Foundations of Classical Electrodynamics
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Lecture-50
Biot–Savart Law

Hello student to the foundation of classical electrodynamics course. So, under module 3 magnetostatics, we have lecture 50 today where, we are going to understand the Biot-Savart law. So, we have today class number 50. So, before going to the Biot-Savart law, so let me quickly remind what we have done so far.

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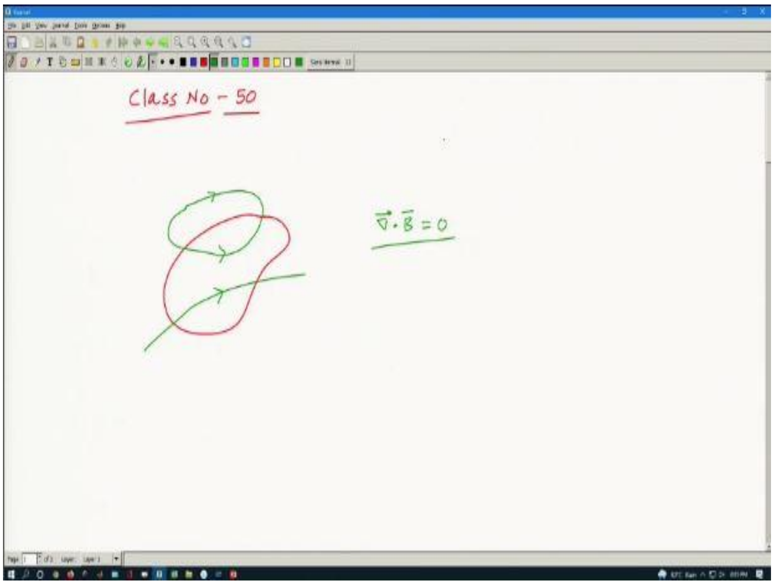

So, if we have a closed volume and try to capture the magnetic field lines, we will see in the last class that magnetic field line is like this. So, it do not begin or there is no like source or sink kind of term we can expect. So, I can write that the $\vec{\nabla} \cdot \vec{B} = 0$, that was the thing we find in the last class. And that means magnetic monopole does not exist, that is why we are getting this result. Well, before going to the Biot-Savart law we need to you know discuss once again this continuity equation.

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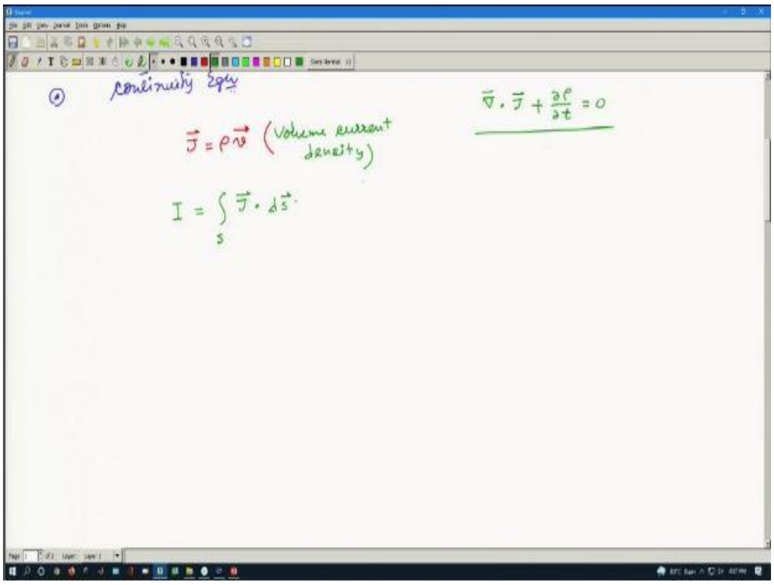

We did it in an initial class. So, before going to the continuity equation let me write down the equation first again we will like to prove it that $\vec{\nabla} \bullet \vec{J} + \frac{\partial \rho}{\partial t} = 0$ that was the relation we have with the current density and the volume charge density. So, we will do it once again. We prove it earlier but we will do it once again. So, $\vec{J}$ the current density is written as ρ multiplied by the velocity.

This is the way, we write the volume current density. This is called the volume current density. And in terms of volume current density, if I want to write down the total current so it should be like this, $\vec{J} \bullet d\vec{s}$. So, I have a cross section and in this cross section the volume current density is flowing and if I integrate over this cross section with entire surface, then I will get the total current.

