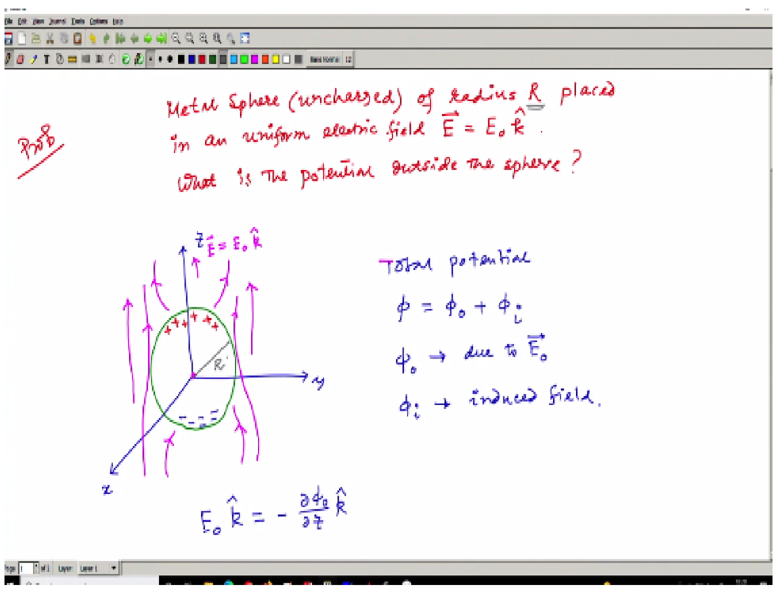
Foundation of Classical Electrodynamics

Prof. Samudra Roy
Department of Physics
Indian Institute of Technology – Kharagpur

Lecture - 47
Electrostatic Boundary Value Problem (Contd.), Image method

Hello students to the foundation of classical electrodynamics course. So, today under module 2, we have lecture 47 and today we will be going to continue the problem that we discussed in the last class that is the electrostatic boundary value problem and also try to do another to solve this kind of problem with another method called Image method.

(Refer Slide Time: 00:40)



So, let us start class number 47 today and we already started the boundary value problem, so, let us continue with another problem. So, I am doing all the problems so, that you can understand that how to deal with this problem by just exploiting the Laplace's equation in different coordinate system. So, the problem is like that a metal sphere, which is uncharged metal sphere means essentially it is a conductor of radius R placed in an uniform electric field will say $\vec{E}$, which is constant and along say z direction $\hat{k}$.

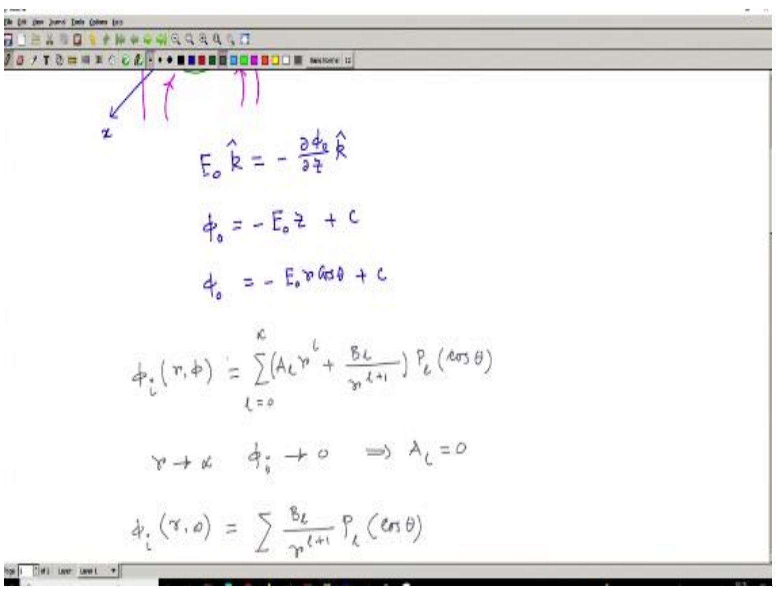
So, the question is what is the potential outside the sphere? That is the question. So, that means, I have let us draw the coordinate system first this is my z direction. So, this is my x and y and suppose we are having a sphere like this and it is in constant electric field. So, the electric field if

I draw say it is along this direction. So, these are the electric field but since it is a conductor so, what happened so, some plus charge will accumulate here and some minus charges are going to accumulate here.

