Foundation of classical electrodynamics
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology - Kharagpur

Lecture - 48
Image Method (Contd.,)

Hello students to the foundation of classical electrodynamics course. So, under module 2 today we have lecture 48 and in today's lecture, we will continue the few problems regarding the image method that is started in the last class.

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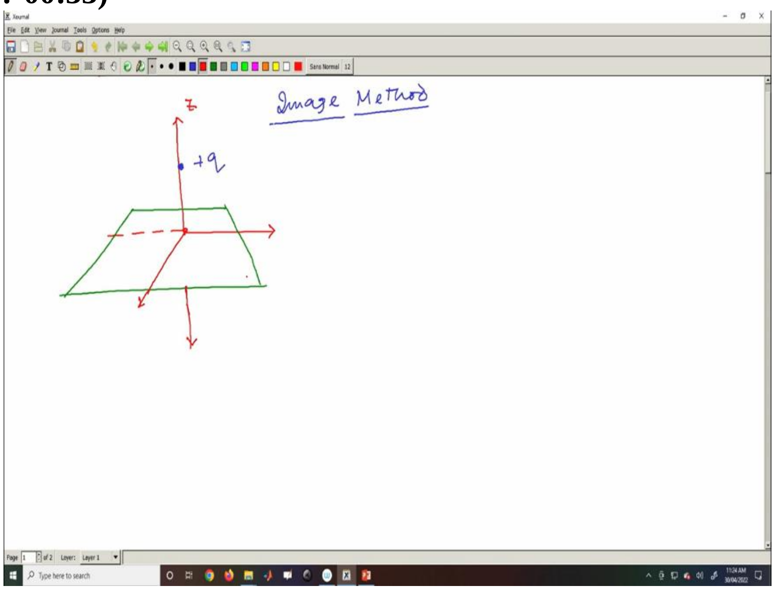

So, today we have class number 48 and today we will be going to continue this image method technique to solve few typical problems. This is also one kind of boundary value problem and effective when we have a system where we place a point charge near the conductor and try to find out the potential or electric field nearby and in that case what happened that for example, in the last day’s problem, we had a coordinate system like this and we had a plane like this.

And one charge was placed here and it was asked that what should be that was the z direction what should be the electric field at some point at r above this conductor. So, the problem is a bit difficult because in order to understand the potential we need to know the charge here, but also it will be going to induce some charge here over this plate and in order to know the potential you need to know what is the charge distribution, what is the amount of charge induced etc.?

So, the problem is simple when you consider an image charge a fictitious charge that will give us an equivalent effect as far as the boundary conditions are concerned and as a consequence of the uniqueness theorem, we will get the same result. So, that problem we discussed last day. So, we will go to extend this.

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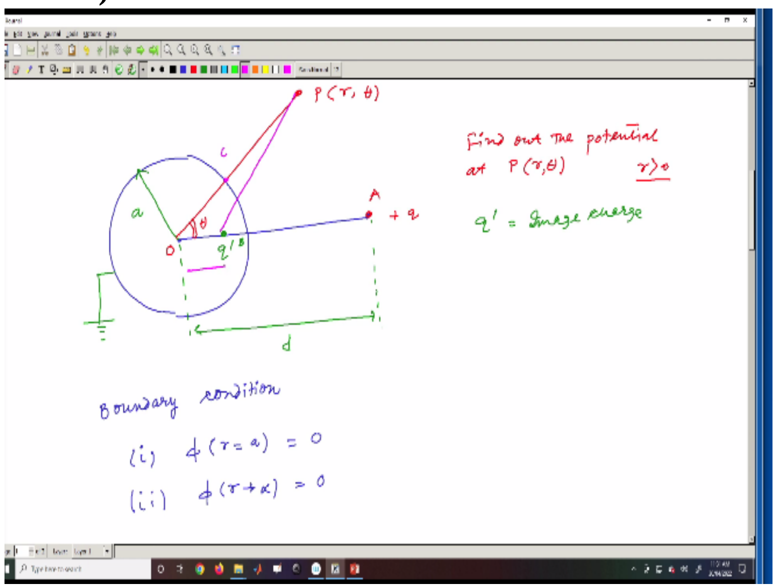

Now we have another problem say example 2, so in this example here we have a point charge and a grounded conducting sphere so, this is the system. So, let us first draw the system, so we have a conducting sphere like this and suppose a point charge is placed somewhere here this is the point charge that is placed. And this is grounded that means the potential over this sphere should be 0 this is grounded and this is the system.

Let us consider the radius of this system to be say a, the sphere a and it is placed suppose a distance from here to here and distance d. Now, let us go back to the problem. So, that is the system I just draw now the problem is this. The problem is to find out the potential at some point P whose coordinates a r, θ and r should be greater than a. So, outside the conductor, I want to find out and it has to be because inside the conductor the field we know it is 0.

So, some point P what should be the potential? So, let us draw that point P suppose, this point here is my P, which is a function of r, θ and I want to find out the potential at that point P and in order to find this potential now, like the problem we discussed before last day that it is partly due to the q is potential here is partly due to the point charge place here say +q so this point is A and this is origin or let us put some name.