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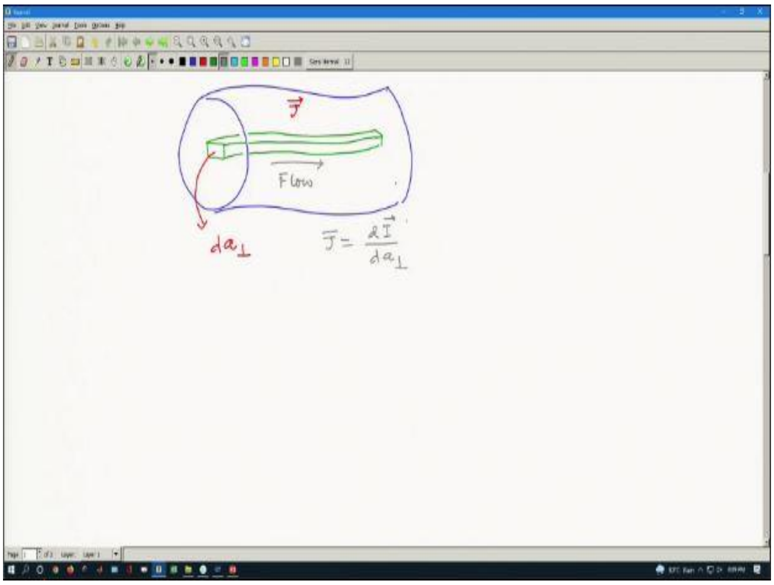

So, the figure can be understood like, suppose I have a wire like a cross section of the wire like this. And I have a small strip here. Suppose let us make this strip like different colour, like this. So, this area I write a perpendicular, because I am making a perpendicular cross section. And the current density is $\vec{J}$ and this is the direction of the flow of the current, say this is the flow. So, $\vec{J}$ here is simply $\frac{d\vec{I}}{da_\perp}$.

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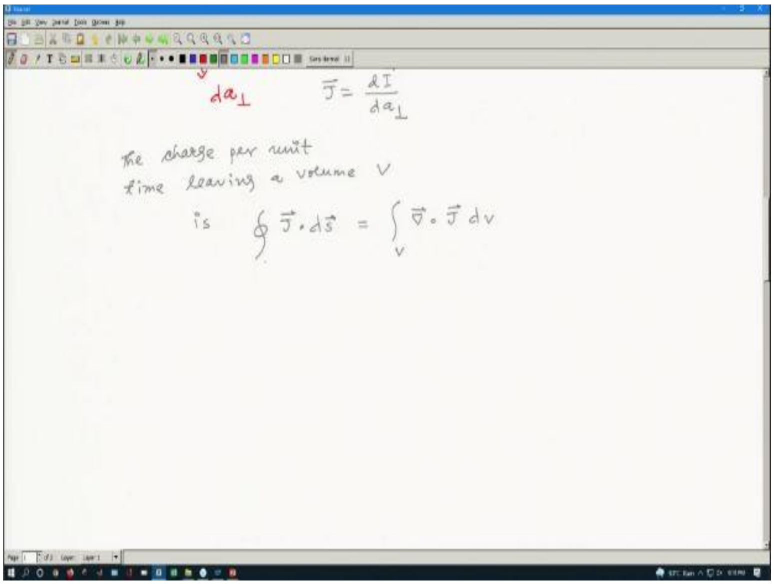

So, the charge per unit volume, now charge per unit time let us leaving a volume v is this quantity, which is simply integration of the $\vec{\nabla} \cdot \vec{J}$ dv. So, this quantity $\vec{J} \cdot \vec{I}$ is basically the current, but I am saying that charge per unit time, that means, the current, which is leaving in a volume v and that means there is a flow of the current and that we can write in this way. And this is simply the corresponding volume integral; because this is a surface integral I am using just the Gauss's law to execute that.

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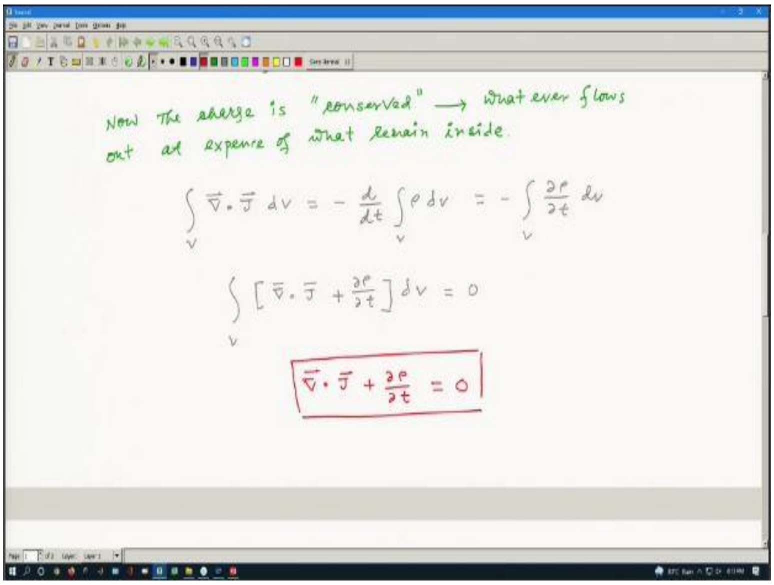

Now total charge now the charge is conserved. So, that means, whatever flows out at expense of what remain inside. So, something is flowing out, but charge is conserved. So, that means, whatever is flowing out that should be as expense of whatever is there already in the inside. So, there will be decrement of the charge inside. So, if I try to write a mathematical language. Then the volume integral of the $\vec{\nabla} \cdot \vec{J}$ over dv that should be the reduction of the charge density per unit time, the reduction of the charge per unit time rather.

