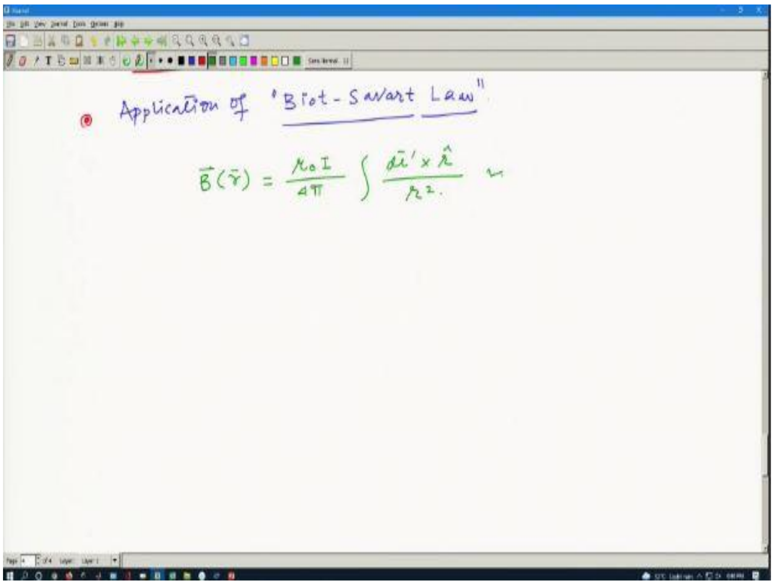
Foundations of Classical Electrodynamics
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology-Kharagpur

Lecture-51
Application of Biot-Savart Law

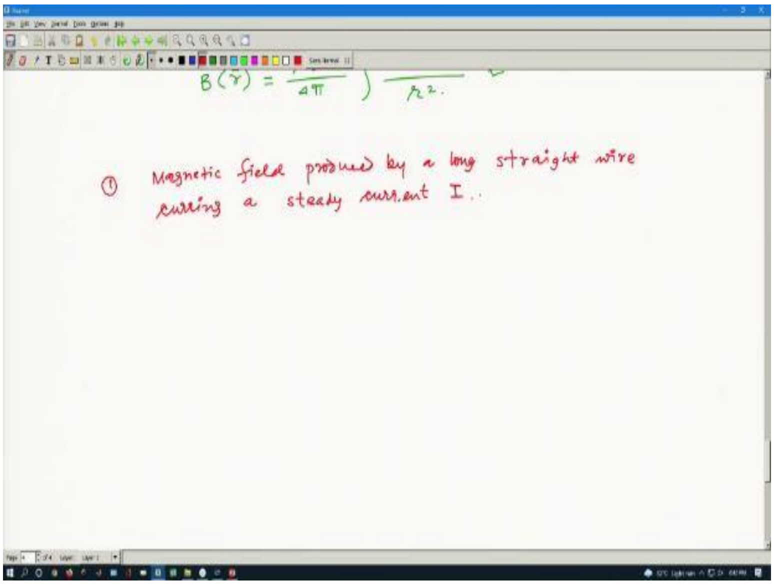
Hello student, to the foundation of classical electrodynamics course. So, we are in module 3. And under module 3, today we have lecture number 51, where we try to understand some application of the Biot-Savart law.

(Refer Slide Time: 00:32)



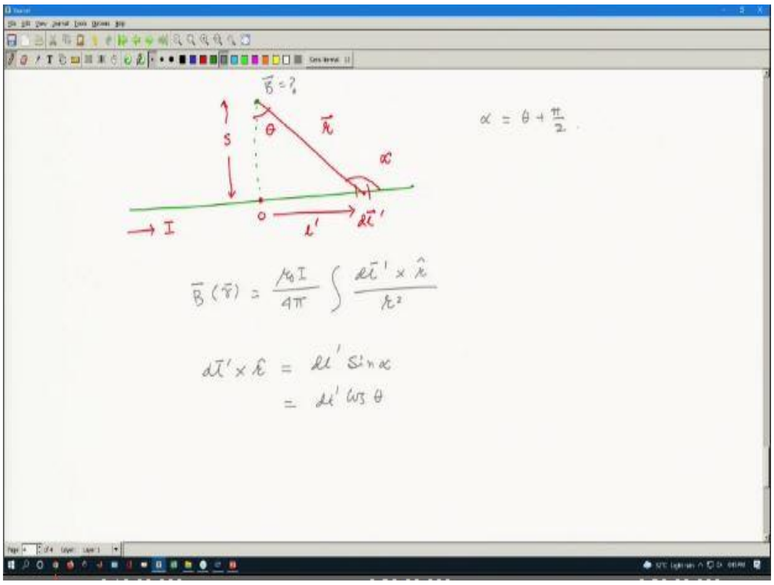
So, today we have class number 51 and today's topic is applications of Biot-Savart law. So, formally the Biot-Savart law, in the last class we have written that at some point r the magnetic field can be written in this way. The amount of magnetic field due to the current carrying wire is written in this way, $\frac{d\vec{l}' \times \hat{r}}{r^2}$. That is the mathematical form of the Biot-Savart law in general. And also we wrote the Biot-Savart law in terms of the current density by exploiting the Helmholtz theorem. So, we will be going to use that later.

(Refer Slide Time: 02:22)



So, let us first do the problem say 1 or case 1. So, the magnetic field what we do that here, the magnetic field produced by a long straight wire carrying a steady current I.

(Refer Slide Time: 03:35)



So, suppose I have a very long wire like this. And I want to find out the magnetic field some point here. And the steady current I is flowing that gives us the magnetic field. So, what should be the magnetic field here that is the problem, very standard problem? So, let us take this distance from here to here as s. And let us consider small segment here dl and from here to here this is say $\vec{r}$.