So, the potential here at point P is due to partly the point charge +q, which is already there and due to the induce charge over this sphere and so, there are 2 contribution like before so, instead of having a sphere now we can consider if an image charge due to which whatever the potential we get that satisfy the boundary conditions. So, let us assume somewhere here by just considering the symmetry somewhere here we have my image charge and that image charge is say q '.

So, q ' by symmetry I put this q ' over this line and this q ' is the effective or the corresponding image charge for which I want to find out so, eventually I have 2 charges +q and whatever the contribution we have here as a induced charge I replace everything to this point charge q ', which is located the line joining to the origin o and the point A and that is purely because of the symmetry of the system.

And whatever the potential will get due to the system is equivalent to the potential I will get due to the placement of +q charge at q ' so, q ' is my image charge. So, now, the boundary condition let us see that this point is say B. So, the boundary condition, which is the most important here that is why it is called the boundary value problem. The boundary condition is saying that ϕ at r = a, that is over the sphere the potential is 0 because it is grounded that is the first boundary condition.

What is second boundary condition? The second boundary condition is saying that ϕ at r tends to infinity has to be 0 and that is the trivial boundary condition that if I go to very far distance the potential should not be there because it falls at $\frac{1}{r}$ normally this way. So, these are the boundary conditions. Now, let us because now we have 2 points charge +q and q '. So, we need to find out what is the distance in terms of this system.

So, suppose this is my θ and suppose I join this line here from here to here. So, this is an also I can join this A and so here this point says C this point is C. So, the potential here at C point is 0. So, let us one by one calculate suppose from here to here this distance is d ' and this d ' we need to evaluate. So, I just joined the point q to B and over this we have a C and over the C the potential is 0. So, let us now write one by one.

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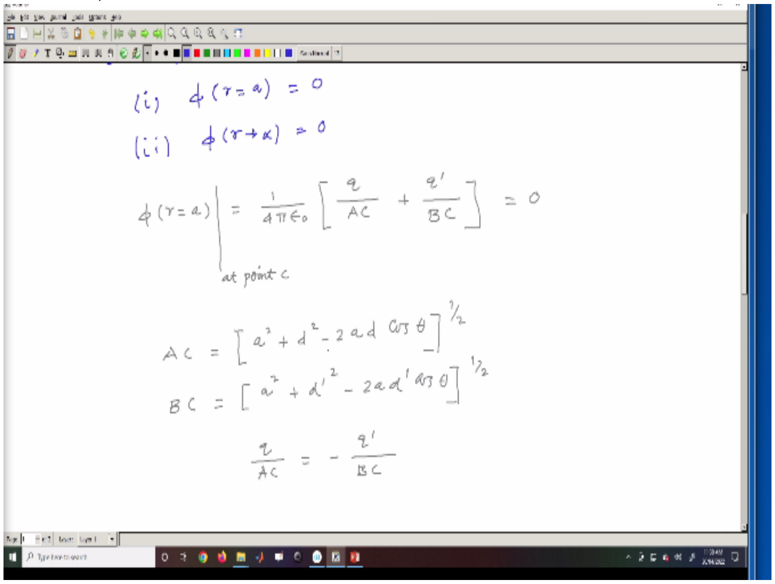

So, what should be my ϕ at r = a due to the +q charge and q ' it is $\frac{1}{4\pi\epsilon_0}$ and then q divided by the distance from that point so, this is at point C so AC and + $\frac{q'}{BC}$ so, that quantity is 0 now AC and BC I can write in terms of whatever the parameter is there. So, let me do that first so, AC the distance is simply $(a^2 + d^2 - 2ad$ and angle $\cos\theta)^{1/2}$. Obviously, this is because OC and AC so, this is d OC is a.

So, $d^2 + a^2$ and this angle is θ so, 2d into $a \cos\theta$ so that is the thing I am having here what about BC? BC is $(a^2 + d'^2 - 2ad'$ and $\cos\theta)^{1/2}$ because now it is this and I have BC. So, BC is eventually this, point this length and AC is this length so that we are drawing in the dotted line. So, now if this is equal to 0 that means $\frac{q}{AC} = -\frac{q'}{BC}$ so that is the condition we are having.

So, $\frac{q}{AC} = -\frac{q'}{BC}$ now this is true for all the θ value whatever the θ value takes over all this point over all these periphery what happened that the value of the potential is 0. So, it is true for all the θ values.

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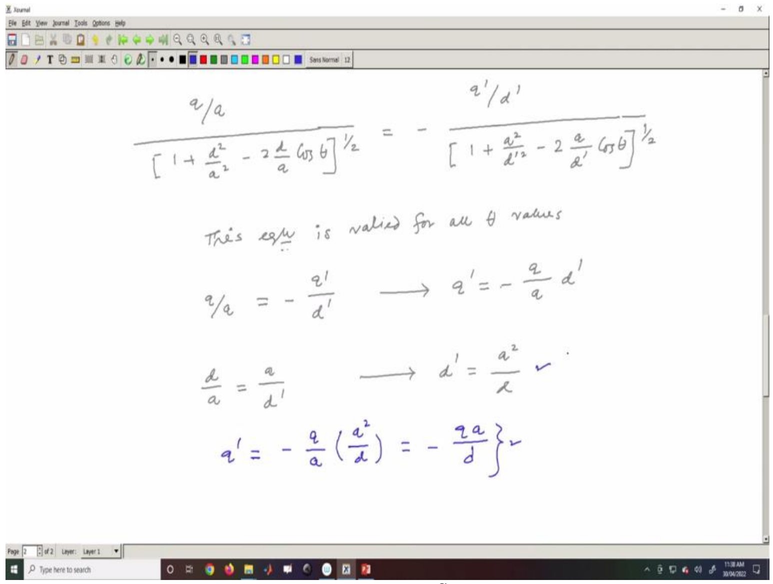