And the field lines if I look it will just go like this field line I am drawing in different color so let us like this. So, this is the along this direction we are having our electric field, which is constant k. So, this is the scenario we are having right now. So, the total potential so, we need to find out the potential so, the total potential will be ϕ that is equal $\phi_0 + \phi_i$, what is ϕ_0? ϕ_0 is a potential due to E_0, which is already there and ϕ_i is due to the potential due to the induced field because there is a charge separation so, some field will be induced.

So, due to induced field so, $E_0\,\hat{k}$ that is simply $\frac{\partial \phi_0}{\partial z}\,\hat{k}$. So, ϕ_0 because I consider the constant electric field the potential is due to the constant electric field related to the constant electric field is ϕ_0. So, this should be the equation.

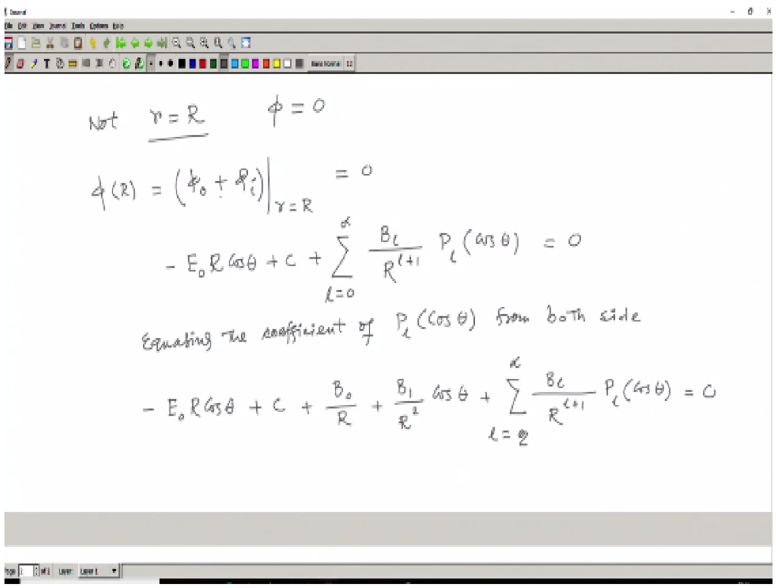
(Refer Slide Time: 06:56)



So, I can so, E_0 is known, so, I readily find out what is my ϕ_0? So, ϕ_0 if I integrate it should be $-E_0\, z$ + a constant C and this z , I can replace in terms of r, θ coordinate because this is spherical symmetry. So, this is simply $- E_0\, r \cos\theta + C$. Now, what is my ϕ_i that I can calculate because my the value of ϕ_i the potential due to the induced electric field the solution because that leads to the I mean the Laplace’s equation leads to that potential.

So, I can have the direct solution for that and this direct solution we derived in the last class it is $A_l\, r^l + \frac{B_l}{r^{l+1}}$ and then it should be $P_l \cos\theta$ with l goes to 0 to infinity. So, that is the general solution we already derive that. Now, r tends to infinity my field need to be 0, that condition readily gives us that $A_l = 0$ that is the boundary condition that I am putting now. So, my field ϕ_i, which is a function of r and θ simply becomes summation over l $\frac{B_l}{r^{l+1}}$ then $P_l \cos\theta$ this is the value.

