\mathfrak{r}}$ is measured from some coordinate; some coordinate system is here. So, from that I can measure this $\vec{\mathfrak{r}}$ the location of this point from this coordinate system.

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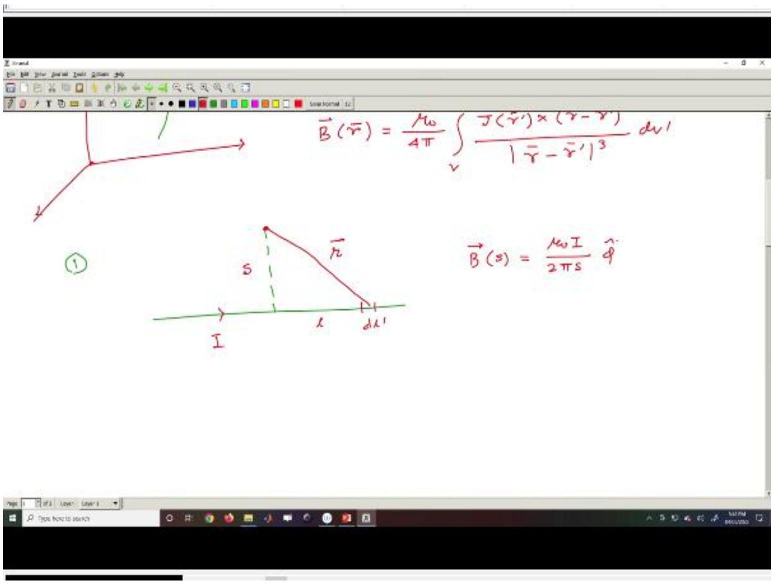

So, in terms of the current density $\vec{J}$ also one can find out this expression and if I do the expression should look like this way, it is $\frac{\mu_0}{4\pi}$. This is the direct consequence of the Helmholtz theorem that we discussed in the last class $\frac{J(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$ dv'. This is the volume integral. When we apply this Biot-Savart law some standard problems.

Say in the first problem that is if I have infinite wire and if I want to find out the magnetic field where this wire is carrying a current I and if I want to find out what is the magnetic field at this point then we take a small section d$\vec{l}$', this is the length $\vec{l}$ and then try to understand this distance where say $\vec{\Pi}$ and do some calculation, but eventually we get this value. So, my $\vec{B}$ say this is distance S. So, $\vec{B}$ at S that was $\frac{\mu_0 I}{2\pi S}$ and then ϕ, this is for infinitely long wire.

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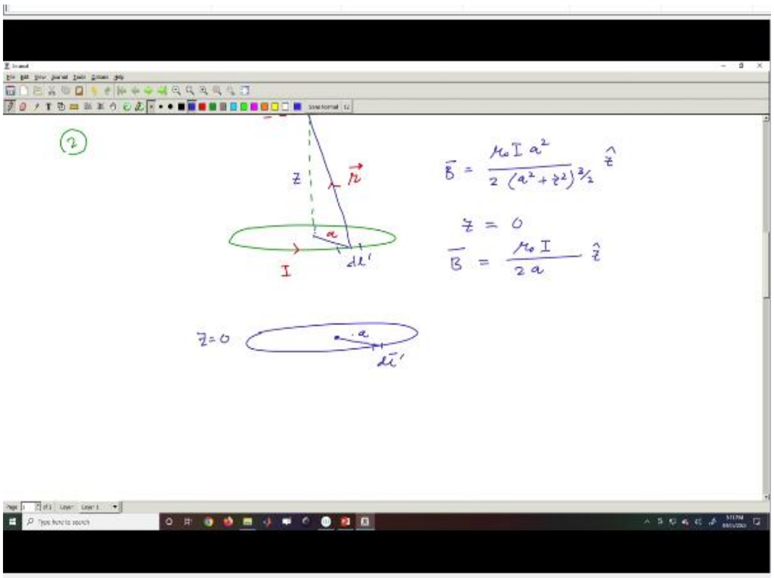

Then another case we discuss where we have a current carrying ring and this is the axis of symmetry and over the axis of symmetry we tried to find this was d$\vec{l}$' and the current was I and if I want to find out the magnetic field here then the magnetic field. So, if this is $\vec{\Pi}$ and if this is a, then the magnetic field should produce along some distance like a direction like this but they will form like cone and this is my d$\vec{B}$' and $\vec{\Pi}$ will be along this direction, this is $\hat{\Pi}$.

So, we calculated and we find that the value magnetic field is μ_0 I and then this is z then $\frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}}$ and that was in z direction because the horizontal component will cancel out and we have only one direction along this. This is the $\hat{z}$. So, that was another problem and if z = 0 then we had $\vec{B} = \frac{\mu_0 I}{2a}\hat{z}$ that is the centre of the ring.

So, that was the current carrying ring and if I want to find out the magnetic field here at the center then that was the value, $d\vec{l}'$ and we calculate and this is a. So, at z = 0 this is the point where the magnetic field will be this amount and then we calculated the force. So, that was case 3.

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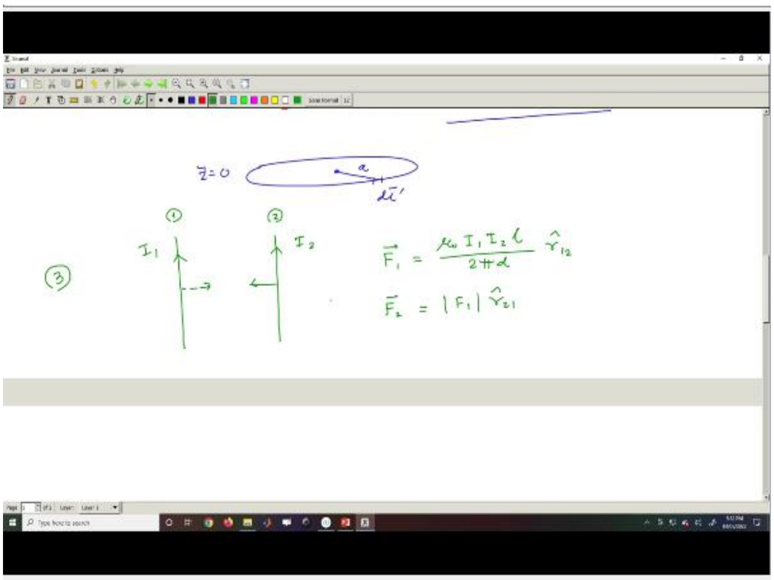

And two wires were there like this carrying current I_1 and I_2 that was say wire 1 and wire 2. The force between these 2 we also calculated and the value that we got is say $\vec{F}_1$ the force is μ_0 I_1 I_2, if the length is l then it is l that is $2\pi d\ \hat{r}_{12}$. That was the value and $\vec{F}_2$ was $|F_1|$ the amplitude is same only the unit vector was from this side to this side that we all calculated. So, now in today's class what we do is suppose go back to the same problem.