So, that means, it is $\frac{d}{dt}$ of total charge I can write as ρ dv here**.** So, that is the expense of this amount of charge we are getting the flow in terms of this divergence form. So, that quantity is nothing but say $\frac{\partial \rho}{\partial t}$ over this volume integral dv. From this equation, I can write that the volume integral $(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t})$ dv = 0. This is a general for any volume it is true.

So, that is why I can have the desired relationship, a very, very important relationship rather with the charge density and the current density $\vec{J}$ and this equation is called the continuity equation. But why this continuity equation is required here? So, let us now consider the steady current.

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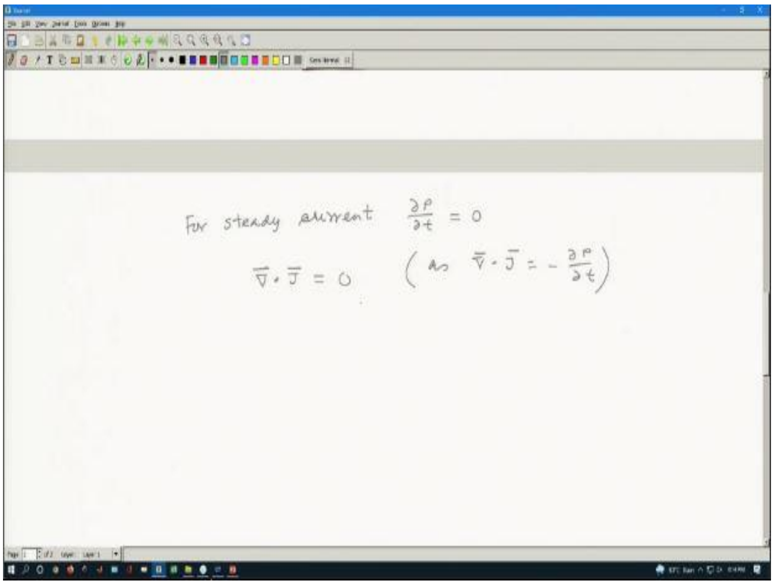

So, for steady current what happened it is 0. So, for steady current it is 0. So, that means under steady current I have a condition, from the continuity equation that this has to be 0, as we know this is equal to $-\frac{\partial \rho}{\partial t}$. So, this quantity has to be 0. So, let us now try to understand this.

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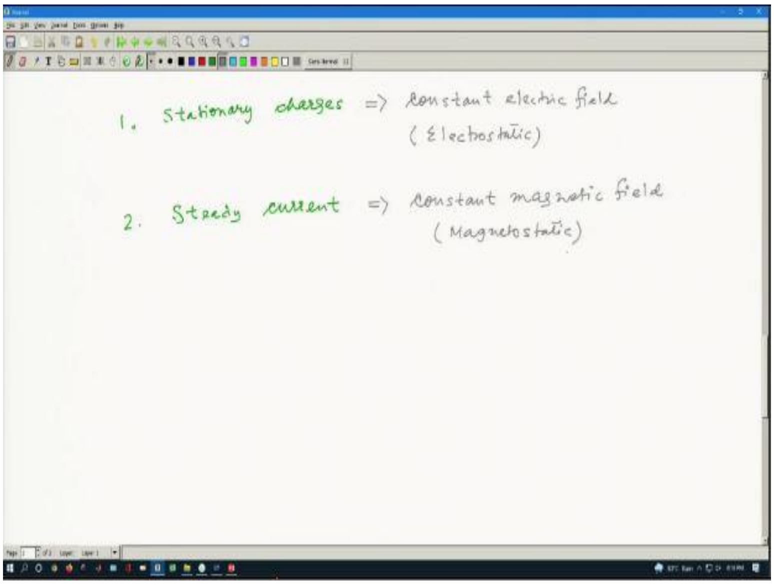

When there is the charge is in stationary. So, for stationary charges what we get is the constant electric field. The stationary charges leads to the constant electric field, there is no change with respect to time. That is why it is called constant electric field and that basically our electrostatic study. On the other hand, when we have steady current that leads to constant magnetic field that is magnetic field, that is not changing with respect to time and that topic is under magnetostatic.

This is the topic of magnetostatic, where we deal with the situation where the current is steady. And in electrostatic we have already discussed the entire part where this charged particle is always in a stationary position it is not moving. So, now after having this idea let us now, try to understand the Biot-Savart law.

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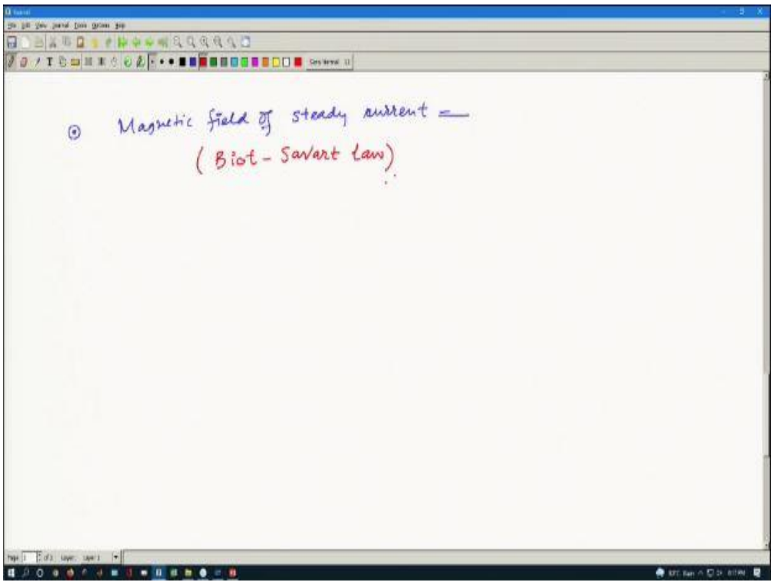