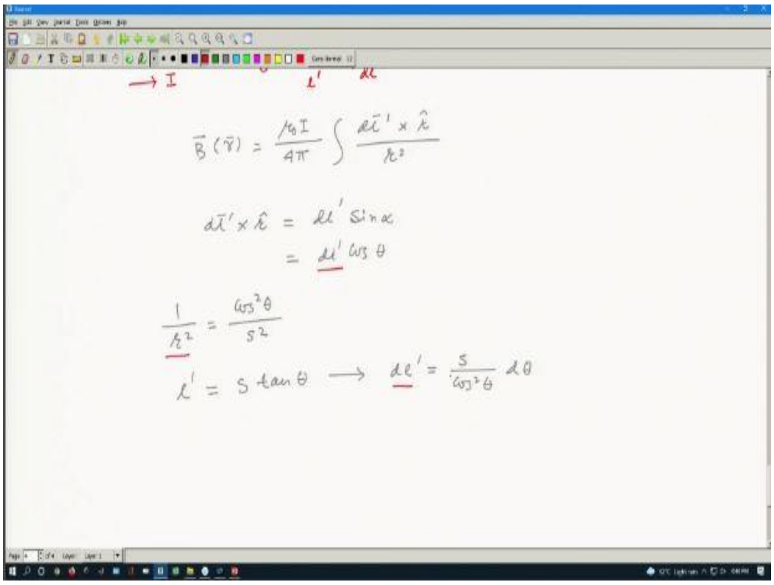
So, this angle I will be requiring that so say α and from here so if this is origin say o, so from here to here the length is so say this is $d\vec{l}'$ and $\vec{l}$ is $\vec{l}'$. So, this is the source point, so that is why always we represent it in prime. So, this angle also we will be requiring so, let us put this angle as θ. So, this is the geometry. So, we want to find out what is the amount of magnetic field at this point this is the question mark.

So, already we wrote down the amount of magnetic field that can be produced and based on the Biot-Savart law the value, I am writing it once again is $\mu_0 I$, I is a steady current, so, I can safely put it outside the integral and then $\frac{d\vec{l}' \times \hat{r}}{r^2}$. So, that is the thing. So, you can see clearly, that if I integrate over dl then this every point the r, this distance r will also be going to change.

So, to solve this problem what we need to do is to try to find out the relationship between l and r and then just simply integrate from minus infinity to plus infinity this length. So, before that let us execute, what is $d\vec{l} \times \hat{r}$, which is simply $d\vec{l}'$ and the angle between these two and we know that angle is α. And if you look carefully this α is related to θ.

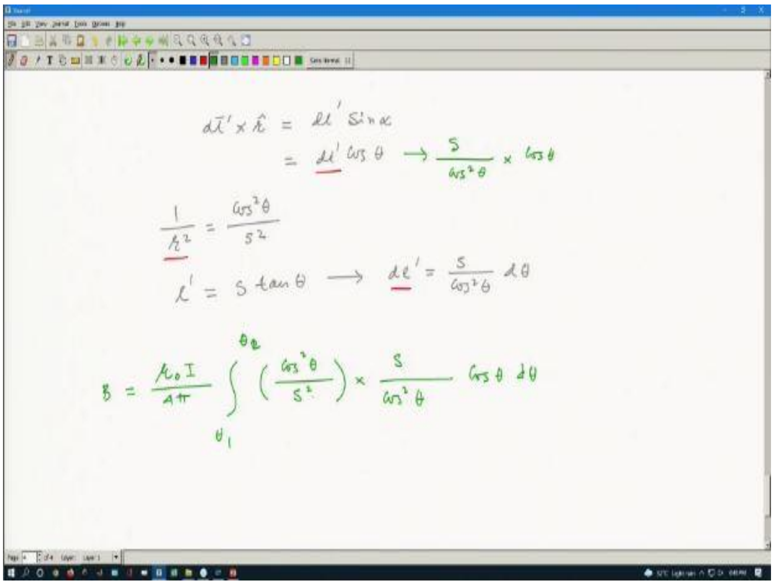
So, this is 90 degree. So, that means I can simply write $d\vec{l}'$ cos θ, because here from this geometry I have $\alpha = \theta + \frac{\pi}{2}$. This is 90 degree, so this plus this should be equal to this α.

(Refer Slide Time: 06:55)



Then $\frac{1}{r^2}$ is simply $\frac{cos^2\,\theta}{s^2}$, because this length is s from here to here it is s, this angle is θ. So, this I can write in terms of θ 1 variable and this constant, s is constant. So, that should be my so $\vec{l}'$, again I can simply write as s into tan θ, which gives me $d\vec{l}'$ to be $\frac{s}{cos^2\,\theta}$ dθ. Everything is now in my hand dl I write in terms of θ, $\frac{1}{r^2}$ I write in terms of θ and $d\vec{l} \times \hat{r}$ is also I write in terms of θ and $d\vec{l}'$. So, here $d\vec{l}'$ I write in terms of θ cos θ is there, here $\frac{1}{r^2}$ I write in terms of θ and also dl is written in terms of θ. So, everything is in terms of θ.

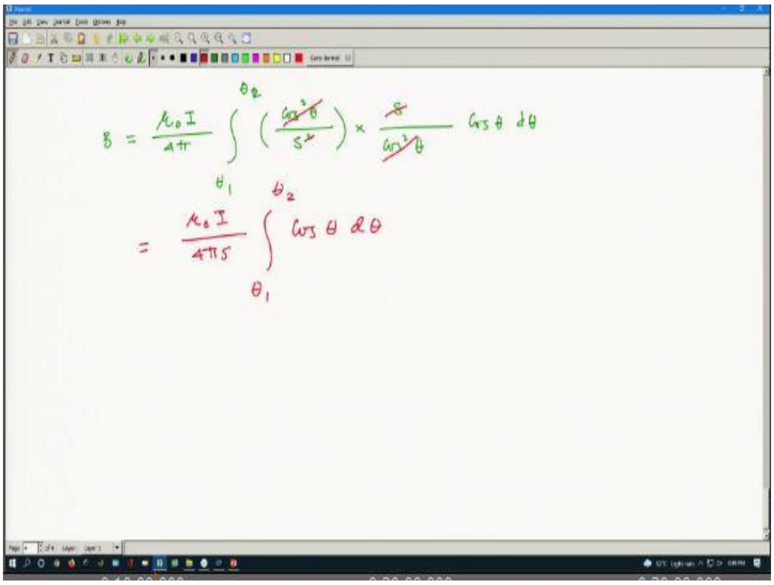
(Refer Slide Time: 08:16)



So, now I just put this value. So, $\vec{B}$ simply $\frac{\mu_0 I}{4\pi}$ and let us try to find out for two θ say, θ_1 to θ_2 these two range and then we put the condition with so, now we have $\frac{cos^2\theta}{s^2}$ and then we have $\frac{s}{cos^2\theta}$, so, $\cos^2\theta$ seems to be cancelling out and cos θ dθ. Let us check, how it is there. Because I have $\frac{\mu_0 I}{4\pi}$ outside, $d\vec{l} \times \vec{r}$ I just write cos θ multiplied by dl'.