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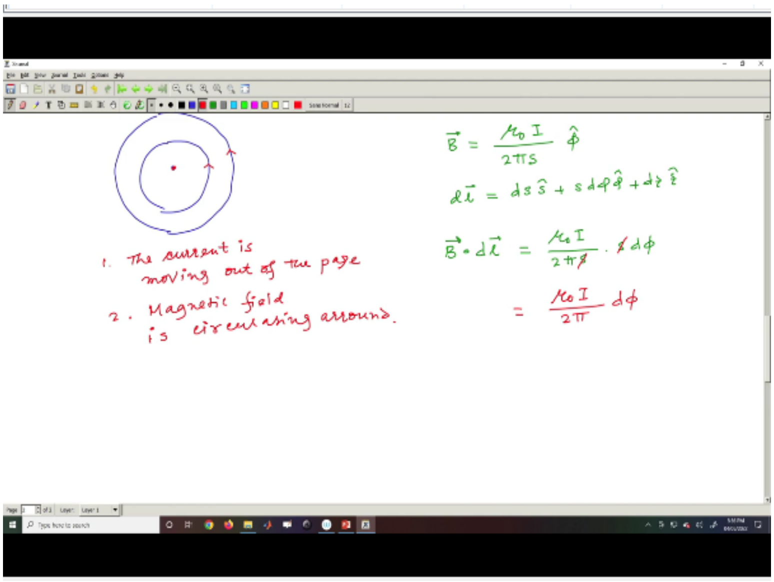

Suppose I am having a wire and suppose this is the wire and which is perpendicular to the plane and the current is flowing, if the current is flowing the magnetic field that will produce is surrounded like this and the current is moving upward. So, here the current is moving out of the page and second thing is that the corresponding magnetic field is circulating around. So, the value of the magnetic field we know for this case because if the wire is infinite.

So, I will be going to use the result that we got here, the first result. So, this will be the amount of magnetic field. So, if I write it down. So, it should be $\vec{B} = \frac{\mu_0 I}{2\pi}$ and if the length is s then s and $\hat{\varphi}$, $d\vec{l}$ if I calculate the line element here then it should be ds $\hat{s}$ + s dφ $\hat{\varphi}$ + dz $\hat{z}$. So, that should be the line element. So, that means if I want to integrate over this circle line then what is the value?

So, that precisely we will do. So, if I want to find out the $\vec{B} \cdot d\vec{l}$ this quantity then we can see that the contribution of the $d\vec{l}$ should have with this φ and eventually we have $\frac{\mu_0 I}{2\pi s}$ and then s dφ and s will be going to cancel out here and here. So, we have something like $\frac{\mu_0 I}{2\pi}$ and then dφ. Now what we do we will be going to integrate over the φ. So, let me draw it once again.

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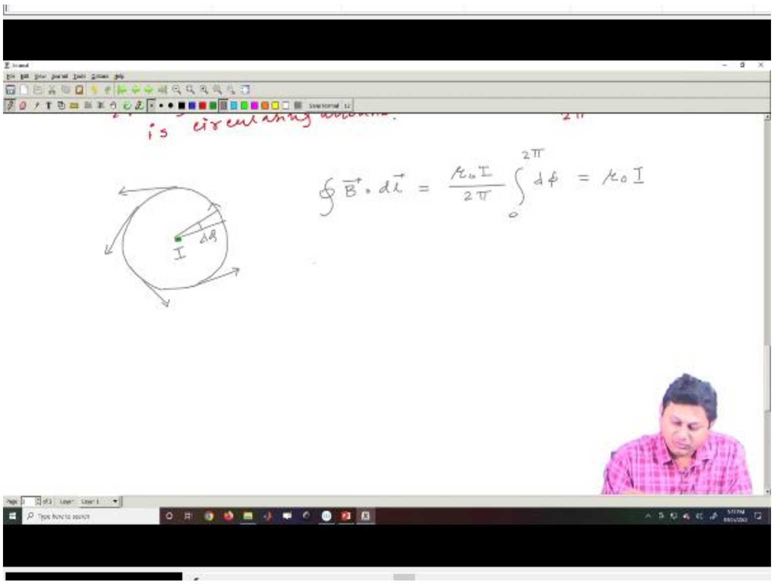

So, I have the current, which is coming out the page and then I have the magnetic field surrounding this current. So, this is the current that is flowing and this is the magnetic field, this is the way magnetic field is generating. Now I want to integrate this closed line. So, if I make a tangential in each point. So, that should be the direction of the magnetic field. So, magnetic field is this every time the tangential line is a momentarily direction of the magnetic field but it is revolving like that in this way it is circulating.

So, I integrate the close line integral if I do of $\vec{B} \bullet d\vec{l}$. So, what we get is $\frac{\mu_0 I}{2\pi}$ and then I integrate this quantity 0 to 2π because this is my dφ and I am integrating over entire circle. So, that quantity is simply 2π. So, eventually I have μ_0 I. Now $\vec{B} \bullet d\vec{l}$ integration $\vec{B} \bullet d\vec{l}$ integration is. So, let me first write it this is a very important expression.