So, Biot-Savart law is nothing but the magnetic field of steady current. So, the magnetic field of the steady current if I want to calculate, so there is a law and this law is associated. So, this law is basically, the Biot-Savart law. We all know this law, because this is the topic, which is there for a long time in the syllabus in class 12 levels.

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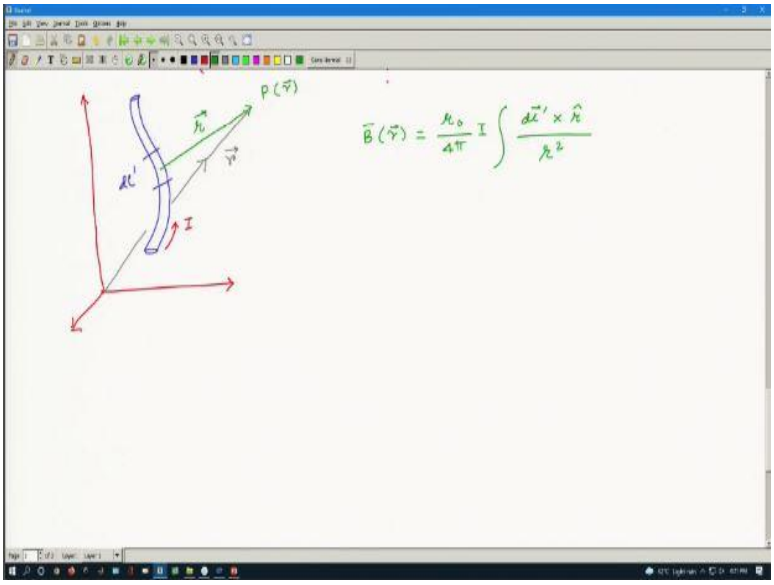

So, let us now consider a coordinate system as usual. And then I draw a wire here. Suppose, this is a piece of wire and from here to here, this length is say dl’. And the steady current is flowing say I. And if a current is flowing, so the law suggests that we will be going to get a corresponding magnetic field and we know that when the current is flowing through a wire around the wire, some magnetic field is produced.

So, the amount of magnetic field that is produced will follow certain law. So, that is eventually the Biot-Savart law. So, suppose this is the point P, where I want to find out the magnetic field and from here the distance is same r and the coordinate of this is say $\vec{r}$. So, from here to here this is $\vec{r}$. So, the location of this point is $\vec{r}$**.** So, now if this is the case, so I just draw the geometry.

So, what should be that the point is what should be the magnetic field at point P? So, my magnetic field $\vec{B}$ at $\vec{r}$ should be proportional to the amount of current flow, proportional to the amount of this length and inversely proportional of r^2. So, that we know. So, that I am writing in terms of this. So, $\frac{\mu_0}{4\pi}$ then amount of current and then for the entire wire I should have dl' cross very important. Then $\frac{\hat{\jmath}}{\jmath^2}$. So, that is the form of the magnetic field that one can expect at this point P.

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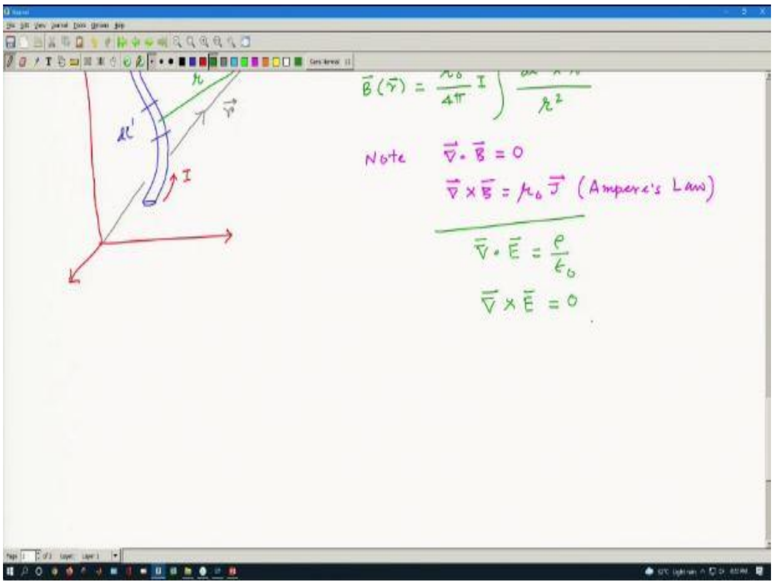

So, here I like to note that, because I am going to use these things. That divergence of the magnetic field is 0 and curl of the magnetic field that we will be going to show. Let us just write it. It is μ_0J, we will going to prove that later. So, this is called the Ampere's law. And we will show that in detail. But you should note that, for like electrostatic we have a divergence and curl of magnetic field here.