(Refer Slide Time: 09:29)



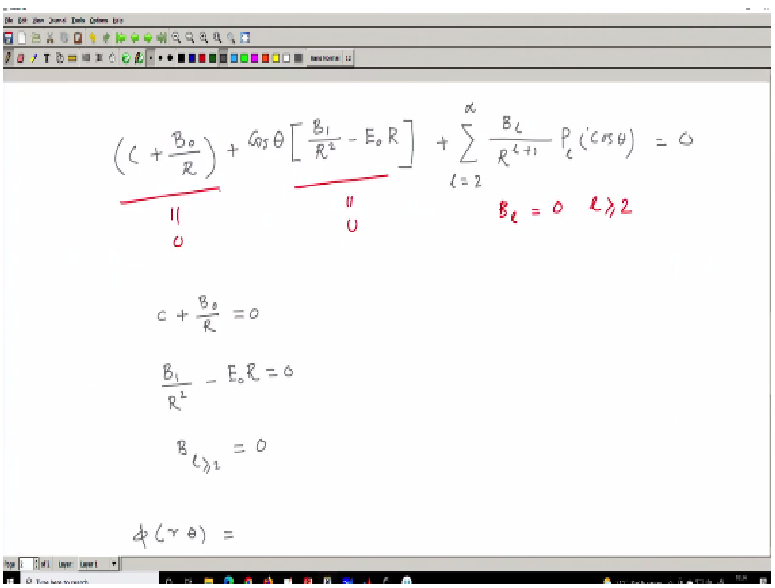
Now at r = R that is over the sphere because the radius of the sphere is R, I think it is given the problem. Here it is given the radius is R. So, this is R so over the sphere, we know the potential has to be 0, and if that is the case, then I just simply put the $\phi_i + \phi_r$, $\phi_0 + \phi_i$ at r = R is 0 because this plus this is my total ϕ. So, ϕ at R means this plus this at R = 0 is 0. So, what is my ϕ_0 it is $E_0\, r \cos\theta + C$.

So, I just simply put this value by replacing r to R then it becomes minus of $E_0 R \cos\theta + C$ plus I should have summation l 0 to infinity then I have $\frac{B_l}{r^{l+1}}$ just replace small r to R, then $P_l \cos\theta = 0$. Now, the next thing is straightforward because in the left-hand side we have a polynomial of cos θ and right-hand side it is 0. So, if I managed to find out the coefficient of cos θ for both the side then I can get the result. So, what I do I will go to equate.

So, equating the coefficient of $P_l \cos\theta$ from both the side we have so, left-hand side what we have minus E_0 R cos θ then C as usual then I have let us expand few terms. So, I have $\frac{B_0}{R}$ that is the l = 0 term because P_0 is 1 then I have plus $\frac{B_1}{R^2}$ and then cos θ because P_1 is cos θ and then rest of the term I should put as usual l now goes to 2 to infinity because I already take l 0 and 1 outside.

So, it should be $\frac{B_l}{R^{l+1}}$ and P_l (cos θ) = 0. Now, rearranging few terms because here we have cos θ and another cos θ is sitting here.

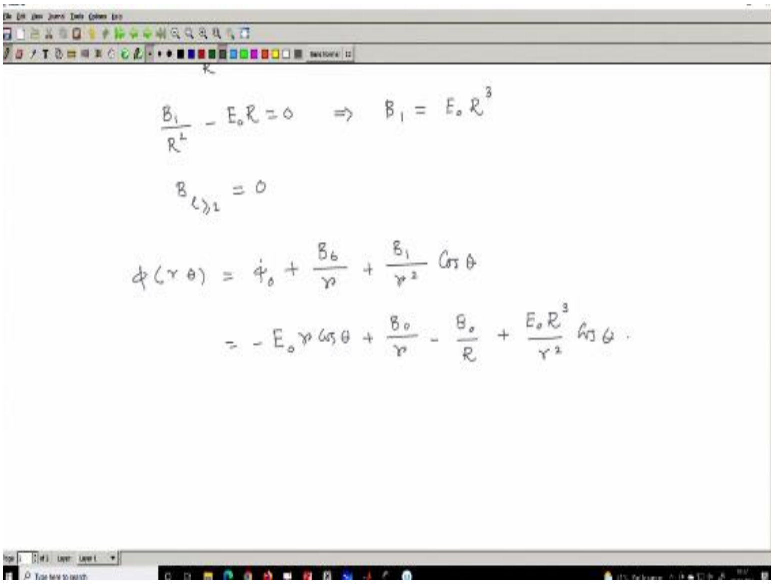
(Refer Slide Time: 13:49)



So, I just rearrange the term, so, let's put all the constants together. So, C + $\frac{B_0}{R}$ and then cos θ then $\frac{B_1}{R^2}$ - E_0 R then plus rest of the term l goes to 2 to infinity then $\frac{B_l}{R^{l+1}}$ P_l (cos θ) = 0. So, now, you can see that in the right-hand side we have 0. So, that means, and it is true for all the cos θ I mean any value of cos θ any value of θ or any value of cos θ.