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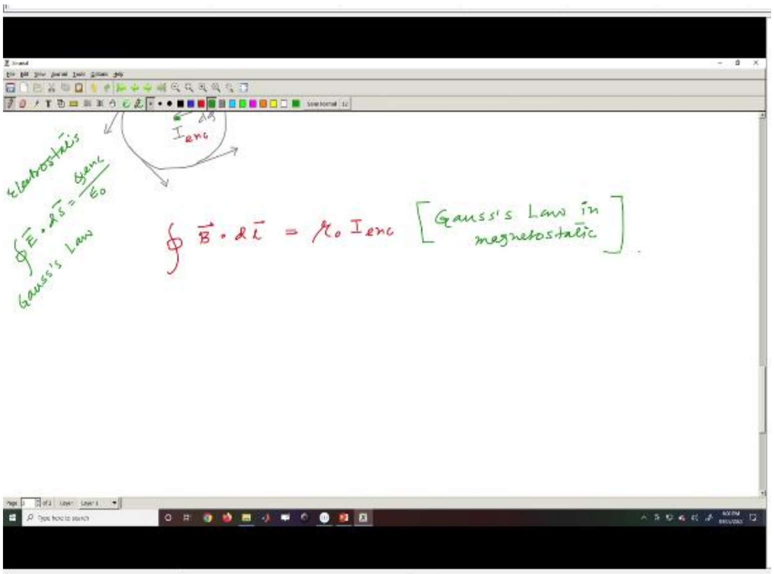

So, in closed line integration of $\vec{B} \cdot d\vec{l}$ is μ_0 I, but now I write as I enclose because the current whatever the circular line I am drawing here the current is inside. So, that is why I write this current is enclosed. So, it is indeed a Gaussian kind of law. So, sometime it is also called because in electrostatic we had a similar law where we had $\vec{E} \cdot d\vec{s}$ is ρ I $\frac{Q_{enc}}{\epsilon_0}$.

So, here we are having the similar kind of. So, electrostatic what we got is this in electrostatics, what we had is this for electric field $\vec{E} \cdot d\vec{s}$ that was $\frac{Q_{enc}}{\mu_0}$ and that was the Gauss's law in electrostatics. Here we have exactly similar kind of things but instead of having the surface integral I have a close line integral and instead of having $\frac{1}{\epsilon}$ we have μ_0 and instead of having Q I have I.

Q was the static charge there, now here is a steady current. So, for every time when we are dealing with the magnetostatic problem I need to have a steady current last day we mentioned that. So, this is Gauss's law in magnetostatics. Now I enclose; what is I enclose let us now look carefully what is the meaning of this I enclose.

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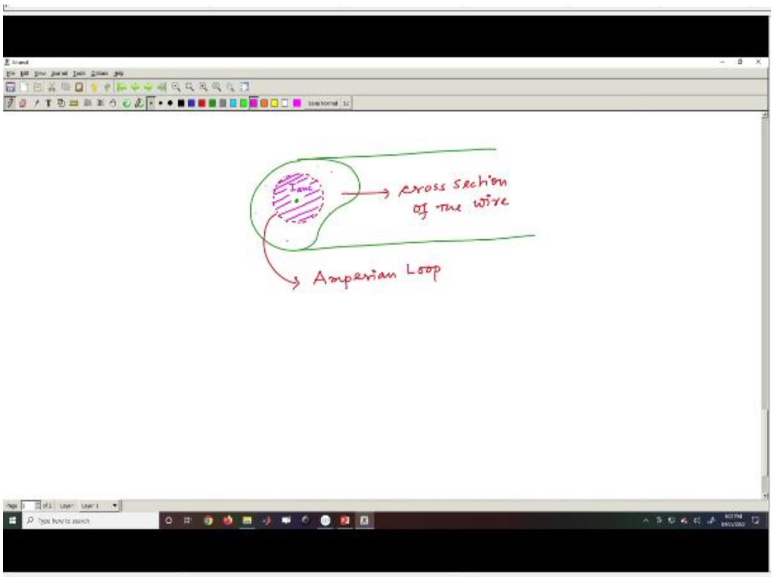

That is the current density if I have some current density $\vec{J}$ then that $\cdot d\vec{s}$, suppose I have a current carrying wire like this, suppose this is the wire I make a cross section of this wire and this is the axis any arbitrary shape wire and I have the axis and if I now draw a circle around this point. So, this is like the Gaussian surface now I have a Gaussian circle like this sometimes it is called Amperians loop.

We will discuss this later. So, this is as I mentioned is called like the Gauss's surface here I just draw Amperian loop and this is the cross section of the wire and the current that is flowing through is the amount of current that we call the I_{enc}. So, I should write it here the I_{enc} as this one. So, current is flowing here. So, this amount is I_{enc}, which is here in this cross section.

In this circular under inside the Amperian loop whatever the current is flowing that is I_{enc}. Mind it also some amount of current is flowing because through the entire wire the current is flowing but we are only considering the I_{enc} only for this region, but other region also having the flow of the current. The current is also flowing outside this region. Here also the current is flowing these regions the current is flowing. But inside the Amperian loop whatever the current I am writing is the amount of current that we call the current enclosed.

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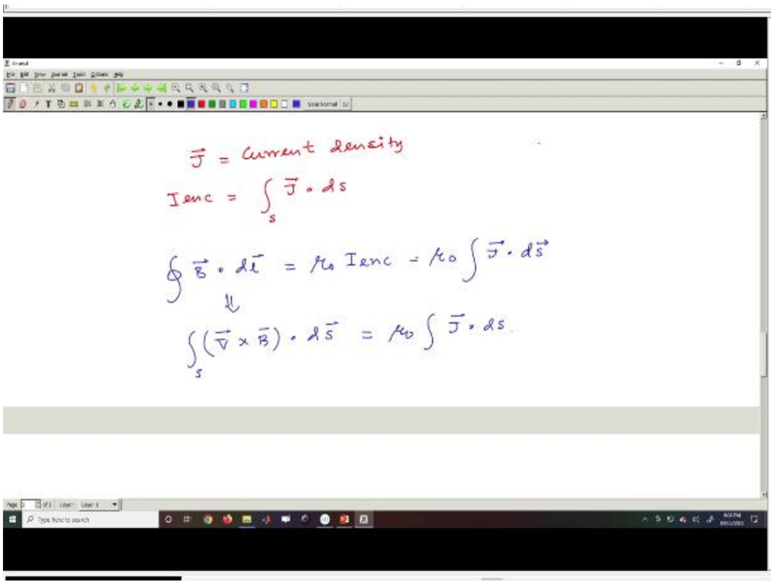

Now $\vec{J}$ is current density and obviously if I know the current density I can write the enclosed current is nothing but the integration of $\vec{J} \cdot d\vec{s}$ over this region and that is precisely I wrote here. Now let us exploit this expression to find out the another form, the integral form is there. So, I will be going to exploit this. So, we find $\vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$. And I_{enc} I can write in terms of the current density $\vec{J}$ and if I do it should be $\vec{J} \cdot d\vec{s}$. Now applying the Stokes law this I can write as surface integration $(\vec{\nabla} \times \vec{B}) \cdot d\vec{s}$ and that amount here is μ_0 integration of $\vec{J} \cdot d\vec{s}$. Now this is true for any kind of surface.