So, now if that is the case so we can readily see that $\frac{q}{a}$ let us simplify the expression and then I take a common and then $\frac{1}{[1+\frac{d^2}{a^2}-2\frac{d}{a}cos\theta]^{1/2}}$. And the right-hand side I should have -q' divided by I take d' common so $[1+\frac{a^2}{d'^2}-2\frac{a}{d'}\ cos\theta]^{1/2}$ this is the condition we are getting.

So, as I mentioned this equation should valid for all θ values if that is the case then coefficient wise it should be same. So, that means $\frac{q}{a}$ I can write $-\frac{q'}{a'}$. So, from that I had an expression of q', which is the image charge in terms of the given information and that is q a d' my d' is still unknown and another equation if it is true, then the ratio of these cos θ has to be same and also this one so, $\frac{d}{a}$ these things should be equal to $\frac{a}{d'}$.

So, from here I have another equation that d' = $\frac{a^2}{d}$ so now q' I can know explicitly terms of all the known quantity, So, q' is simply becomes $-\frac{q}{a}$ and d' I calculated it is $\frac{a^2}{d}$. So, it is minus of q and $\frac{a}{d}$. So, that means this is the amount of induced charge or the corresponding image charge if I put here at the distance d', where the d' is also calculated that it is $\frac{a^2}{d}$.

Then the first boundary condition that the potential at r = a should vanish is satisfied. So, that means, I satisfy the first boundary condition with this and also with this potential you can see that the second boundary condition we will see that how the second boundary condition can also satisfy because it is $\frac{1}{r}$ form, so potential so I know now the q ’ and all these things so what should be the potential that r that I need to write.

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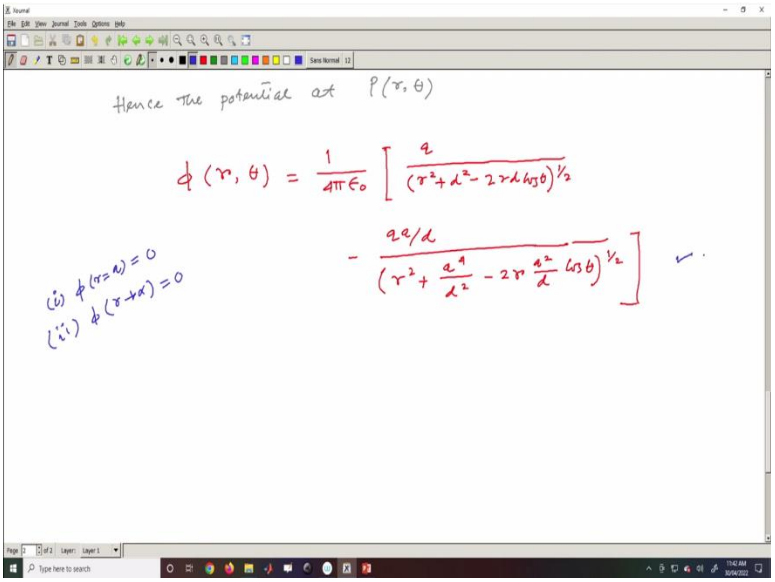

So, hence the potential at P (r, θ) at the point P, which is potential at the point P function of (r, θ) that value is how much ϕ (r, θ) = $\frac{1}{4\pi\epsilon_0}$ and then it should be q divided by these distance mind it. Now, this is from here to here this distance I can write in terms of r and θ and d and it simply comes out to be this $(r^2 + d^2 - 2rd\cos\theta)^{1/2}$ that is for the charge q.

And for charge q ’, q ’ is $\frac{qa}{d}$ so, I simply put it divided by $r^2 + d'^2$ now d'^2 also we know what is d ’? d ’ is $\frac{a^2}{d}$ this quantity so, this is my d ’ so I will put it here $\frac{a^4}{d^2}$ the minus sign not plus the may have a minus sign 2r then $\frac{a^2}{d}$ and cos θ whole to the power 1/2 that should be my potential.

And you can see that for this potential both the boundary condition is satisfied. So, the boundary condition again I should write the first boundary condition ϕ at r = a is 0 if you put r = a here then you should see that this quantity has to be 0 and also second one ϕ (r tends to infinity) is 0 because if you put r tends to infinity here then you can see that the ϕ is 0, so both the boundary condition is satisfied.