So, that means, these is individually 0 this is individually 0 and all B_l should be 0, when l is greater or equal to 2. So, what do I get? I get C + $\frac{B_0}{R}$ = 0, I get $\frac{B_1}{R^2}$ - E_0 R = 0 and B_l where l greater than equal to 2 is 0 in this kind of problem in most of the cases, you find that initial few terms are there, but rest of the term are normally 0. So, if that is the case, I simply have my ϕ(r, θ) to be ϕ_0, what I get here $\phi_0 + \phi_r$.

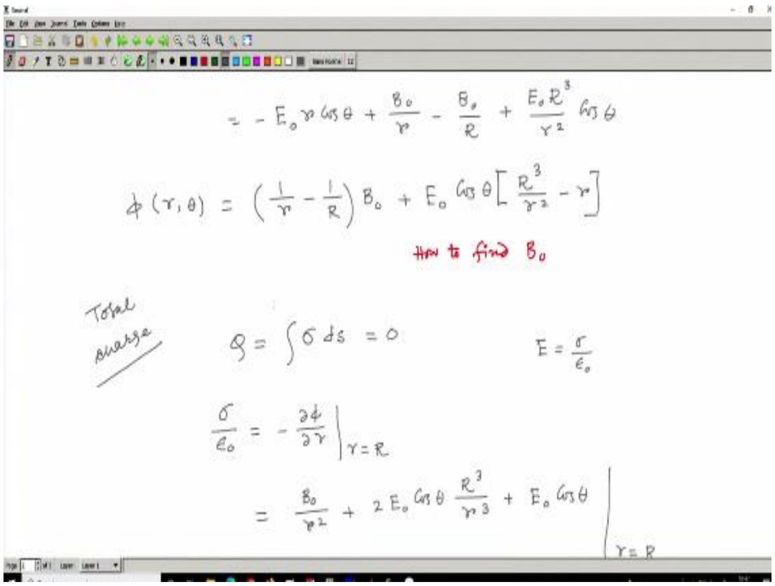
(Refer Slide Time: 16:18)



So, I simply have ϕ_0 and then $\frac{B_0}{r} + \frac{B_1}{r^2}\cos\theta$ because all the B_l, l greater than equal to 2 is 0. So, that should be my term. Now, I know that what is then I put simply - E_0 r cos θ and $+\frac{B_0}{r}$ and -B_1 I can find it from here that we want let us put B_1 how much it should be E_0 R^3 and my C is minus of so, here we got a C mind it. So, this C is from here my C was $-\frac{B_0}{R}$.

So, that term I now put here the C is $-\frac{B_0}{R}$ that is one term and another term is $\frac{B_1}{R^2}$ so +B_1 is $\frac{E_0 R^3}{r^2}$ and cos θ. So, everything I tried to write in terms of B_0 and only one constant now, I eliminate all the other constants putting the boundary condition.

(Refer Slide Time: 18:35)