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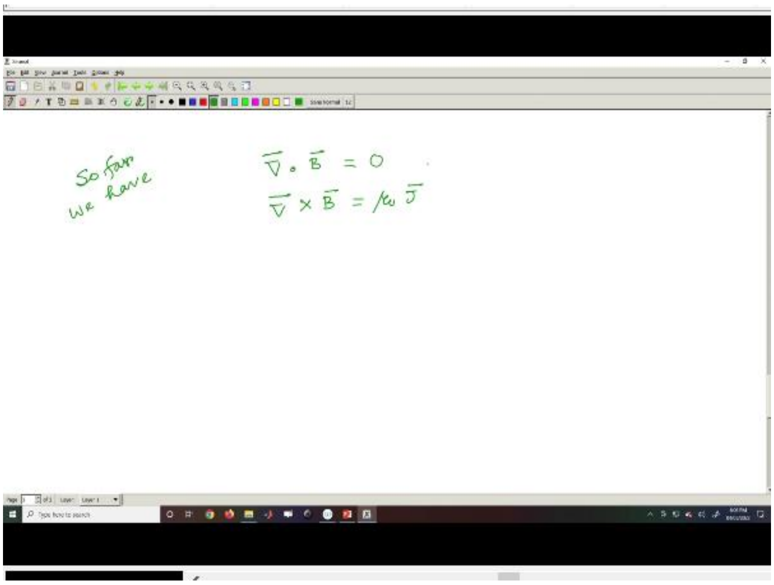

So, I can have the expression $\vec{\nabla} \times \vec{B}$ is simply $\mu_0 \vec{J}$, which is nothing but the Ampere's law. So, far we have I already wrote that today. So, far we have $\vec{\nabla} \cdot \vec{B} = 0$ that we proved few class ago and now today we write $\vec{\nabla} \times \vec{B}$, which is a non-zero quantity is $\mu_0 \vec{J}$ that way. Now let us now look carefully these 2 equations. These two very, very important equations.

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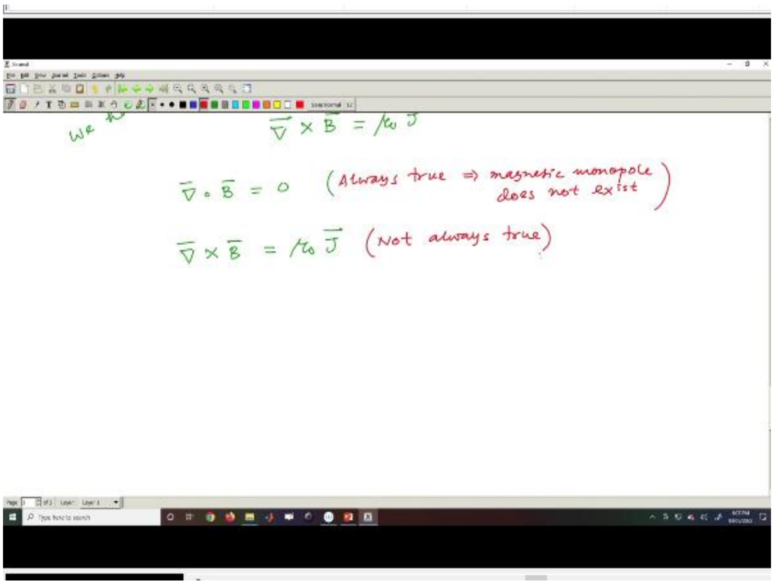

So, this is always true where $\vec{\nabla} \cdot \vec{B}$ is 0, this is always true and it reflects the fact that magnetic monopole does not exist, but is this always true $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$. Is this always true the question? And the answer I should write here is not always true. This is Ampere's law, this is not always true.

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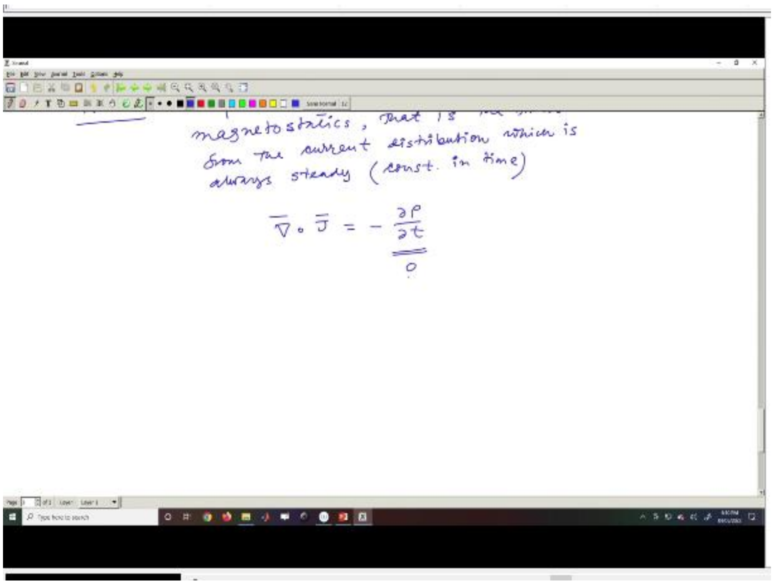

So, I should write here a note that the Ampere's law is only valid for magnetostatic that means or that is the field from the current distribution, which is always steady. What is the meaning that means constant in time. So, the Ampere's law is only valid for the magnetostatic and what is the condition of the magnetostatic in the first class exploiting the continuity equation that was $\vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$ that was the condition.