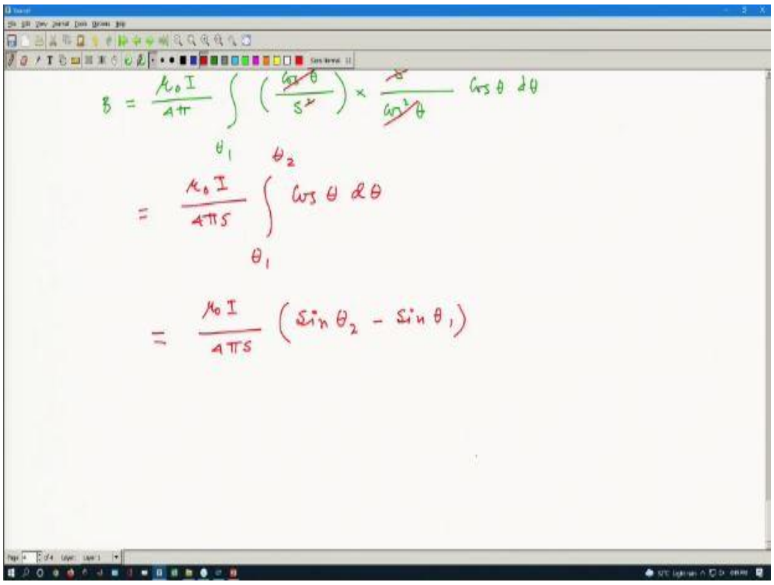
So, this is eventually, dl is $\frac{s}{cos^2\theta}$ and then I have a cos θ sitting here. And $\frac{1}{\text{л}^2}$ is $\frac{cos^2\theta}{s^2}$ so, that I put here. And then $d\vec{l} \times \vec{r}$ is $\frac{s}{cos^2\theta}$ multiplied by $\cos^2\theta$. So, this I write here.

(Refer Slide Time: 09:52)



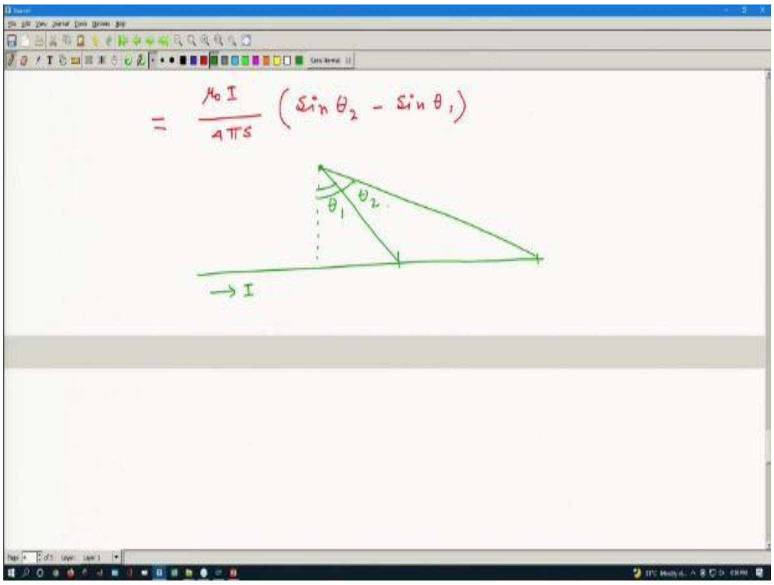
So, eventually $\cos^2\theta$ is cancelling out here and s_1, s is also cancelling out, so simply I have a s, which is constant. So, I can take it outside. So, it is $\frac{\mu_0 I}{4\pi s}$ integration θ_1, θ_2 and then $\cos\theta \, d\theta$.

(Refer Slide Time: 10:31)



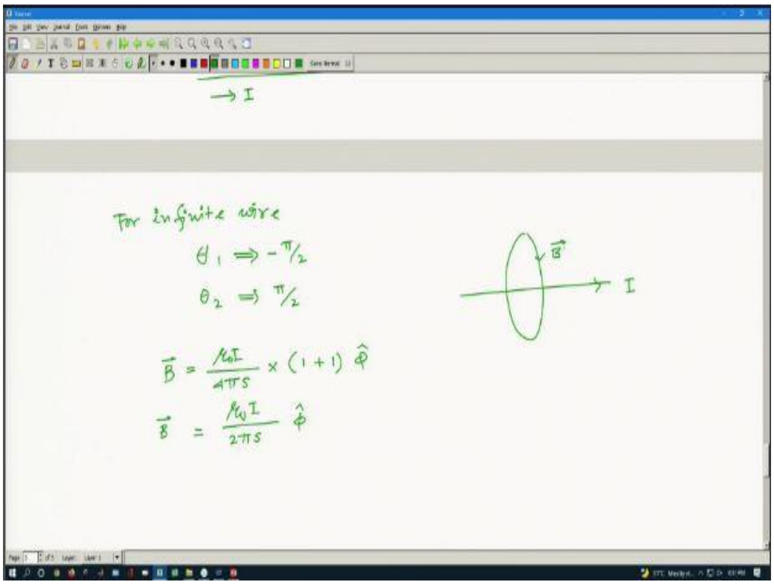
And this integration is very much doable one. So, $\frac{\mu_0 I}{4\pi s}$ and I have $\sin\theta_2 - \sin\theta_1$. So, what is $\sin\theta_2$, $\sin\theta_1$ and what is the picture quickly if I understand.

(Refer Slide Time: 11:03)



Then this is my wire and this is the magnetic field. So, I can have a length from say here to here with these 2 angles. So, this is the steady current I is flowing. Suppose, this is my θ_1 and this is my θ_2.

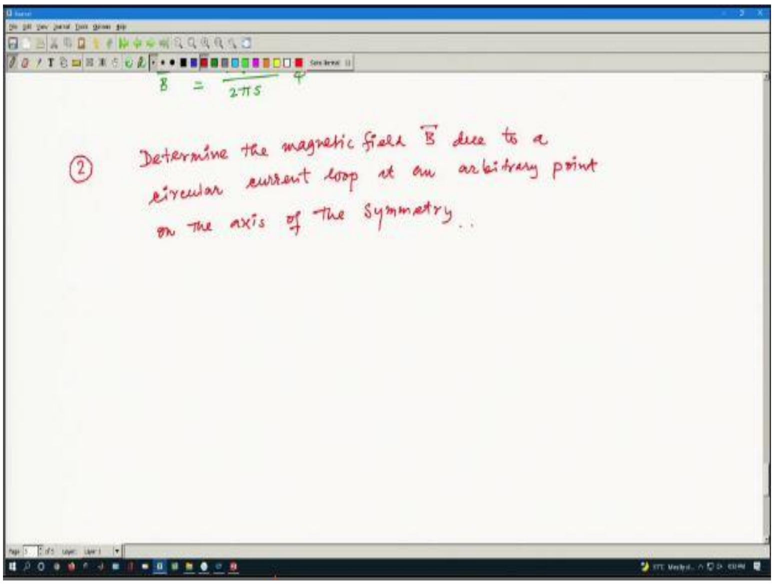
(Refer Slide Time: 11:36)



So, now for infinite wire what happened? For infinite wire, this limit θ_1 will be simply goes to $-\frac{\pi}{2}$ and θ_2 goes to $\frac{\pi}{2}$, because I go to minus negative side and plus side. Then if I put these 2 values there I simply get my $\vec{B}$ as $\frac{\mu_0 I}{4\pi s}$ then multiplied by (1 + 1). And also I put the vector sign, so vector should be the direction of the φ, because it is revolving. So, it is revolving.