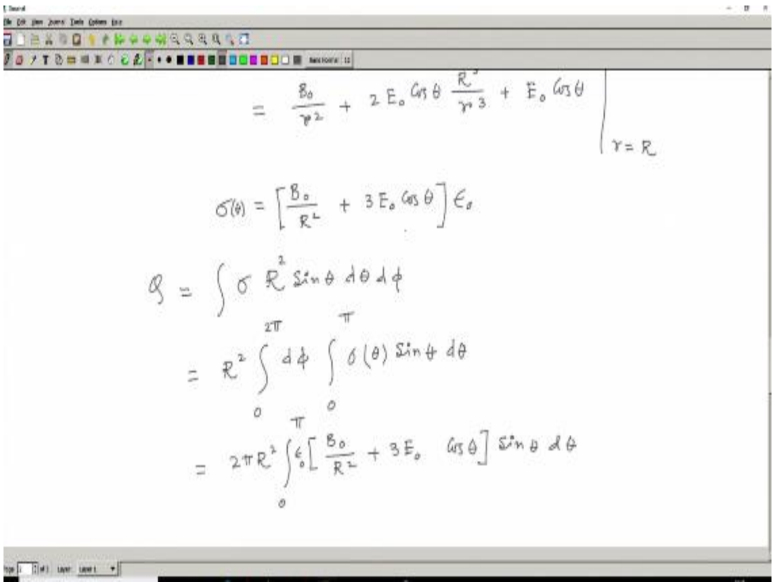


And eventually I have ($\frac{1}{r}$ - $\frac{1}{R}$) and then I have B_0 + E_0 cos θ if I take common then I have say $\frac{R^3}{r^2}$ - r. So, that is my potential at any point (r, θ) but still B_0 is there I need to you know determine B_0. So, one of the way to determine the B_0 is the total charge because the total charge here over this sphere has to be 0 and that condition may lead to the value of B_0. So, let us try to find out that how to get that. So, the total charge of this sphere so, the problem is how to find B_0?

So, the total charge will be Q equal to integration of σ ds, because σ is the charge density, and that is 0. And we know the relationship with the charge density and the potential because E for conductor just above the conductor the value of the electric field is this and we can we will go to exploit this expression and that simply gives us $\frac{\partial \phi}{\partial r}$ at r = R that should be the expression where we can relate the surface charge density with the potential we know ϕ is known, so, if I make a derivative with respect to r, so, this is my ϕ.

So, I can make a derivative with respect to r and if I do I will go to get this $\frac{B_0}{r^2}$ with a plus sign because r is in denominator and then +2 E_0 cos θ and then $\frac{R^3}{r^3}$ and then +E_0 cos θ I need to evaluate everything at r = R point.

(Refer Slide Time: 21:41)

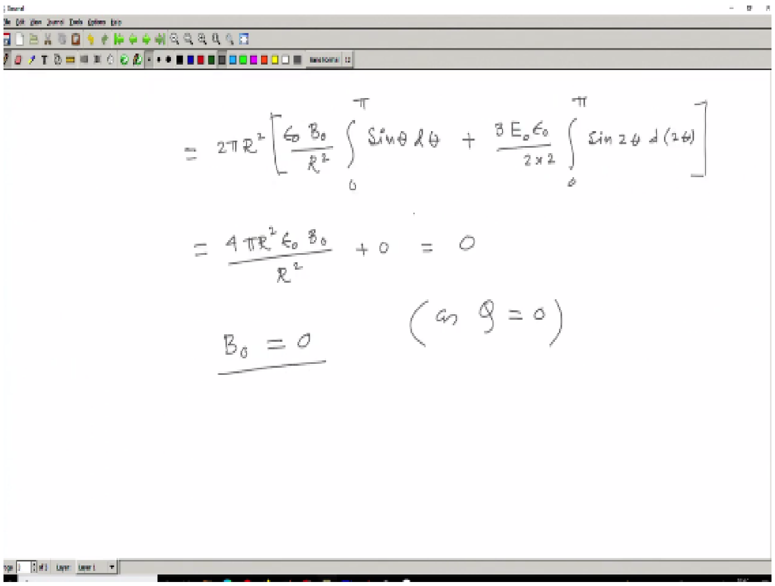


So, that eventually gives me $\frac{B_0}{R^2}$ plus this will be 1 and we have 3 E_0 cos θ that we will get. So, from here I have my σ. So, now the total charge if I calculate for this σ it should be integration of σ and ds in the spherical coordinate system over the surface it should be R sin θ dθ dφ R sin θ at over the surface we are calculating. So, that should be the value so, well.