So that means if I have a structure like this a conducting sphere and if I place a charge nearby and then if I want to find out what is the potential at some point P that potential should not be simply $\frac{q}{4\pi r}$ rather this is this one this is this one. With that satisfying these 2 boundary conditions that is the main thing. Now, we like to you know check the potential is there now, exploiting the value of the potential we like to check that the induced charge that we calculated is really the value of this quantity or not $\frac{qa}{d}$ this value or not.

So, in order to understand that so, that was standard technique is first to find out the surface charge density. So, first we calculate the surface charge density and after that what we do that I will calculate the total charge. So, the surface charge density how the surface charge density will be going to change because I am placing a +q charge here. So, the surface charge density over the surface will change and that we first calculate.

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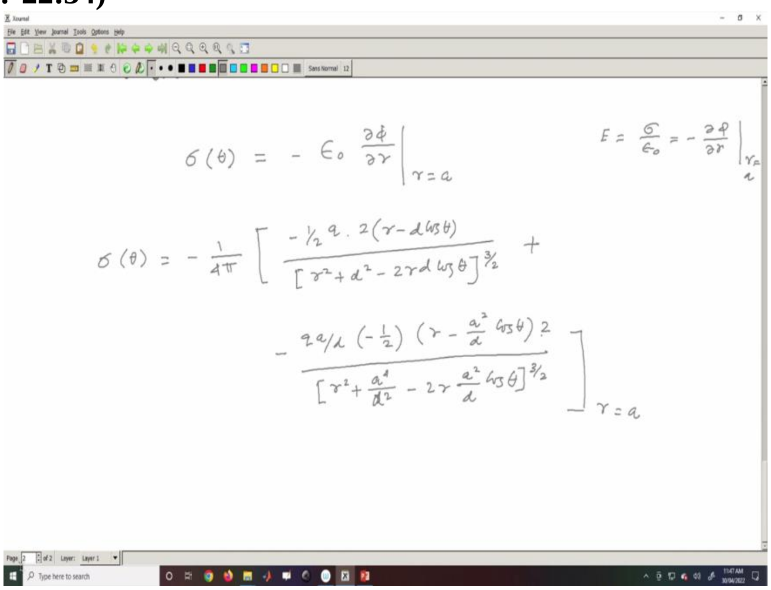

So, this is regarding the calculation of induced charge so, before going to calculate total induce charge, let us find out the surface charge density, which seems to be a function of θ, because it will not you know the surface charge density may not be uniform over θ and that quantity is $-\epsilon_0$ $\frac{\partial\phi}{\partial r}$ evaluated at the point r = a this is the standard expression we are using from last class onwards and we know that why it is because the electric field very near is $\frac{\sigma}{\epsilon_0}$, which is $-\frac{\partial\phi}{\partial r}$ in this case.

So, from that we know the expression is this and that is over r = a over the sphere. So, now ϕ we know because I just calculate ϕ, so we know what is $\frac{\partial\phi}{\partial r}$ so, that we need to calculate it is exactly like the previous problem. So, if I do then $\sigma(\theta)$ should be equal to $-\frac{1}{4\pi\epsilon_0}\ \epsilon_0$ we will be going to cancel out and then derivative with respect to r of this quantity this first one this one.

So, first we need to make a derivative of this one and second we need to make a derivative of this one. So, if I do I will get say minus of half and then q was there and then I have 2 of r - d cos θ divided by whatever the term we had this $[r^2 + d^2 - 2rd \cos\theta]^{3/2}$ that is my first term what is my second term? My second term is $-\frac{qa}{d}$ and then again I have another minus half here and then simply r - $\frac{a^2}{d}$ cos θ.

And then multiplied by 2 whole divided by whatever we had $[r^2 + \frac{a^4}{d^2} - 2r\frac{a^2}{d}\cos\theta]^{3/2}$ looks a very lengthy calculation but very straightforward evaluated at r = a.

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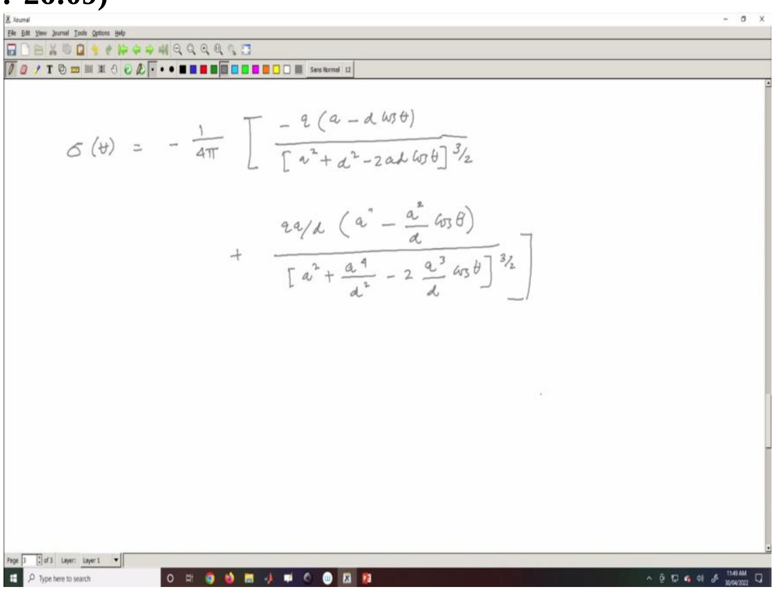

Now, the next thing is at r = a, what is the value? So, r = a so that quantity gives us $-\frac{1}{4\pi}$ and then I just replaced r = a and if I do it should be -q (a - d cos θ) whole divided by $a^2 + d^2 - 2ad\cos\theta$ just replacing r to a, $\frac{3}{2}$ and then plus of $\frac{qa}{d}$ and then $a - \frac{a^2}{d}$ cos θ whole divided by $[a^2 + \frac{a^4}{d^2} - 2\frac{a^3}{d}\cos\theta]^{3/2}$.