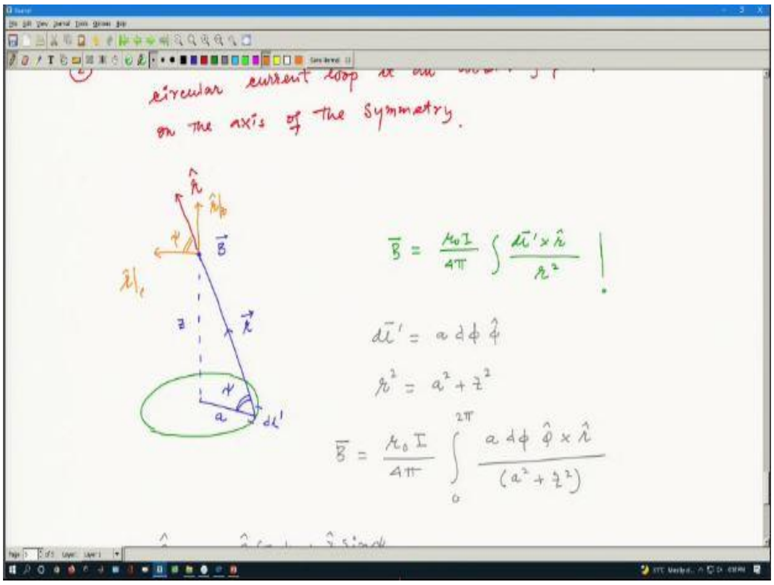
So, I will get finally the value $\frac{\mu_0 I}{2\pi s}$ with $\hat{\varphi}$. So, that should be the value of the $\vec{B}$ at that point here, because it is revolving. So, once we have the current wire along this direction the magnetic field will be revolving around this, in this direction. So, this is my current and this is the corresponding magnetic field that we are producing. So, this is one problem, very standard problem.

(Refer Slide Time: 13:20)



Let us now do another problem; problem 2 or case 2. In case 2, it is saying that determine the magnetic field $\vec{B}$, due to a circular current loop at an arbitrary point on the axis of the symmetry. So, now we are supposed to find out the magnetic field.

(Refer Slide Time: 15:05)



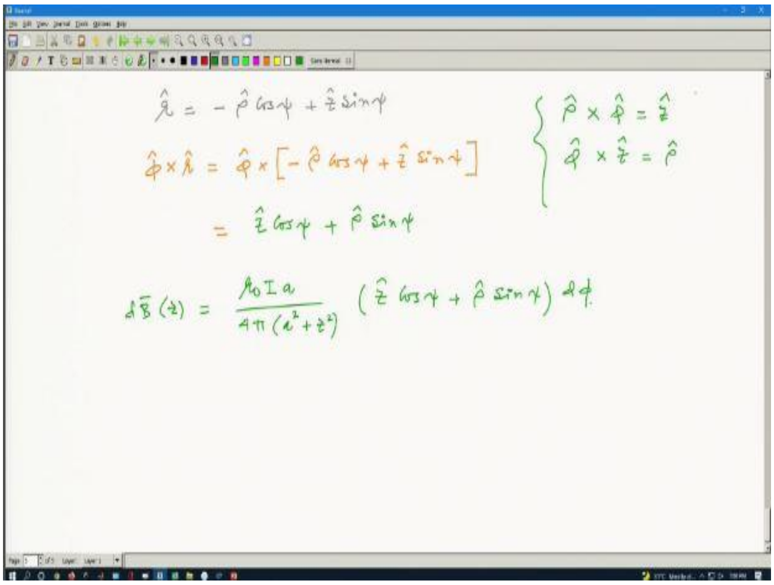
Suppose, this is the current carrying loop and this is the axis of symmetry and we need to find out because the current is flowing here, so we need to find out. Suppose a small section is here. So, let me join this. If I given from here to here so, at this location I want to find out the magnetic field. So, this is the location $\vec{B}$ I need to find out.

Suppose this is r as usual and this is z. Over z from here to here this length is z, which is fixed. This radius is say a and this angle whatever the angle is making is a Ψ. We want to be required this. And this length as I mentioned is dl′. So, again we are going to exploit the Biot-Savart expression, let me write it down first. So, $\vec{B}$ is $\frac{\mu_0 I}{4\pi}$ integration $d\vec{l}'$ × $\frac{\hat{Л}}{Л^2}$ that is the form of the magnetic field.

One should get this amount of magnetic field there. So, from the symmetry, let us exploit this. So, dl′, this is a dφ and then $\hat{\varphi}$. The line element in polar coordinates simply and then $Л^2$ is simply $a^2 + z^2$. So, you can see that this Л the magnitude of Л is not going to change, whatever the point you have over this loop, the value of $Л^2$ will not going to change.

So, $\vec{B}$ then simply $\frac{\mu_0 I}{4\pi}$ that is already there and then I need to integrate over this entire so that means, this φ should go to 0 to 2π and it is a dφ then $\hat{\varphi}$ × $\frac{\hat{Л}}{Л^2}$, which means, it is $a^2 + z^2$.