So, now, we have so, it should be R^2 because it is the surface so it will be R^2 sin θ dθ dφ so, because it is a surface so, then I have R^2 0 to 2π and that is dφ and then integration 0 to π. Now, you can see σ is a function of θ. So, the integration the σ should be inside the integration of the θ apart from that we have a sin θ term here.

So, sin θ dθ σ(θ) σ function of θ sin θ dθ so, that value is from here we have 2π so, we have 2π R^2 and then integration 0 to π σ(θ) if I write that we already calculate and that value is B_0 what was σ? $\frac{B_0}{R^2}$ σ I need to multiply it because it is $\frac{\sigma}{\epsilon_0}$. So, σ has to be this multiplied by ϵ_0. So, $\frac{B_0}{R^2}$ and then plus what we get 3 E_0 ϵ_0 and then cos θ and then I have one ϵ_0 multiplication here and then sin θ dθ.

(Refer Slide Time: 25:34)

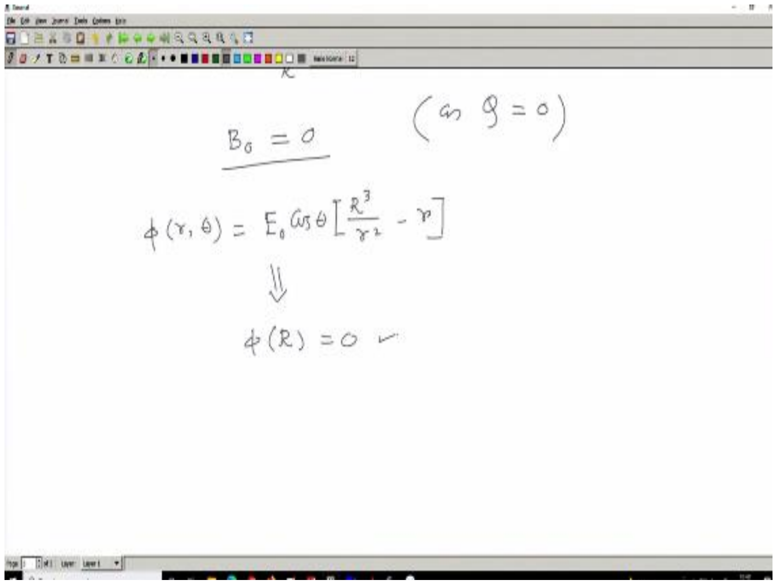


So, finally, if I execute this integration we have $2\pi R^2$ and then if I took this ϵ_0 for the first integral, let us make like this $\epsilon_0 \frac{B_0}{R^2}$ if I put outside, then the integration simply give us integration 0 to π sin θ dθ and another case +3 E_0 ϵ_0 I take common. So, I should not have ϵ_0 anymore here because I write ϵ_0 outside. So, it should be 3 E_0 and then ϵ_0 divided by a note that here we have a cos θ sin θ.

So, we can write that it is divided by say 2 and then it becomes sin 2θ and if make d(2θ), then another 2 I will put here integration 0 to π, because sin θ cos θ multiplication, I make a half and then it should be 2 sin θ cos θ then it should be sin 2θ, that is the result. So, now, we have $2\pi R^2$ so, these 0 to π sin θ if I integrate I should not have any value. So, here this will be cos θ and I have 2 here.