Now, this next term $\frac{qa}{d}(a - \frac{a^2}{d} \cos\theta)$ divided by that quantity, that quantity let us simplify because if I am not able to simplify, if I am not going to simplify it, then I am not going to do further.

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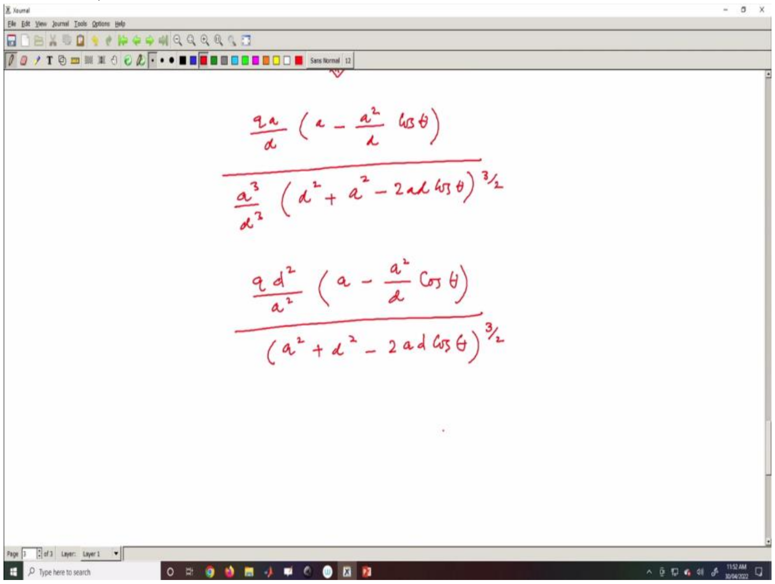

So, if I only consider this term and simplify so let us do that first for this term I have $\frac{qa}{d}$ and then I should write $(a - \frac{a^2}{d} \cos\theta)$ what is there I just write it denominator I will make a change I say write a^3 I take common d^3 if you take this then I have say $(d^2 + a^2 - 2ad \cos\theta)^{3/2}$. So, you can check it that whatever we had written is correct or not.

So, we have one a^3 and also so here we just so this is nothing here. So, from the denominator I just take this common whatever I take common is I need to make these $2ad$. So, that means I take $\frac{1}{d}$ common and I also I take the square of that thing. So, if I do that, then I have $\frac{3}{2}$ so these $\frac{3}{2}$ become this quantity. So, I just take $\frac{a^2}{d^2}$ common. So, from here I just take $\frac{a^2}{d^2}$ common and I will get this.

So, I mean what is the point of having this the point is, I can have the denominator both the cases same this one and this one that is the reason why did so, let us simplify this more. So, if I put this upward, then I simply have q and this d goes to upward so I should have $\frac{d^2}{a^2}$ and then multiplied by

(a - $\frac{a^2}{d}$ cos θ) whole divided by (a^2 + d^2 -2ad cos θ)$^{3/2}$. Now we are set because we are in a position to calculate because the denominator is same.

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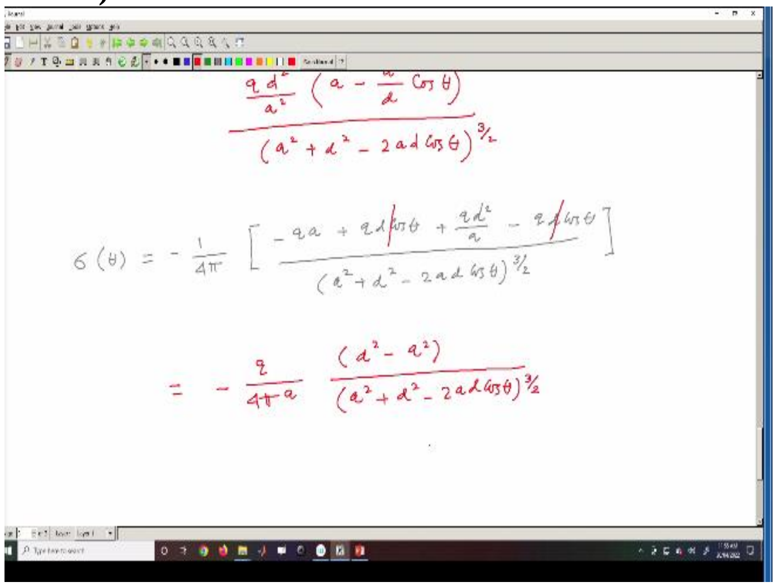

So, my σ now becomes σ(θ) is $-\frac{1}{4\pi}$ and I should have -qa + qd divided by cos θ. I just simplify the first here this one -qa then take a not divide by I take my divide, it is just qd cos θ it I just multiplied it. So, it is just qd cos θ that is the first what about the second term? Second term sounds like plus q$\frac{d^2}{a}$ and then minus of q a^2 a^2 will cancel out qd cos θ, one thing we are getting here and I have the common denominator that is (a^2 + d^2 -2ad cos θ)$^{3/2}$.