(Refer Slide Time: 18:27)



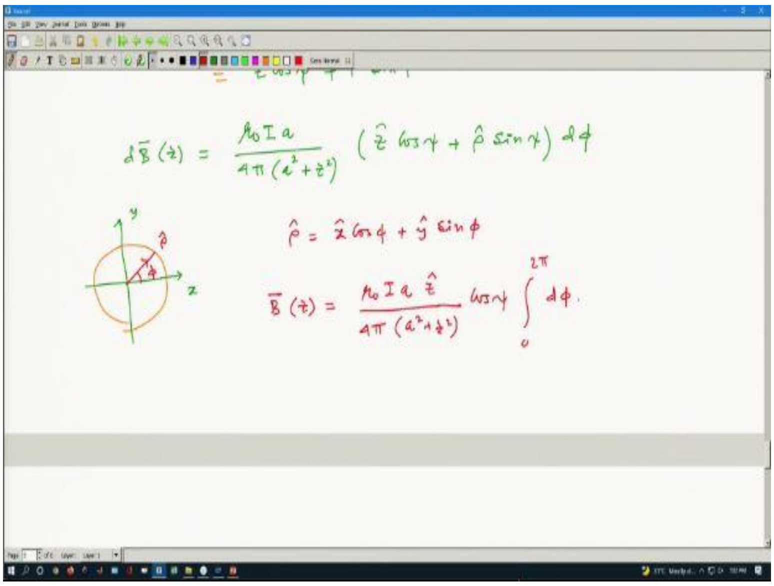
Now what is $\hat{Л}$? $\hat{Л}$ is $-\hat{\rho}$ cos Ψ + $\hat{z}$ sin Ψ. So, I can divide the $\hat{Л}$, so if I draw here, so $\hat{Л}$ should be along this direction. So, this is my $\hat{Л}$. So, I can divide this $\hat{Л}$ in two components. One is say this one; this is the z component of the $\hat{Л}$. So, I should say r vector z and another component is this one, which is the ρ component of $\hat{Л}$.

This is ρ component and this is. So, if this is Ψ, this angle; so this angle has to be Ψ and then the ρ component will have cos with a negative sign, because it is in opposite direction of whatever the ρ we measure and rest of the part is z. So, then the $\hat{\varphi} \times \widehat{Jl}$ is simply $\hat{\varphi}$ cross this quantity, which is - of $\hat{\rho}$ cos Ψ + $\hat{z}$ sin Ψ. And now we know, so let me write it down here.

Our whole knowledge will now, I need to use and that is $\hat{\rho} \times \hat{\varphi}$ is my $\hat{z}$ and $\hat{\varphi} \times \hat{z}$ is $\hat{\rho}$. So, that we know and that will be going to use here. And if I do, I will get $\hat{z}$ $\hat{\varphi} \times \hat{\rho}$ is $\hat{z}$. So, $\hat{\rho} \times \hat{\varphi}$ is $\hat{z}$, but $\hat{\varphi} \times \hat{z}$ is -$\hat{z}$, so this - z is going to absorb, so $\hat{z}$, then cos Ψ + $\hat{\rho}$ and then sin Ψ.

So, if I calculate d$\vec{B}$ at z then it is simply μ_0 I and then I have *a* here. So, *a* I take it outside, whole divided by 4π that is also there $a^2 + z^2$ and we have $\hat{z}$ cos Ψ + $\hat{\rho}$ sin Ψ and then dφ over the φ I am going to integrate. Now $\hat{\rho}$ in terms of φ if I write, because this is the value and then I need to integrate over φ but, ρ should have some value.

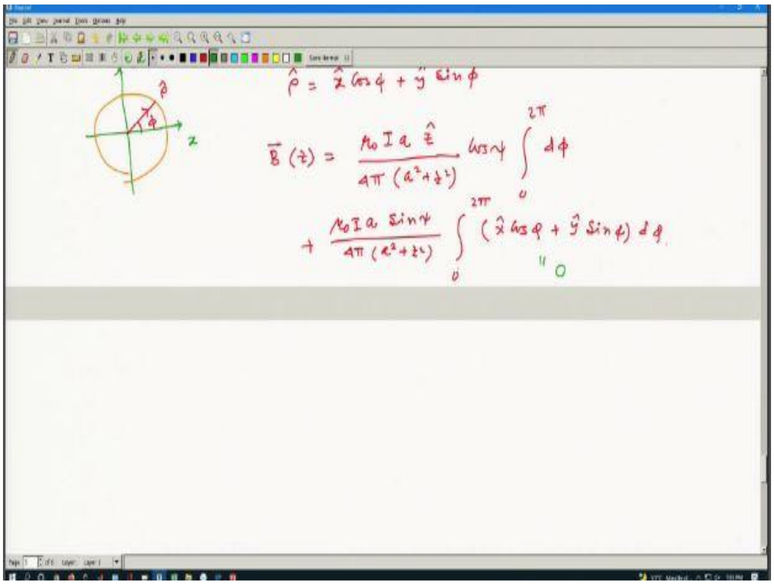
(Refer Slide Time: 22:20)



So, let me draw it here, too so how so this is the coordinate. So, this is my x and this is my y and z is perpendicular to that and ρ I am calculating from here to here this is my ρ. It is a $\hat{\rho}$ if this is making an angle φ. So, then the $\hat{\rho}$ should have $\hat{x}$ cos φ + $\hat{y}$ sin φ. And then my $\vec{B}$, once we know that my $\vec{B}$ at point z is μ_0 I *a* and then $\hat{z}$ because, I want to just find out the z component.

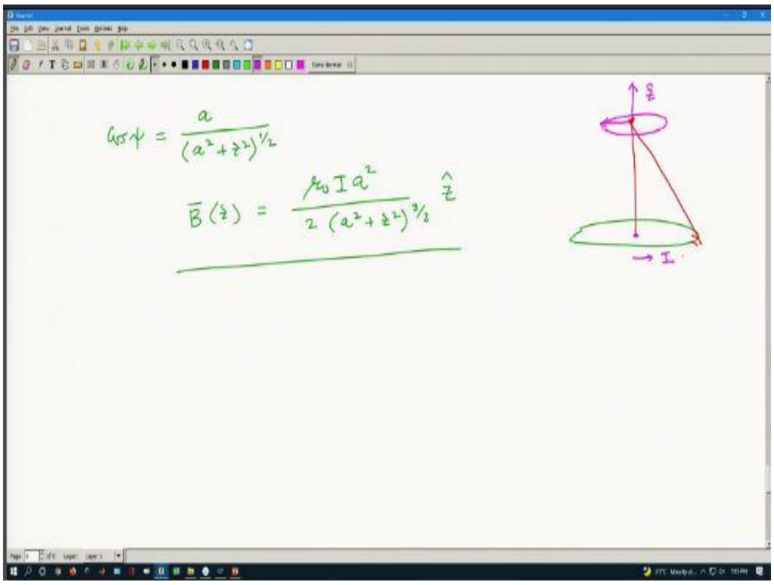
The rest of the component will be going to cancel out. So, I should not bother about the rest of the component at all, which is this one divided by 4π (a^2 + z^2) and then we have a cos Ψ here and then I integrate from 0 to 2π dφ. And the rest of the part I am not going to consider, because it will simply cancel out. And even if you do the integral then so this is the first part.