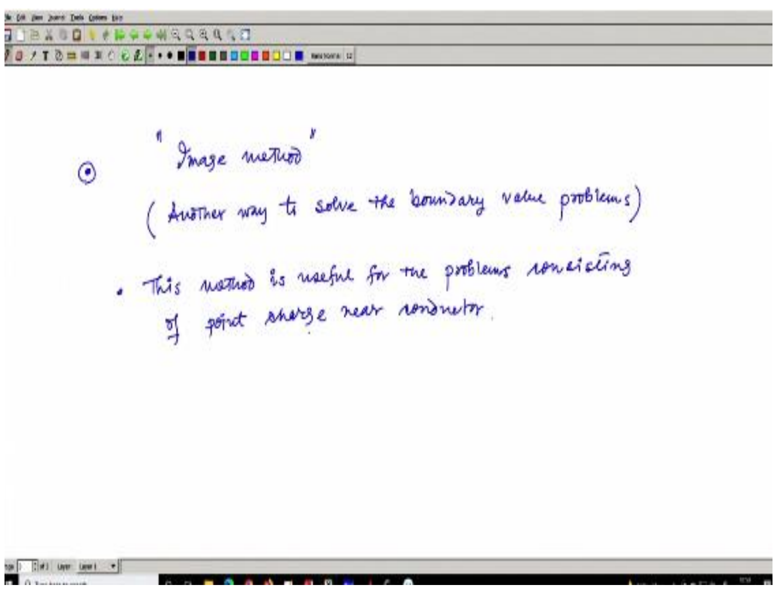
So, eventually this will be 4. So, I have a $4\pi R^2$ then $\epsilon_0 \frac{B_0}{R^2}$ + 0 because this integration and that is the total charge and that is 0. Since this is 0 after doing all these calculation, we find that B_0 is nothing but 0 as Q = 0. So, finally, I will put here the value B_0 because B_0 is 0 again. So, my result will be only this part.

(Refer Slide Time: 28:49)



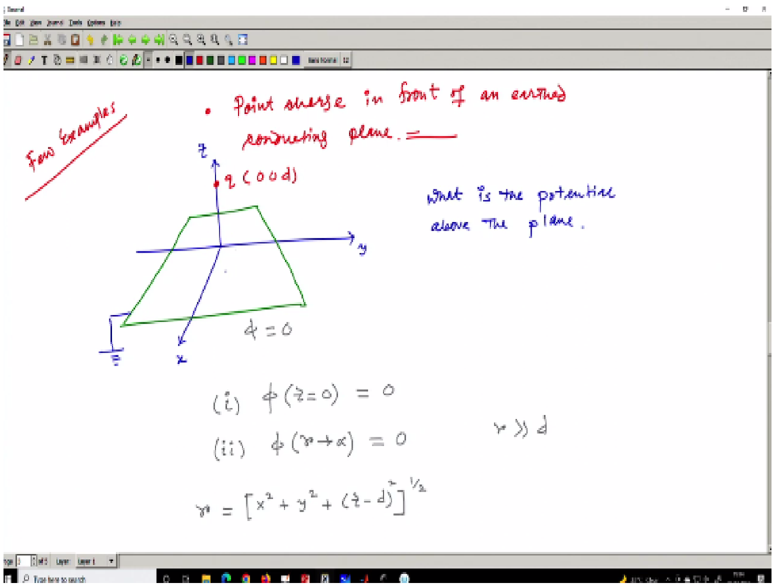
So, finally, I can get the value of ϕ as my final result $\phi(r, \theta) = E_0 \cos\theta \left[\frac{R^3}{r^2} - r\right]$. Note that at from here you can check it that $\phi(R) = 0$, which should be because if you put r = R here and then it should be r - R so, it should give you 0. So, it seems that my potential whatever I calculated is correct.

(Refer Slide Time: 29:46)



So, next we will extend these boundary value problem and we will go to learn another method another important method in few cases also we are going to use this method and that is called the image method we call it image method. So, image method is another way to solve the boundary value problem so, this method is generally useful for the problems consisting of point charge near conductor. So, when a conductor is there and again you place a point charge near that and then if somebody asks what should be the potential and all these then these kinds of image method the procedure is very useful.

(Refer Slide Time: 31:50)