And we mentioned that in magnetostatic we have always a steady flow of current. So, $\frac{\partial \rho}{\partial t}$ that condition has to be 0. So, eventually the $\vec{\nabla} \bullet \vec{J}$ comes out to be 0. The same thing we will be going to show here but it is not always true.

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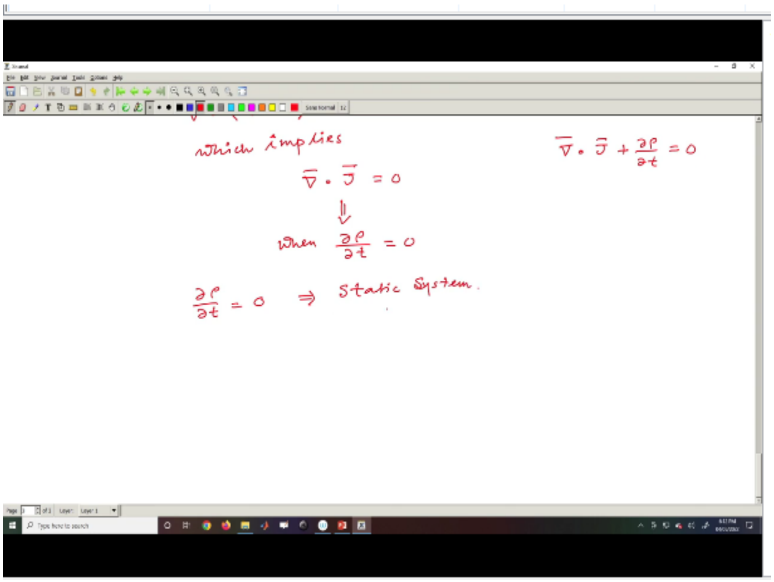

So, $\vec{\nabla} \times \vec{B}$ we say it is $\mu_0 \vec{J}$ but I am saying that this is not always true because if I make a divergence both the side what happen? Let us check. I make a divergence of both the side. So, divergence of curl of something has to be 0, it is identically 0, there is no condition this is unconditionally 0, what about the right-hand side? In the right-hand side I have $\mu_0 \vec{\nabla} \bullet \vec{J}$.

Now as I mentioned this quantity is unconditionally 0 that means this quantity has to be always 0 but. So, which implies these has to be 0 but this is not always because we know that from the continuity equation that this is not 0 rather this is $-\frac{\partial\rho}{\partial t}$. So, under which condition this 0 when $\frac{\partial\rho}{\partial t}$ is 0 that is precisely we are saying. So, this is 0 means that when $\frac{\partial\rho}{\partial t}$ is 0 that is the condition.

So, left-hand side is always 0 but right-hand side inside is conditionally 0. So, that creates some issue. So, we will be going to solve this later on. So, that is the contribution of the Maxwell and $\frac{\partial\rho}{\partial t} = 0$ is simply the condition for this static case. So, this is the static system. For static system we have this to be 0. Well in time dependent system that means the Ampere's law need to be modified. So, in time dependent system.

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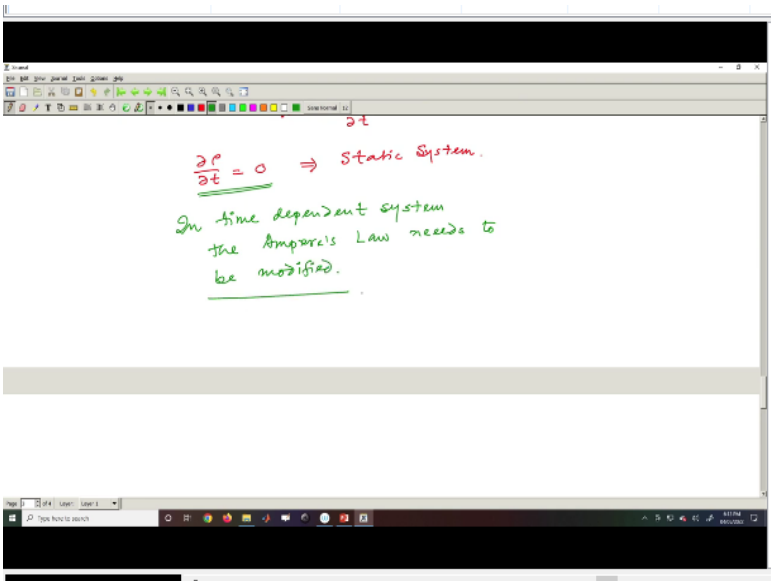

So, that means in time dependent system the Ampere's law, which is true only under this condition needs to be modified. So, we will see later how the modified. We can modify that. So, these two very important things now we derive directly from the Biot-Savart law. So, we already have.

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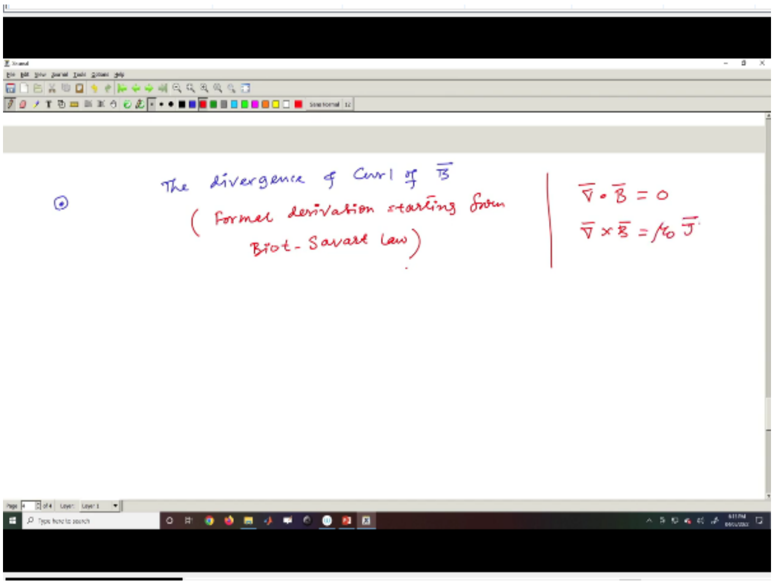

So, next thing is the $\vec{\nabla} \cdot \vec{B}$ and $\vec{\nabla} \times \vec{B}$, which we already derive, but here we do it, this is a formal derivation starting from Biot-Savart law. So, the result is already known that $\vec{\nabla} \cdot \vec{B}$ we should get 0 and $\vec{\nabla} \times \vec{B}$ in magnetostatic it should be $\mu_0 \vec{J}$. But exploiting the Biot-Savart law the formal derivation is there. So, we will try to do that here.