(Refer Slide Time: 24:20)



The second part if you just let me add, you will find that this value will be 0 under that limit. So, μ_0 I a and then the next part is sin Ψ and I have $4\pi(a^2 + z^2)$ and inside the integral now 0 to 2π. I have two term like $\hat{x}$ cos φ + $\hat{y}$ sin φ over dφ. But this is even if you go the integral from 0 to π, so this component will eventually cancel out. So, this will get 0 in this limit.

(Refer Slide Time: 25:17)

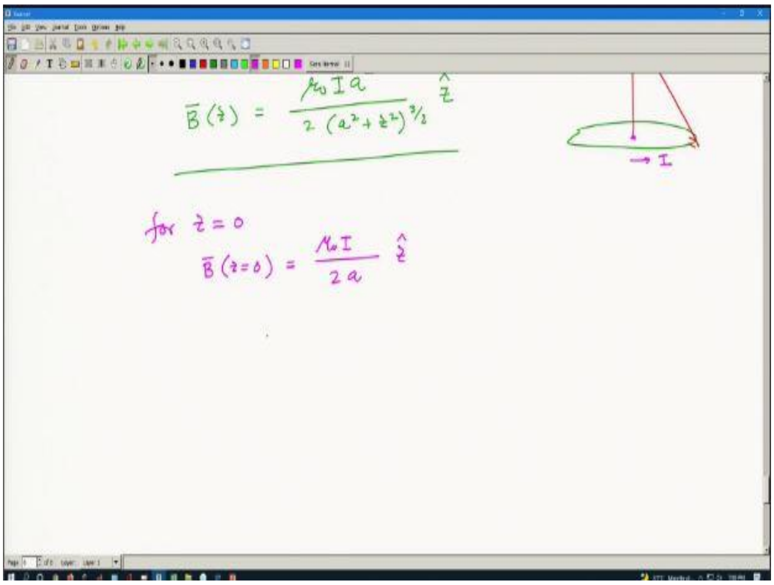


So, that makes and also cos Ψ is simply $\frac{a}{(a^2+z^2)^{1/2}}$. So, that makes simply $\vec{B}$(z) = μ_0 I, I have 2 a here, so $\frac{a^2}{2(a^2+z^2)^{3/2}}$ because, I just write cos θ cos Ψ here. So, this cos Ψ I just write. And also this integration gives us 2π, so this 2π and this 4π will give me 2 μ_0 I 1 a^2 is there.

So, I just put a^2 and that should be in the direction of $\hat{z}$ that is all. So, that should be the value of the magnetic field over here. So, if I look carefully. So, this is the loop and I try to find out the magnetic field somewhere here over the axis. So, I take this section, this is the length and if I divide the magnetic field here into 2 parts. Say one part will be along this direction and another part will be around this direction.

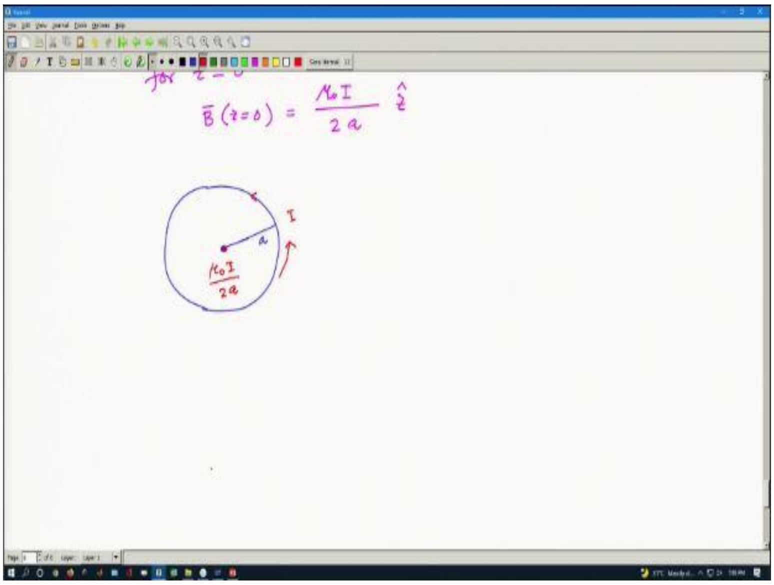
So, this direction will cancel out, because of the symmetry. This will cancel out. So, I will only have this magnetic field along this direction for the current that is flowing here I. Now if z = 0 that means, what about the magnetic field here.

(Refer Slide Time: 26:58)



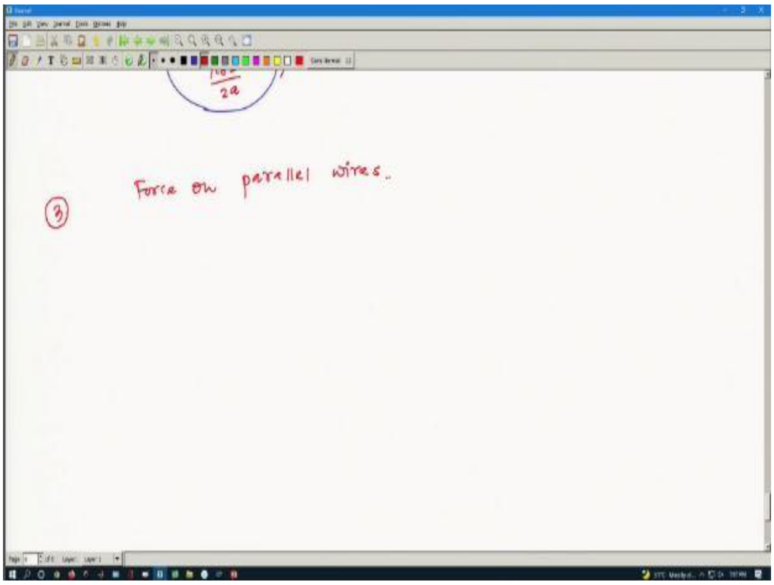
So, for z = 0, I have $\vec{B}$ at z = 0 = μ_0 a very simplified expression I divided by so z = 0, 1 a you are going to cancel it, 2 a and the direction will be along z direction.