So, the $\vec{\nabla} \bullet \vec{B}$ is 0 and $\vec{\nabla} \times \vec{B}$ is μ_0 J. So, in comparison in electrostatic if I write here. So, the $\vec{\nabla} \bullet \vec{E}$ that was the Gauss's law, was $\frac{\rho}{\epsilon_0}$ and $\vec{\nabla} \times \vec{E}$ was 0, that was the electrostatic thing and it is the magnetostatic thing. So, you should just note that here. After that what we do interesting, because we are going to recap the Helmholtz theorem.

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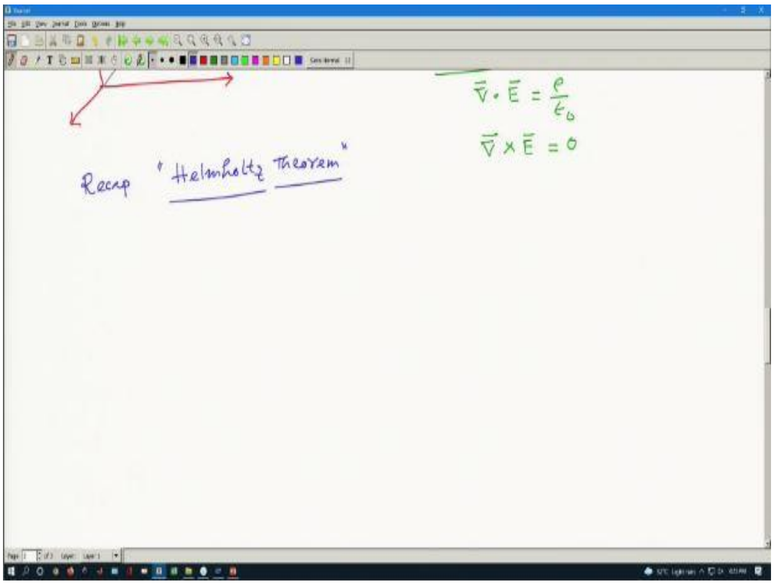

The famous Helmholtz theorem, we have been using this theorem in several cases. So, let me recap that. So, recap what was there? Recap of Helmholtz theorem and what we are having here in Helmholtz theorem.

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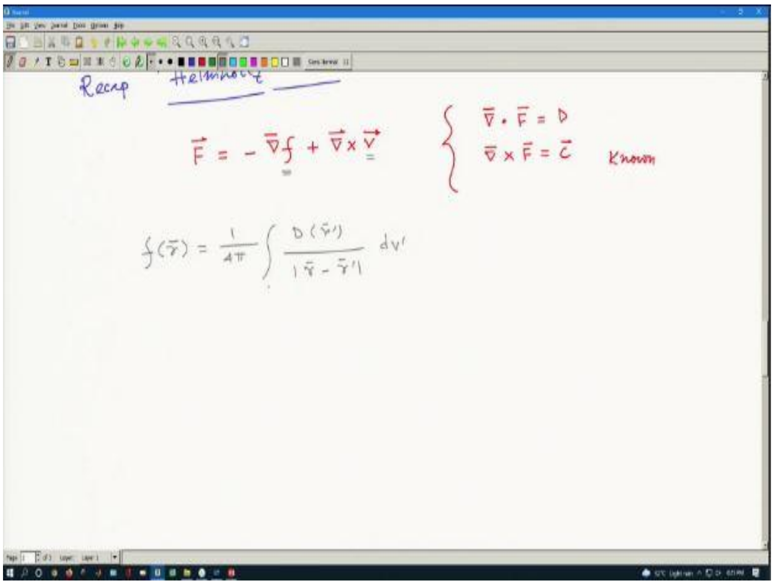

So, any vector field $\vec{F}$ can be decomposed in terms of a scalar field f and a vector field $\vec{v}$. Any vector field $\vec{F}$ can be written in this way, that the $\vec{\nabla} \bullet \vec{F} + \vec{\nabla} \times \vec{B}$. Now the question is if the

$\vec{\nabla} \bullet \vec{F}$ is known say it is D and the $\vec{\nabla} \times \vec{F}$ is also known say this is $\vec{C}$. So, these are known. Then would it be possible to construct the value of $\vec{F}$ and the answer was yes, because if I able to construct f and $\vec{v}$ in terms of $\vec{C}$ and D, then eventually I can construct my $\vec{F}$.

And there is a recipe and the recipe was f, which is a function of $\vec{r}$ should be written in this way $\frac{1}{4\pi}$ integration of whatever the divergence we have here, so let us put this divergence. The value of the divergence, which is essentially scalar field divided by $|\vec{r} - \vec{r}'|$ integration over the entire volume dv'. That was the value of the f r through which you can find the first part of this total field total unknown vector field $\vec{F}$.