So, what I do here is I can just simply these things will simply cancel out and eventually what I am getting is this $-\frac{1}{4\pi a}$ and I have d^2 – a^2 here divided by (a^2 + d^2 - 2ad cos θ)$^{3/2}$ that is the value of the θ we are having.

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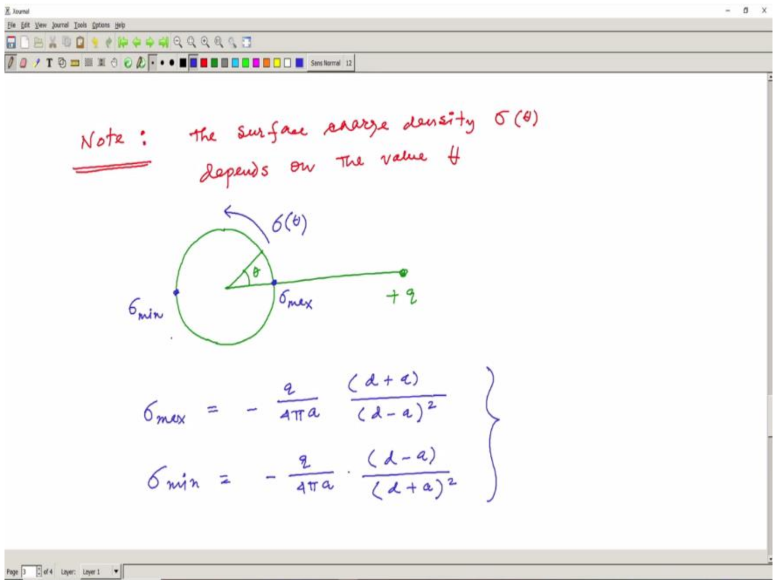

Now, you can see that the charge density is not independent of the θ note induce the surface charge density σ function of θ on very much depends on the value θ. So, eventually that means that I have this sphere here and the point charge was sitting here but the charge density depends on θ. So, whatever the charge density we are having here at θ = 0 at this point the charge density is different. So, the surface charge density is varying over this θ, because it is a function of θ.

Now, we can find out what is the maximum, minimum? So, we can readily do that let us do that as well. So, σ_{max} that if I calculate then it has to be the value when the denominator will going to be minimum when we have you know this is like $(a - d)^2$. So, if that is the case, so, I have $\frac{q}{4\pi a}$ and we have here in the numerator we have $d^2 - a^2$. So, I can write it (d + a) into (d – a) then this d minus will be going to cancel out by one (d – a) that we are having in the denominator.

So, it should be $\frac{(d+a)}{(d-a)}$ and it has to be square because it is cube. So, I this has to be square. So, what is the minimum? So, minimum happen when the downstair is maximum, so, that means it is $-\frac{q}{4\pi a}$ as usual. And then we have d – a, because now in the numerator we have $(d + a)^2$, so this is the way it is distributed. So, that means when θ = 0, we have $(a - b)^2$ so that means the induce charge will be going to be maximized.

When we have the point here, this is the point we have maximum and this is the point when θ = π then cos θ will be -1 and the numerator it will be $(a - d)^{3/2}$ here this point. So that gives me the minimum point here the minimum distribution I know I can now find out the total charge because the total distribution is in my hand.

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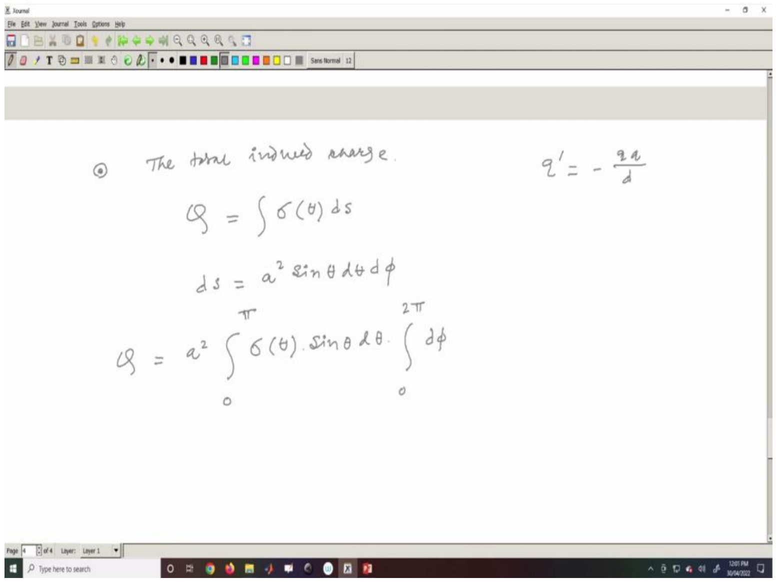

So, the next thing is to find out what should be the total induce charge so, that is the final verification that the total induced charge Q that has to be integration of σ, which is a function of θ and ds. This is the technique we also use in the last problem, but here we should note that q*'* that was the induced charge we already figure out the value to be $-\frac{qa}{d}$. And that we should have here $\frac{qa}{d}$ that value I mean that should be equivalent to the image charge.