(Refer Slide Time: 27:30)



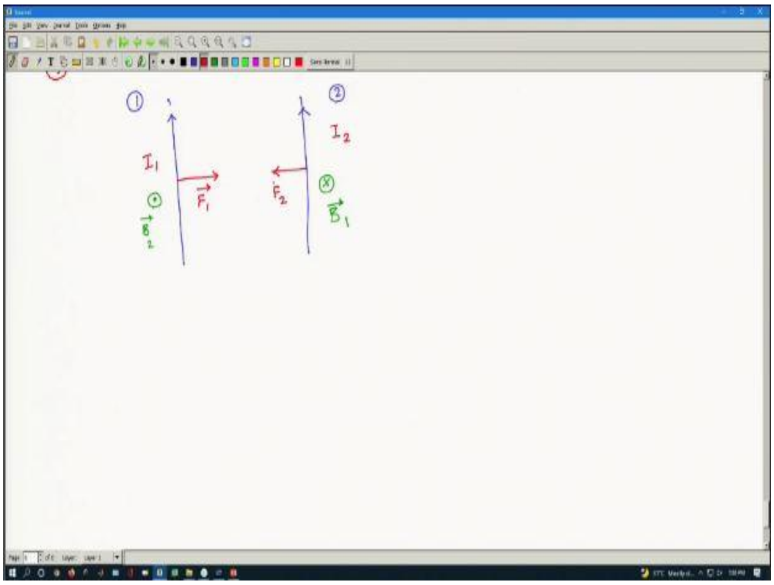
So, when we have a circle here current carrying loops. Having a circle with radius a, the magnetic field will be perpendicular to the plane and the value should be μ_0 I. So, I is the amount of current that is carrying through this wire divided by $2a$. So, if I decrease the value of a the magnetic field will be going to increase.

(Refer Slide Time: 28:19)



Let us go on the next problem. The problem is 3 and a part of this problem, we have done in the first in I think last class. That is the force on parallel wires.

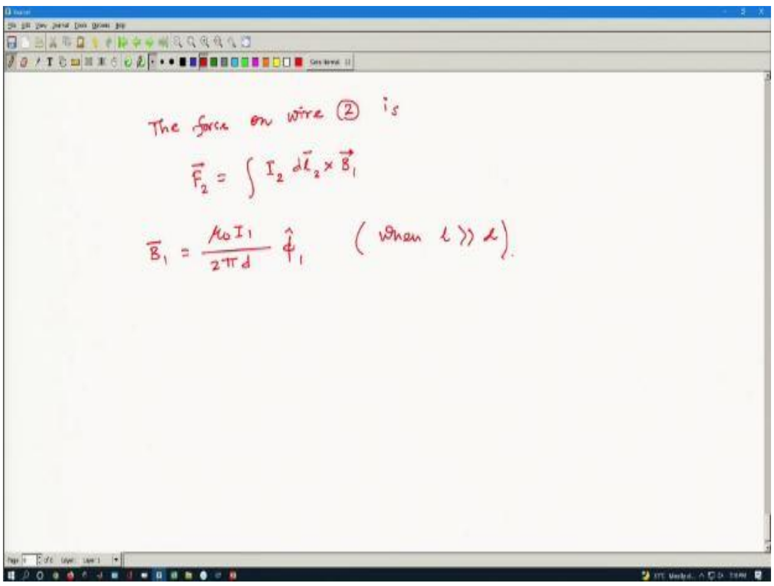
(Refer Slide Time: 28:43)



So, suppose I have 2 parallel wires. One is this, another is this. So, the current is flowing like this. This is 1, this is 2. So, current is here it is I_1 and the current flow here is I_2. So, when the current is flowing in I_2, I have a magnetic field so this wire will going to experience a magnetic field in the upward direction. So, this is the notation I use for upward direction and this value is say $\vec{B}_2$, which is due to the current that is flowing through the wire 2.

In the similar way for the current that is flowing in 1, the magnetic field that is experienced by current this loop. This current carrying wire 2 is along the downward and that is $\vec{B}_1$**.** So, now the force between these I want to calculate. The current is flowing, both the cases in the identical direction. So, there should be some force like this $\vec{F}_1$ and this is say $\vec{F}_2$**.**

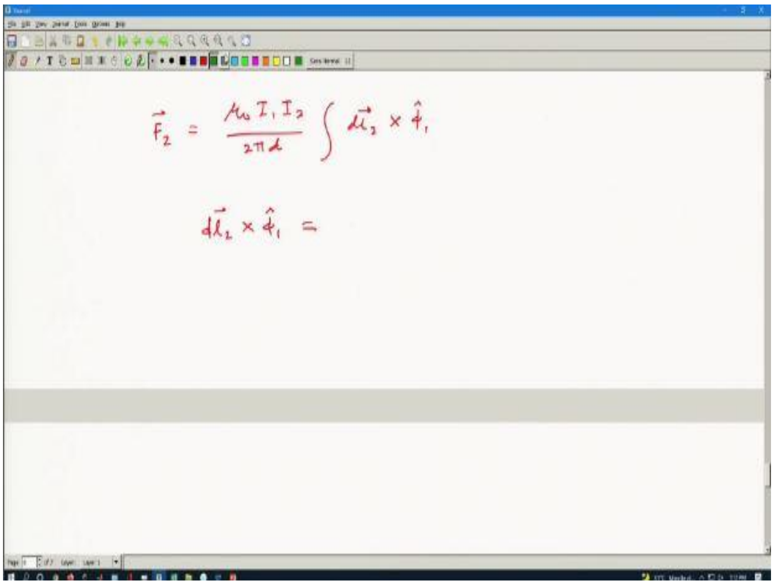
(Refer Slide Time: 30:28)



So, the expression of the force I have already calculated. And the value is the force on wire 2 is simply $\vec{F}_2$ is equal to integration of I_2 $d\vec{l}_2 \times \vec{B}_1$. And $\vec{B}_1$ is a magnetic field due to the current I_1 in wire 1, at the position of $d\vec{l}_2$. So, that is why I write so any position $d\vec{l}_2$, this is the amount of force that the wire will going to experience, because of the $\vec{B}_1$. So, $\vec{B}_1$ if I calculate, this is how much $\frac{\mu_0 I_1}{2\pi d}$ $\hat{\phi}_1$. This I already calculated today. That what about a very long wire or infinitely extended wire what should be the magnetic field?