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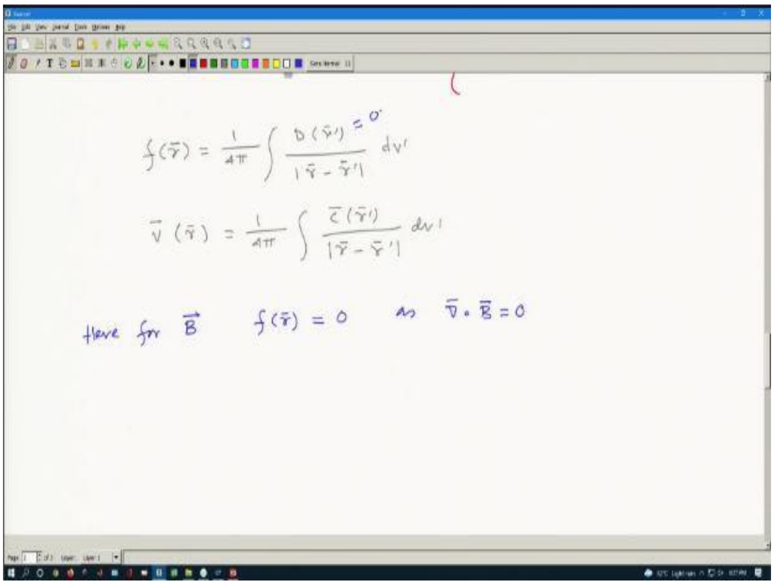

And similarly v can be evaluated exploiting the knowledge of this curl. So, it should be $\frac{1}{4\pi}$ integration, then the curl that should be evaluated say at $\vec{r}'$ and $\vec{r} - \vec{r}'$ integration over this v'. So, now here if I look the value of the thing for.

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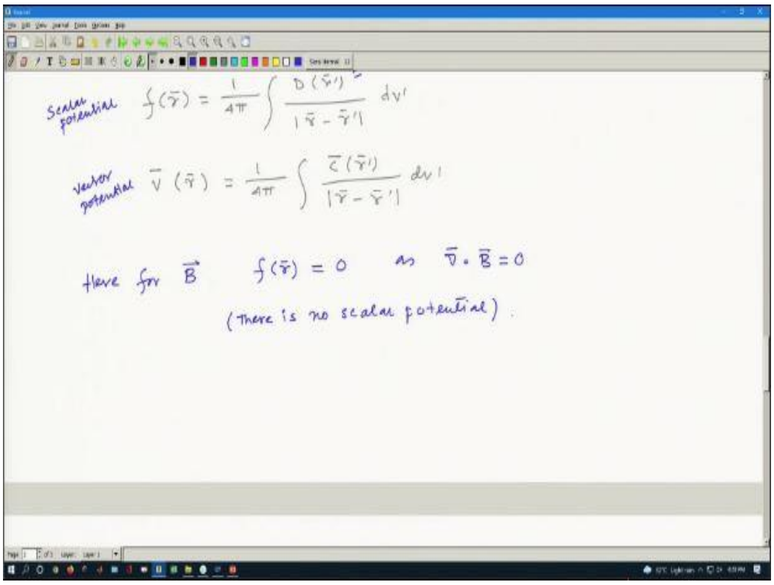

So, here for magnetic field $\vec{B}$ what we see that my f($\vec{r}$) seems to be 0. Why because as the $\vec{\nabla} \cdot \vec{B}$ is 0 and if you see that it related to the divergence of that field. So, d here, so f($\vec{r}$) has to be 0, because this quantity for magnetic field is 0. For electrostatic field in the other way, this was non-zero. So, that means for magnetic field, there is no scalar potential. In general this is called the scalar potential, because it is behave some sort of potential and is a scalar field and this is called the vector potential. So, there is no scalar potential for magnetic field $\vec{B}$, but what about the vector potential. Vector potential is very much there.

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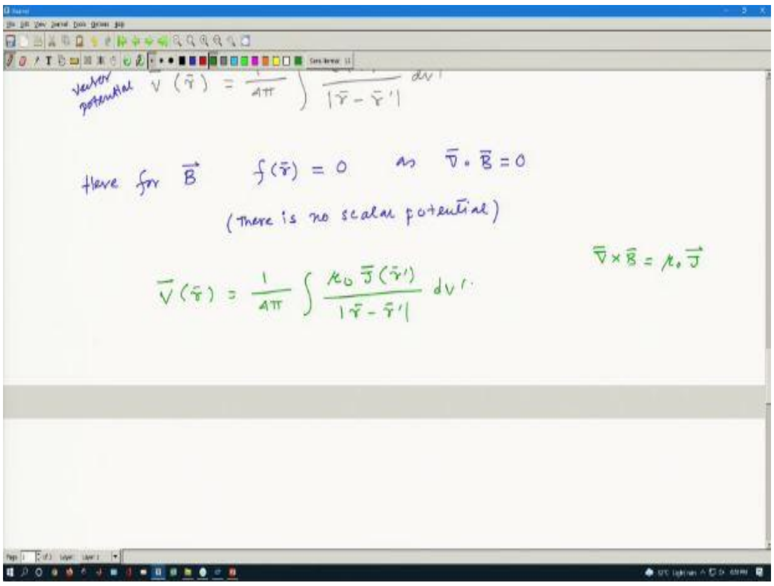