So, that is the image charge we calculated and now, you calculate the total induce charge q by exploiting σ and the σ I calculated from the potential that is calculated. So, that is some sort of cross verification we are making and let us find what value we are getting. So, my ds should be a^2 sin θ dθ dφ and my Q should be a^2 is constant so I can take it outside integration 0 to π, σ(θ) then sin θ dθ this is the integration of θ and I should have another integration 0 to 2π and that integration is over dφ. But there is a φ symmetry so it should have simply 2π out of that.

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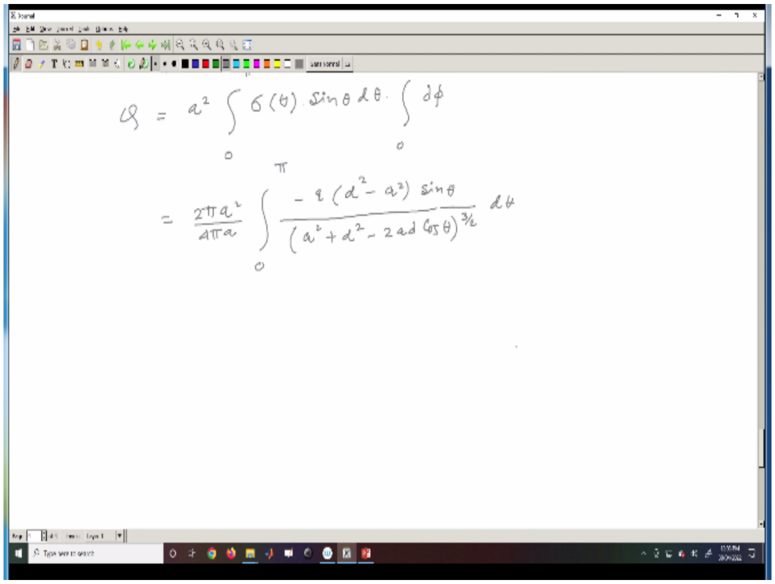

So, I simply write it as $2\pi\ a^2$ now my integration let us explicitly write my σ it is 0 to π and it is minus of q ($d^2 - a^2$) divided by what σ we get? This is the σ we are getting this is a σ we are using σ my σ(θ) is this quantity minus of q ($d^2 - a^2$) $4\ \pi\ a$ is there so I can put this $4\pi a$ outside say I should write is $4\pi a$ and then I have a sin θ, which is already there in the denominator I have (a^2 + d^2 - $2ad$ cos θ)$^{3/2}$ and over dθ.

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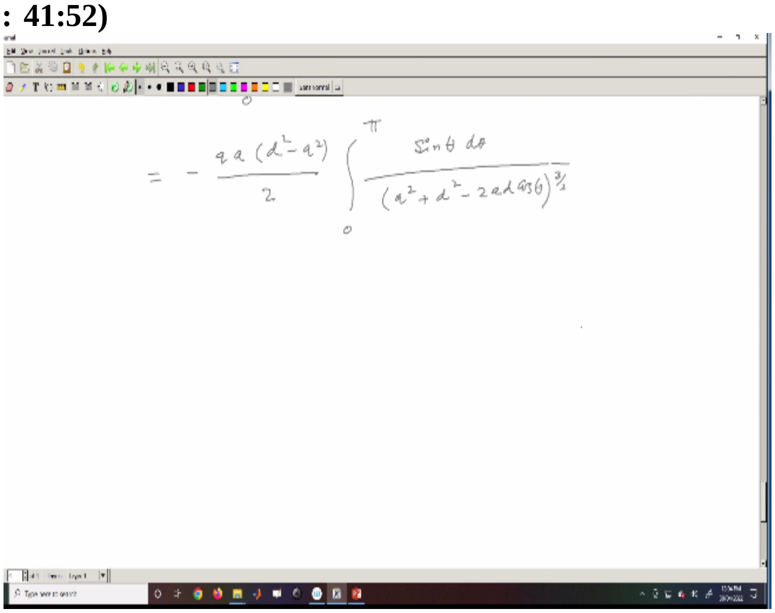

So, this quantity is simply minus of 2π 2π 4π will give me $\frac{1}{2}$ so it should be minus of q and then one a will cancel out ($d^2 - a^2$) will come out and I should have a 2 in the denominator that is this quantity and then 0 to π integration we simply have sin θ dθ whole divided by this term (a^2 + d^2 – $2ad$ cos θ)$^{3/2}$.