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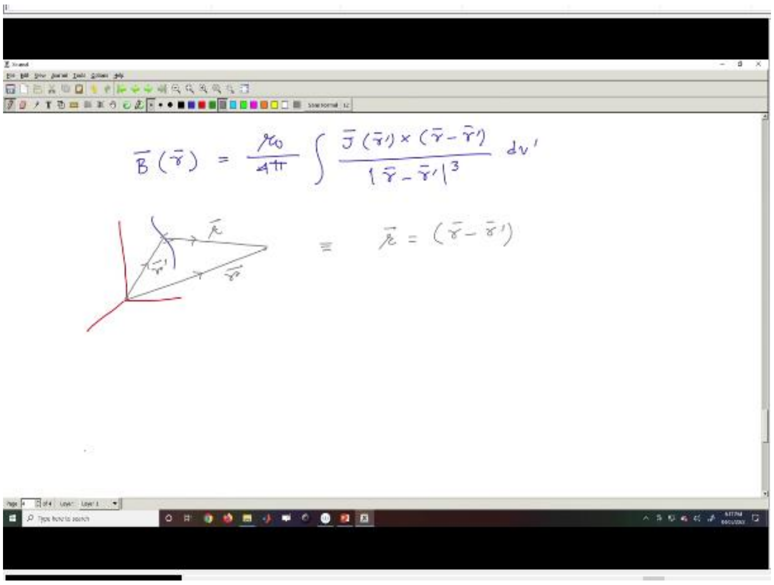

So, the Biot-Savart law says $\vec{B}$ at some point $\vec{r}$ should be $\frac{\mu_0}{4\pi}$ and then $\int \frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$ dv'. That is one form of Biot-Savart law, which includes the current density $\vec{J}$. Biot-Savart law so, what was $\vec{r}$? So, let me quickly draw the coordinate systems. So, here we have a coordinate system and the wire is something here and the magnetic field I want to find out here. So, this is the point.

So, this is my $\vec{r}$ and the location here from which I calculate the magnetic field is $\vec{Л}$ and from the origin to the source point that is my $\vec{r}\,'$ I need to be careful enough in writing the notation. So, this is $\vec{r}\,'$. So, these are the three vectors that was involved here. So, that means from here I can write this my $\vec{Л}$ is $\vec{r}$ - $\vec{r}\,'$ that is my $\vec{Л}$. So, let us replace this in the expression of the Biot-Savart law.

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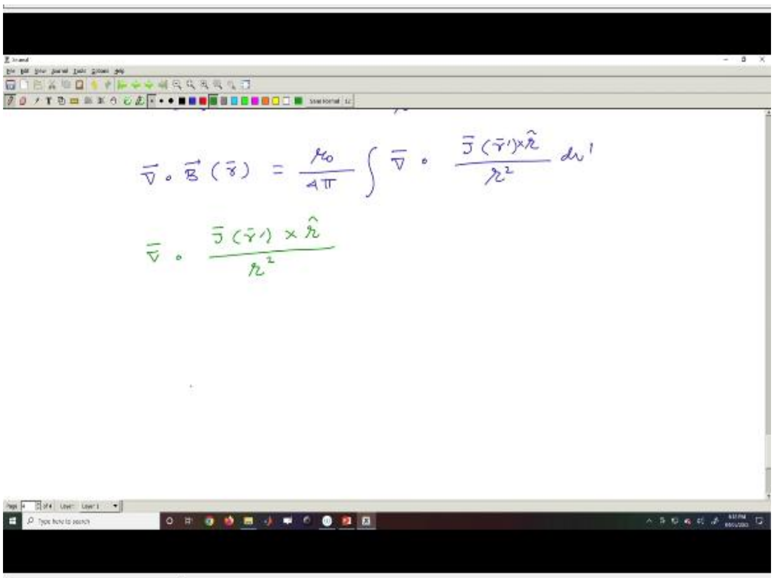

So, the $\vec{B}$ now as a function of $\vec{r}$ is $\frac{\mu_0}{4\pi}$ then integration of $\vec{J}$, which is at $\vec{r}\,'$. Now I write $\vec{r}$ and this $Л^2$ dv'. This is my Biot-Savart law. Now I simply try to find out what is the divergence of this quantity where right-hand side is there. So, if I make a divergence. So, I should write this $\frac{\mu_0}{4\pi}$ and then I put this divergence inside and divergence of this quantity $\frac{\vec{J}(\vec{r}\,') \times \hat{Л}}{Л^2}$ dv' and there I am missing a cross here.

So, cross is always there. So, this is cross. So, I eventually need to evaluate this quantity. So, $\vec{\nabla} \bullet \frac{\vec{J}(\vec{r}\,') \times \hat{Л}}{Л^2}$. This divergence operator is operating over $\vec{r}$. So, if I expand this.