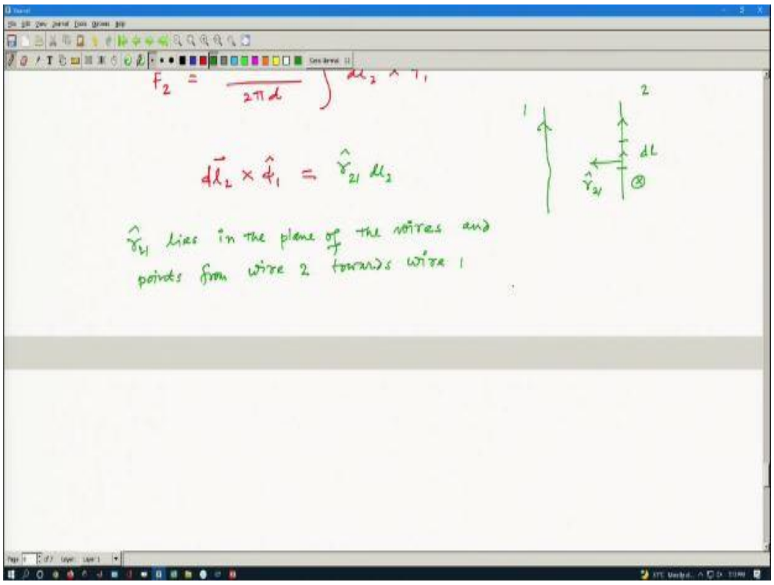
So, I have already calculated this expression. So, I am just using this expression $\frac{\mu_0 I}{2\pi}$. The point where I want to find out the magnetic field here I am just writing that one $\frac{\mu_0 I_1}{2\pi d}$ here. So, and that is when I can write it as when the length of the wire is very, very greater than the location where try to find out the magnetic field. So, infinitely extended wire or length of the wire is very long compared to the distance, where I try to find out the magnetic field.

(Refer Slide Time: 32:32)



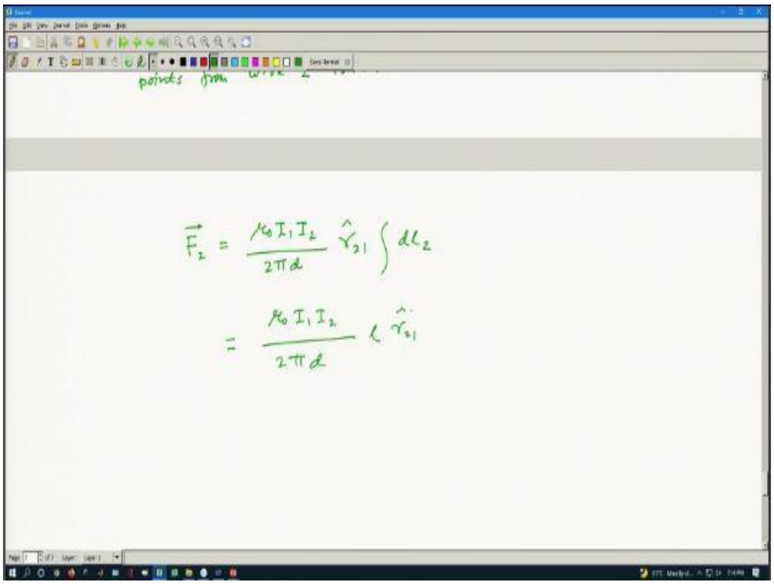
So, $\vec{F}_2$ if I calculate then it simply becomes μ_0 I_1 I_2, because I_2 I can take outside divided by $2\pi d$, d is a constant that is a distance between 2 wires. And then $d\vec{l}_2 \times \hat{\phi}_1$. Now $d\vec{l}_2 \times \hat{\phi}_1$ this is simply so $d\vec{l}$ is so this is 1 where 2 say and dl is along this direction.

(Refer Slide Time: 33:31)



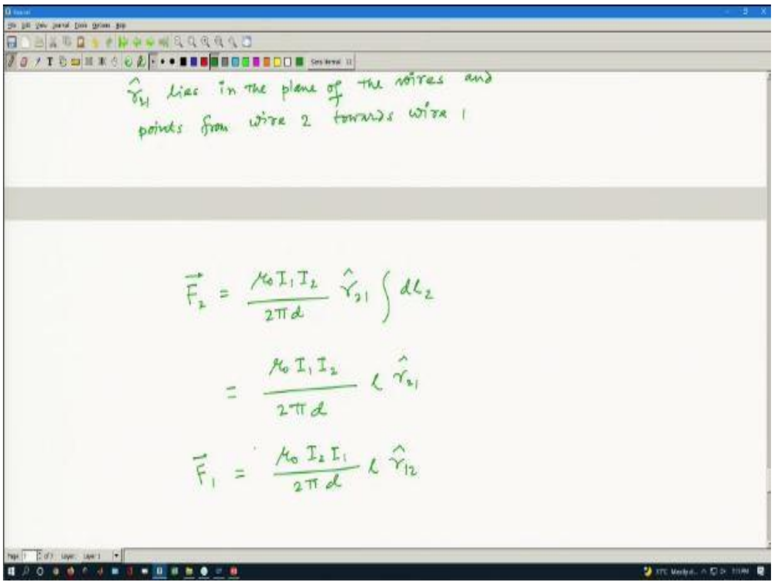
So, this is my $d\vec{l}$, $d\vec{l}$ is also along this direction and what about the $\hat{\Phi}_1$. The $\hat{\Phi}_1$ is along downward. So, this is the direction of $\hat{\Phi}_1$ downward. So, if that is the case then $d\vec{l}_2 \times \hat{\Phi}_1$ should be along this direction. And that is if I write this is $\hat{r}_{21}$ and dl_2. So, this is the unit vector along 2 to 1. So, what is $\hat{r}_{21}$? Let me write clearly. So, $\hat{r}_{21}$ lies in the plane of the wire and points from wire 2 from towards wire 1. This one, wire 1 is here and it is this direction.

(Refer Slide Time: 35:27)



So, $\vec{F}_2$ I simply write here $\vec{F}_2$ is $\frac{\mu_0 I_1 I_2}{2\pi d}$ and $\hat{r}_{21}$ and then dl_2 and if I integrate over this entire length. So, it should be $\frac{\mu_0 I_1 I_2}{2\pi d}$ and then l $\hat{r}_{21}$.

(Refer Slide Time: 36:17)



Similarly, for $\vec{F}_1$ you can see that everything will same except. So, I will have $\frac{\mu_0 I_1 I_2}{2\pi d}$ magnitude wise it is same, length l and then $\hat{r}_{21}$, that will be the difference. So, one is in this direction, another is another direction. So, that means we have an attraction. If they are moving in opposite direction, so in the first day in the last class I guess, we show this figure and we mention that we are going to calculate rigorously.

So, this is the calculation, where you find out. How the force between 2 wires are there. So, today my time is up. So, I like to conclude here in today's class. So, today we have a very straightforward 3 examples or 3 applications of Biot-Savart law. So, I suggest the students please check different books and there may be some other interesting problems or you need to calculate the magnetic field for a given geometry. Some geometry is there, that based on that you need to calculate the value of the magnetic field, for a given point.

But the procedure will be same you need to just integrate you need to judicially calculate the relationship between the parameters and then that is all. In the next class, we will continue with the magneto-static and try to understand more about the Ampere's law and this kind of things and maybe some application of the Ampere's law. So, with that note I will conclude here. Thank you very much for your attention and see you in the next class.