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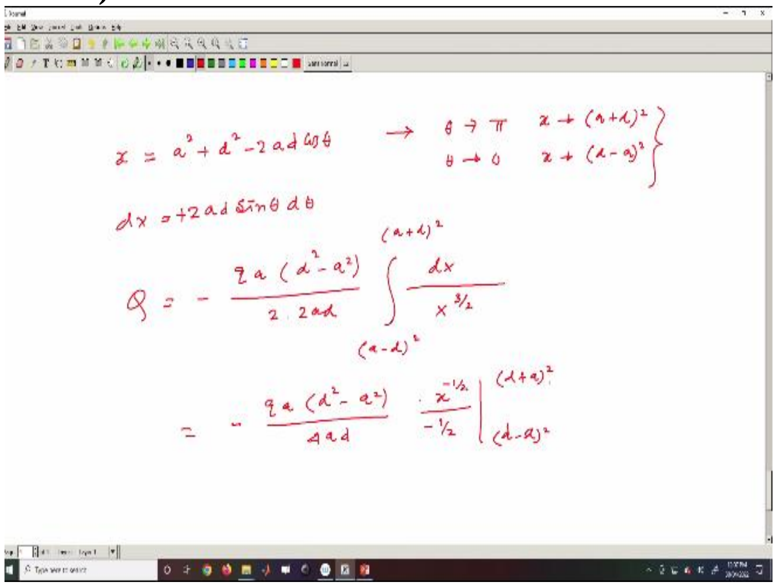

Now, let us write change your variable x consider us $a^2 + d^2 - 2ad \cos\theta$ if that is my case. Then the in terms of x the limit will be going to change so, when θ goes to you know π, x goes to $a + d^2$ because from here, but when θ goes to 0, x goes to $(d - a)^2$ this is the limit, dx if I calculate it should be $2ad \sin\theta \, d\theta$ with a negative sign, already the negative sign is here. So, I should have a plus sign here.

So, eventually my Q is minus of qa then $(d^2 - a^2)$ divided by 2 now, this is dx is this quantity so $2ad$ so, I just put an extra $2ad$ here and then I integrate so these become my dx then and in the denominator I have simply x $\frac{3}{2}$ and the limit will be $(a - d)^2$ and the upper limit will be $(a + d)^2$. So, these is how much minus of qa then $(d^2 - a^2)$ then 4 of ad and I have x to the power $-\frac{1}{2}$ and in the denominator I have minus of half that will be evaluated at $(a - d)^2$ or $(d - a)^2$ and $(d + a)^2$.

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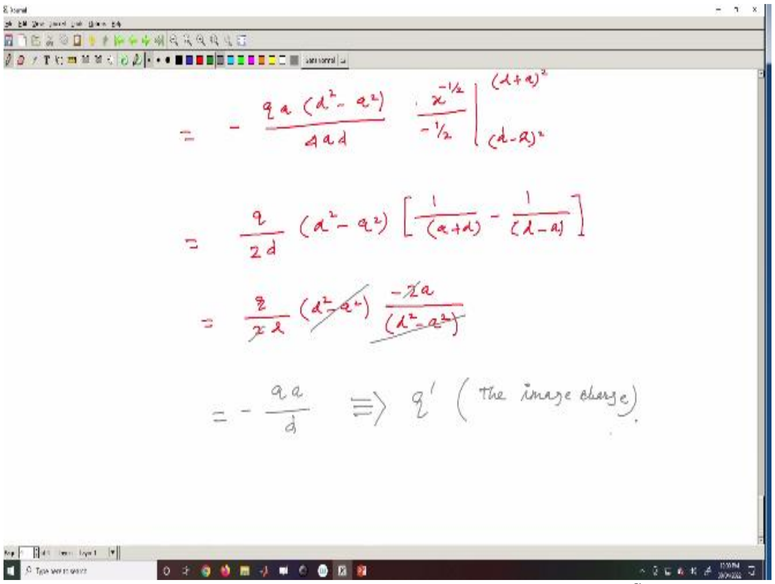

So, this half two minus sign we are going to absorb, so we have $\frac{q}{2}$ will be there so 2 and a is cancelling out here. So, we have $\frac{q}{2d}$ and then ($d^2 - a^2$) and for that limit, I have [$\frac{1}{(a+d)}$ - $\frac{1}{(d-a)}$], that is all. So, this quantity is $\frac{q}{2d}$ ($d^2 - a^2$) and this quantity is ($d - a$), this is ($d - a$), $-a$, -d. So, I should have say in the denominator I have ($d^2 - a^2$) and here we have $2a$, d will going to cancel out so, we have $2a$ with a negative sign.

So that thing is cancelling out and 1 2 will cancel out so whatever I get is $-\frac{qa}{d}$. Now, this is equivalent to q' that is our equal to this is the image charge that we consider so, we cross verify that the image charge can be calculated from the potential with this direct calculation. It is a lengthy calculation, but just to show that whatever the procedure you are doing is proper or not. So, my time is already up here today.

So, I just tried to show a couple of problems regarding the image method to the complete problem. So, I suggest the student to please do the boundary value problem by from some standard book and with the help of the knowledge that you gain in last 2 or 3 class you just need to apply that on that problem to find out the problem generally little bit lengthy, but it is very important for your understanding that how the methods are working. So, with that note, I like to conclude my class here. So, see you in the next class. Thank you.