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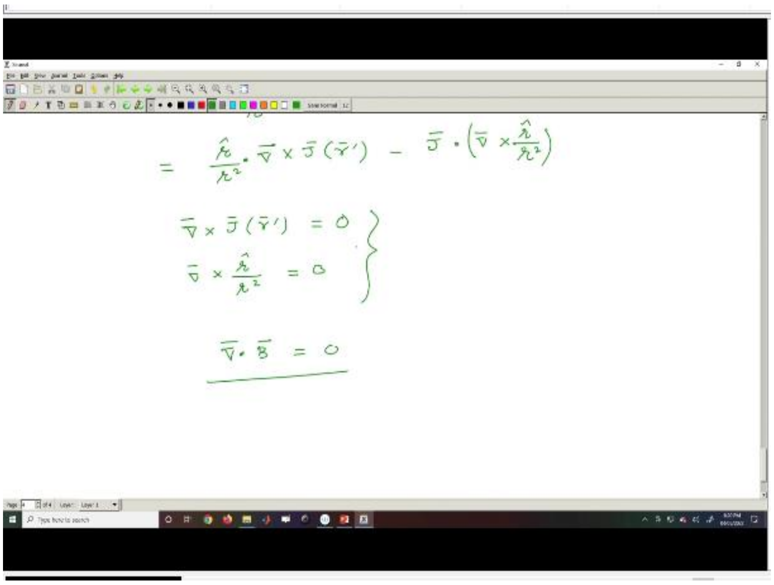

So, this quantity is simply $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ and then $\vec{\nabla} \cdot \vec{J}$, which is at $\vec{r}\,'$ and $-\vec{J} \cdot (\vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2})$, just exploiting the vector identity I can get this. So, I am making a mistake here. So, this is curl because here we have a divergence. So, it should be dot of this quantity and then $\vec{J} \cdot$ of that quantity now it is correct. So, I missed the dot here. So, $\vec{\nabla} \times \vec{J}(\vec{r}\,')$ this has to be 0 because r is J depends on the prime variable and this operator is non-prime.

So, that is and also the $\vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ it has to be 0. So, both the cases it is 0. So, that tells us simply from the left-hand side and the right-hand side I can write that $\vec{\nabla} \cdot \vec{B}$ simply 0 and I calculated that exploiting the Biot-Savart law the formal calculation. Now the next one, which is important.

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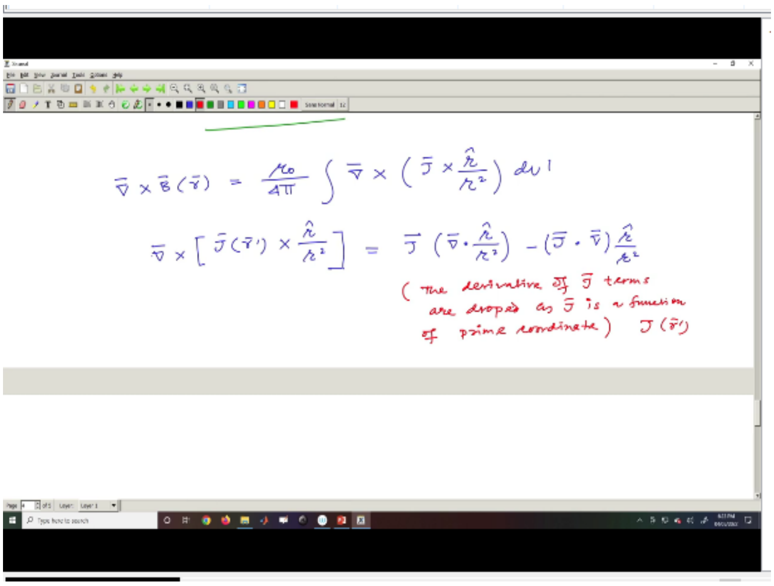

The next one is how to calculate the curl. So, $\vec{\nabla} \times \vec{B}$, which is a function of $\vec{r}$ is simply $\frac{\mu_0}{4\pi}$, $\vec{B}$ I know exploiting the Biot-Savart law I find it. So, it is $\vec{\nabla} \times (\vec{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2})$ that should be the form and dv'. Now again I need to use the vector identity curl of say $\vec{J}$, which is a function of $\vec{r}\,'$ $\times$ $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$. So, it is $\vec{J}$ and then I have $\vec{\nabla} \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ and - $\vec{J} \cdot$ this and that should be operating over $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$.

Please note these vector identities. So, here all the derivatives also we have other few terms. So, please note it, the derivative of $\vec{J}$ term, all the $\vec{J}$ terms are dropped as $\vec{J}$ is a function of prime coordinate, because $\vec{J}$ is function of $\vec{r}\,'$ and whenever this operator is operating over $\vec{J}$ it gives rise to 0 that we already got.

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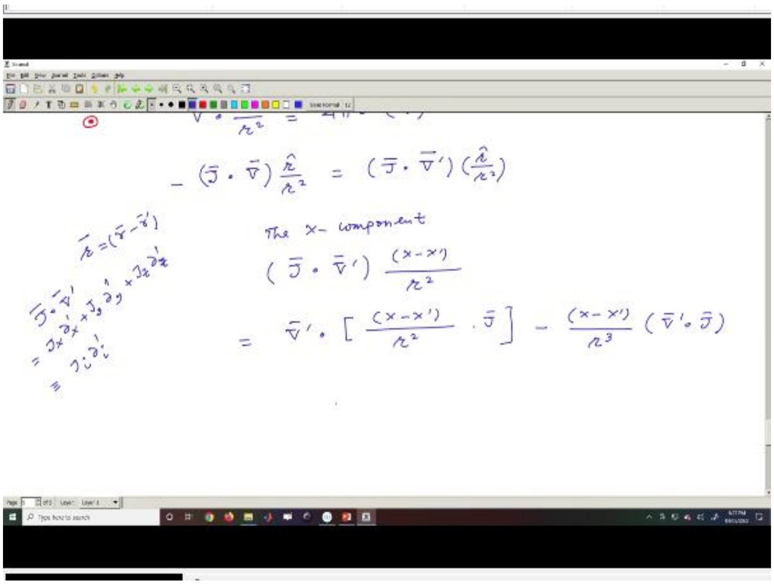

Now let us remind a very important vector identity that we also proved that $\vec{\nabla} \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$, we had it here, here we are having this. So, that thing we know because we derived it in the initial few classes that is $4\pi\ \delta^3(\mathrm{r})$, δ^3 stands for the three dimensional delta functions. So, - of $\vec{J}$ dot this quantity $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ this is also there is now I can write because it is operating over $\vec{r}\,'$, it is operating over $\vec{r}$ and inside the $\vec{r}$ we have prime and non-prime both.