So, the vector potential $\vec{v}$, which is a function of $\vec{r}$ is simply $\frac{1}{4\pi}$ integration. So, now I need to exploit the value of the $\vec{\nabla} \times \vec{v}$, because $\vec{\nabla} \times \vec{B}$ this value I need to put here inside this integral, so it is μ_0 J. We did improve, but let us exploit this. Let us use this. So, I have μ_0 J, but that will be evaluated at the point $\frac{\vec{r}'}{\vec{r}-\vec{r}'}$ and then dv'. So, that should be. So, then if v is known, then the next step is straight forward.

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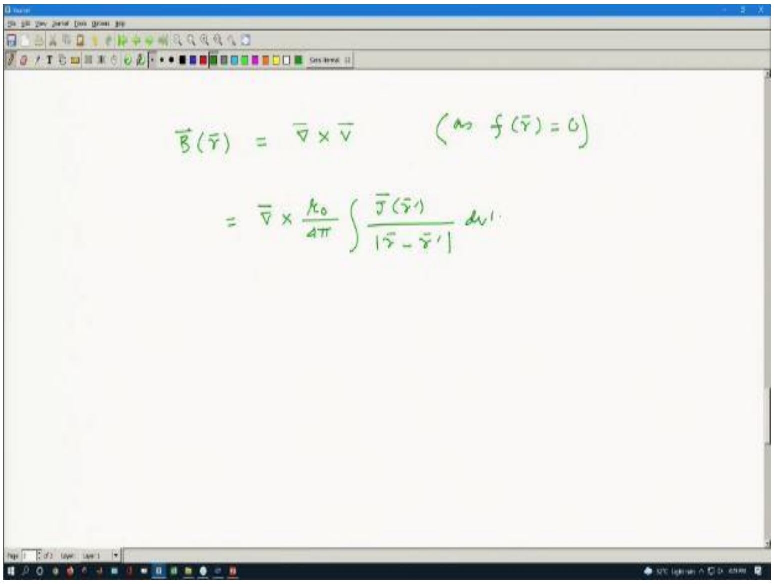

Because I can find out my $\vec{B}$ and my $\vec{B}$ should be simply $\vec{B}$ is at the point $\vec{r}$ that is simply $\vec{\nabla} \times \vec{v}$, because as f($\vec{r}$) is already 0. So, that quantity is simply $\vec{\nabla} \times \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}$ dv'. So, I can have this value and now I can exploit more to find out whether this expression leads to our old Biot-Savart law or not. So, eventually we will find that it leads to the expression like this. Because now I am in a position to find out my $\vec{B}$. So, let us proceed.

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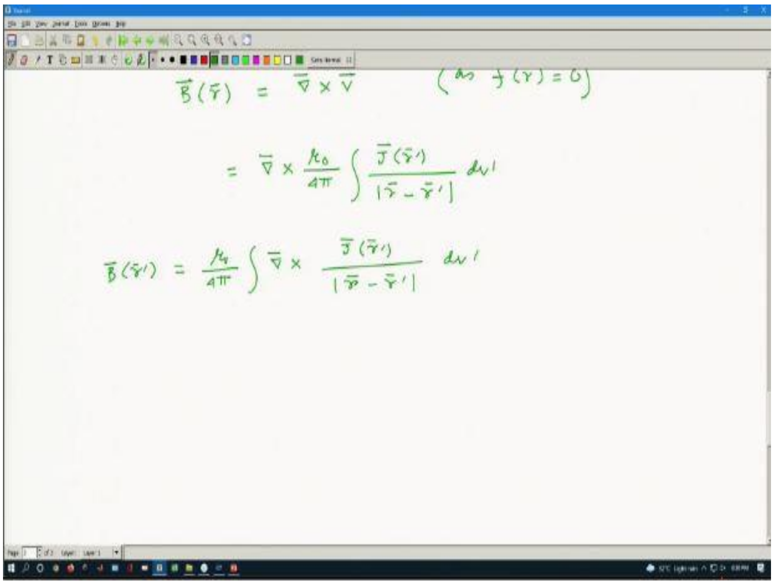

So, my $\vec{B}(\vec{r}\,')$ is simply $\frac{\mu_0}{4\pi}$, let us put this cross inside this integral. So, I should have that this is cross and that is operating over this vector field $\frac{\vec{J}\,(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$ and over the integral dv'. Note that this is the denominator we have the function of $\vec{r}$ and in the numerator I function of $\vec{r}\,'$, but this operator is a function of $\vec{r}$ only. So, if I go to the traditional way by just using the vector identity.