So, I can make it prime and in order to do that. So, $\vec{\mathscr{r}}$ is $\vec{r}$ - $\vec{r}\,'$. So, whenever I change this operator to prime this negative sign is going to absorb. So, I should have $\vec{J}$ dot this prime that is now operating over $Л^2$. Now the x component if I calculate only. So, I have $\vec{J}$ dot this operator prime and the x component of entire stuff is $\frac{x-x'}{Л^2}$.

$Л^2$ is a scalar and that will be going to operate over x. So, what is this quantity $\vec{J}$ dot this operator? So, this operator will be simply J_x ∂_x' + J_y ∂_y' this operator should be something like this J_y ∂_y' and J_z ∂_z' or in short notation it is ∂_i'. This will be going to operate over the entire stuff. So, these simply gives like this operator is operating over this x – x' over $Л^2$ I can split this stuff $\vec{J}$.

So, this operating over that means this minus of plus of these things. So, I can put it here. So, that quantity. So, this operator that is operating over this x component and then over $\vec{J}$. So, that thing I can split into two parts, this one and this one will appear. So, these things I can write in this way. So, just exploiting the vector identity and also this operator how this operator looks I also write in the left-hand side. So, please check it by doing this once.

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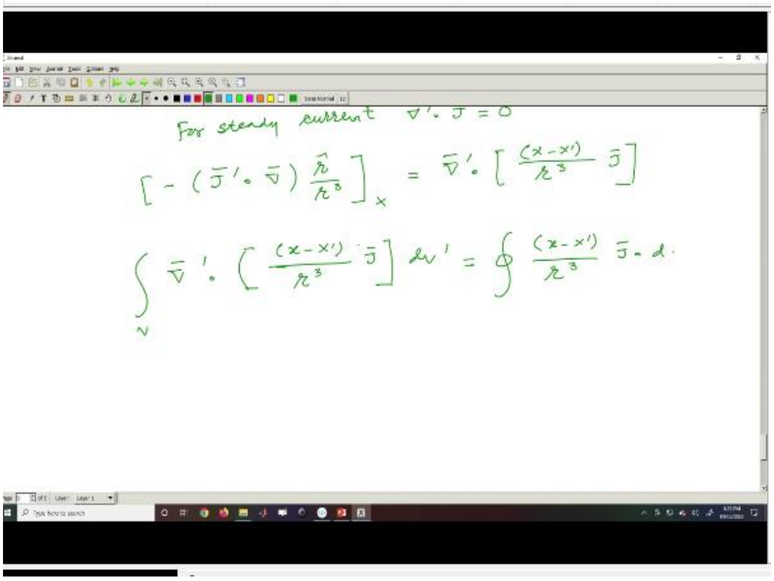

Now for steady current we have this equal to 0. So, for steady current we if we have this 0 then eventually we have - $\vec{J}$' dot this $\frac{\vec{Л}}{Л^3}$, the x component of that stuff is simply $\frac{x-x'}{Л^3}$ $\vec{J}$. So, now I am going to use this into the integral. So, the volume integration of this quantity, which is there, which is $\frac{x-x'}{Л^3}$ and then we have $\vec{J}$ dv'. If I now write in terms of surface integral. So, this is simply $\frac{x-x'}{Л^3}\vec{J} \cdot d\vec{s}$' just using the Gauss's law closed surface integral.

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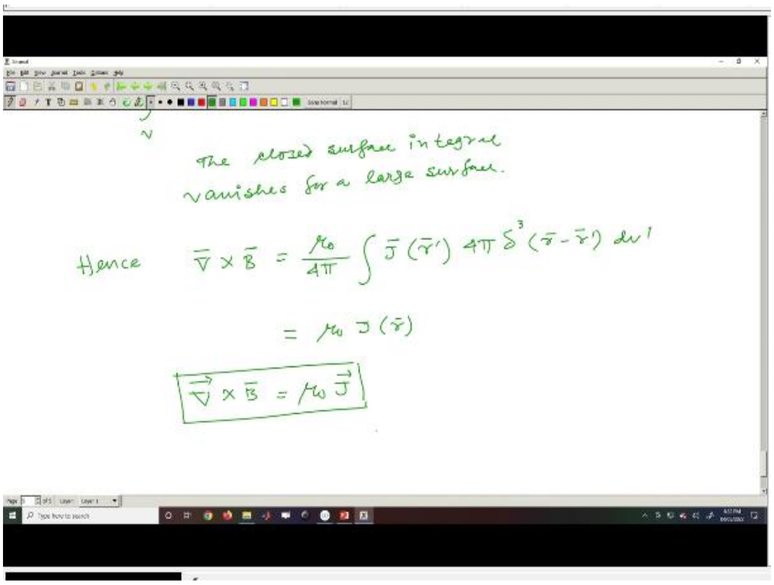

Now this closed surface integral vanishes for a large surface, if I take a sufficiently large surface then this $\vec{r}$ should be this is $\frac{1}{r}$ because the surface here ds will be r^2 and the dependence is $\frac{1}{r}$. So, these things eventually vanishes. So, if this is not there then I can have only one term remaining and hence I can have $\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi}$ and we have the term that we know that $\vec{r}\,'$ and we have 4π $\delta(\vec{r})$. So, it is eventually δ^3.

So, $\vec{r}$ I can write at $\vec{r} - \vec{r}\,'$ and that is over the entire volume integral because rest of the term is 0. So, please check it that I had the curl of both side. So, when I calculate this curl. So, this curl is containing two term, one is this one and another is $\vec{J}$ dot this thing. So, this is my term number 1 and this is my term number 2. So, term number 1 is sustaining because term number 1 is calculated.

So, this is my term number 1 here, whatever the term number 2 this term number is minus of this quantity is term number 2. So, I write it term number 2 here. So, this is my term number 2 and this second term is vanishing if I take a very, very large surface. So, that will not going to contribute here anymore. So, in that case finally we have because this is related to delta function I simply have now μ_0 $J(\vec{r})$.

So, I find that $\vec{\nabla} \times \vec{B}$ is equal to μ_0 J that I wanted to prove and I show that exploiting the Biot-Savart law just the formal derivation starting from the Biot-Savart law we calculated that. So, with that note I like to conclude here. So, I do not have much time today. So, thank you very much for your attention and see you in the next class