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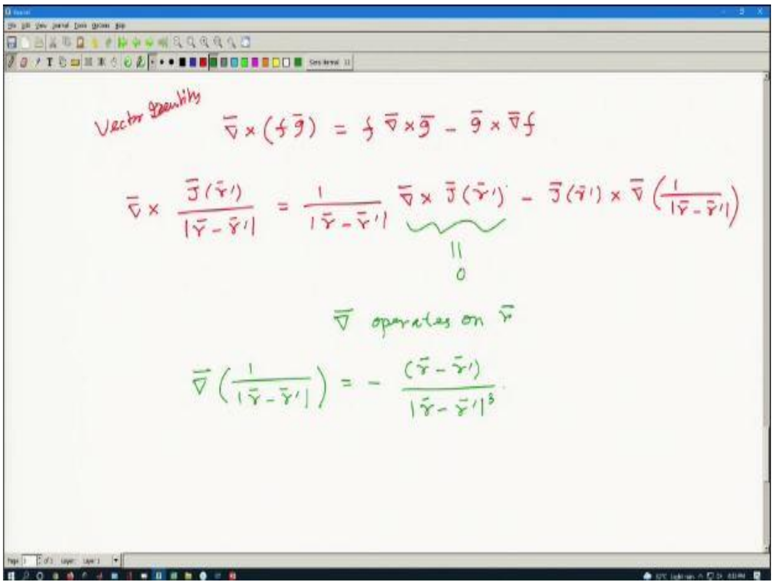

So, our vector identity was curl of a scalar field f and the vector field $\vec{g}$ should be simply f then $\vec{\nabla} \times \vec{g}$ - $\vec{g} \times \vec{\nabla}$f. And curl of whatever I am doing here is operating over $\vec{r}\,'$ and $\vec{r}$ - $\vec{r}\,'$. So, I should write first this and then $\vec{\nabla} \times \vec{J}(\vec{r}\,')$ and then $-\vec{J}(\vec{r}\,') \times \vec{\nabla}(\frac{1}{|\vec{r}-\vec{r}\,'|})$. This is the way, but you can see as I mentioned that this quantity has to be 0 because this operates on $\vec{r}$.

This is a function of $\vec{r}$, so but it is a function of $\vec{r}\,'$. So, when you do the derivative we will see that this is 0. So, then I have only one term left. And if I execute this term, this is also a known term and that thing is simply -$(\vec{r} - \vec{r}\,')$ and then it should be $(\vec{r} - \vec{r}\,')^3$, because this is a known thing, so we have already done this. So, we will be going to put this here in this equation here.

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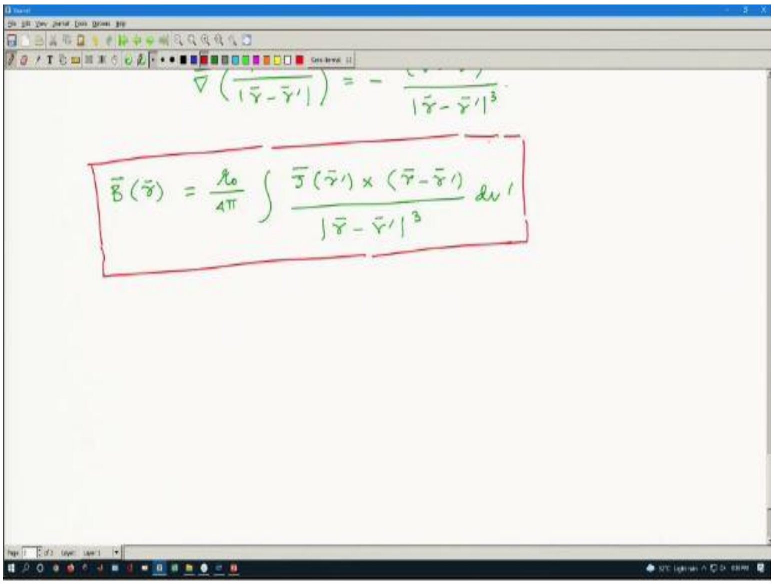

So, eventually I am going to get simply $\vec{B}$ is $\frac{\mu_0}{4\pi}$ and then if I integrate it so, it should be $\frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$ dv'. So, this is nothing but the Biot-Savart law, in using this volume current density. So, this is the form of the Biot-Savart law that we already stated earlier. Here we are writing the Biot-Savart law in terms of the current density $\vec{J}$.

If I go back to the old expression, which we write this one so, here if I include this $\vec{r}$ inside. If I include this current A inside then this current can be written in terms of current density and then this volume integral is eventually appear and we will have the expression something like this. So, today my time is up. So, I like to conclude here. So, in today's class what we do that we write down the Biot-Savart law.

And try to understand the Biot-Savart law in the context of the Helmholtz theorem and just exploiting the knowledge of the Helmholtz theorem and derive the Biot-Savart law once again. And in the next class, we will do some kind of application of the Biot-Savart law and try to understand that if I apply the Biot-Savart law in certain system. For example, infinitely long current carrying wire, so how much magnetic field it will going to produce at some point say $\vec{r}$ and we will do this problem rigorously.

Later we find that such problems will be much simplified when we use the Ampere's law. But now in the next class, we do some rigorous calculation to find out that if a current carrying wire is there. Maybe in different shape, what should be the amount of magnetic field it will produce at some point in space? With that note I would like to conclude here, thank you very much for your attention and see you in the next class.