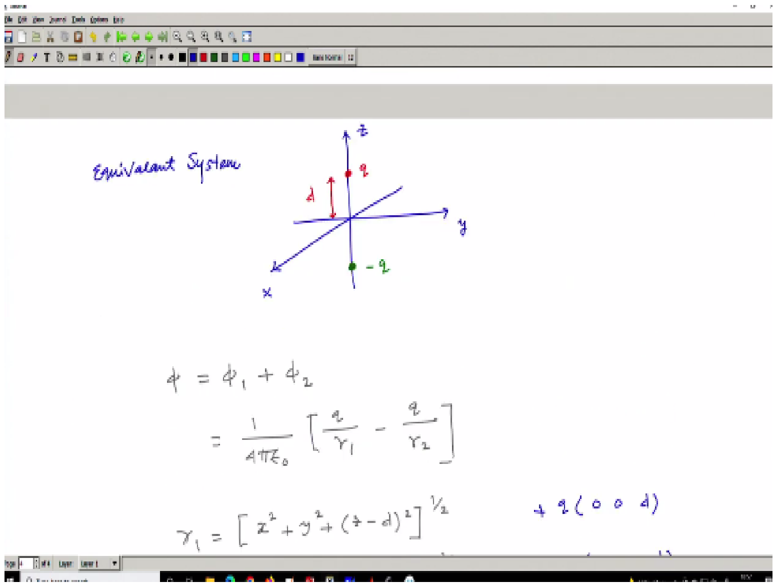


So, let us let us try to do with certain examples a few example so, the first example is the point charge in front of an earthed conducting plane. So, let us put a point charge so let us draw this how? So, suppose I have a coordinate system like this just I have a plane here like this and the point charge is placed somewhere here with coordinate is say 0 0 d where this is my z axis say this is x and this is y. So, this is my plane, plane is in over x y.

And this distance is d from here to here. So, the problem is what is the potential above the plane? That is the problem. And this is grounded mind it since this is grounded. So, the first boundary condition that we have is ϕ on this thing on this plane is 0 because this is grounded. So, let me jot down the boundary conditions that we have here. So, the first one is that ϕ at $z = 0$ is 0 I just wrote it here. Second, the trivial boundary condition that ϕ when r tends to infinity has to vanish when r is very very greater than d.

I am far away if I want to find out what is the potential then that should be 0 what is r here r is simply $[x^2 + y^2 + (z - d)^2]^{1/2}$. So, let us put a third bracket here so, this is my r so, far away from this point from here it is far away point and this is. So, now, the equivalent now in the image method what do we do we instead of having this system I have an equivalent system.

(Refer Slide Time: 36:23)

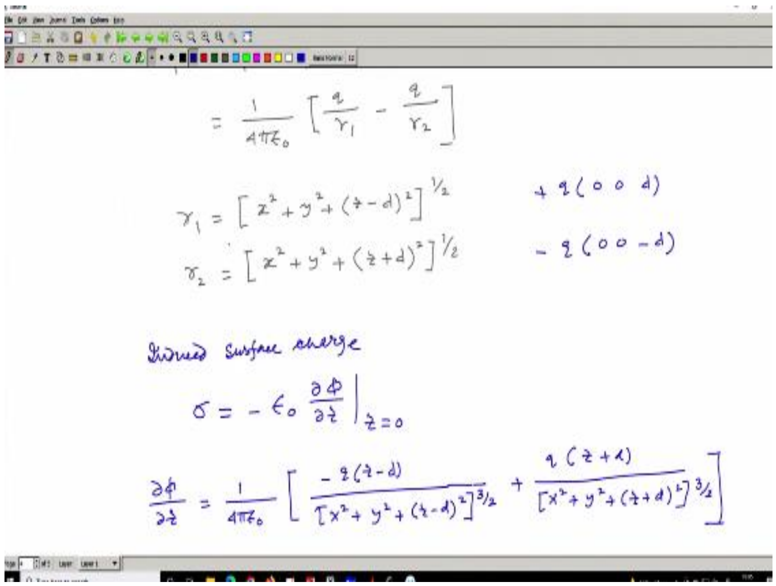


And the equivalent system is this, that is why it is called image we consider an image of this giving charge just below to the plate. So, the equivalent system is whatever the system whatever the problem is given the equivalent system is something like this. So, suppose this is my coordinate this is y axis, this is x axis, this is z axis already this charge is placed here at a distance of d the equivalent system is saying that I let us consider an equivalent charge here in the opposite direction with -q.

So, whatever this system so, this system is such that it you know it satisfy all the boundary condition that is given and for this equivalent system if I want to find out the potential then that should be the same potential if I have the original system, which is given here. So, let us do that. So, if I calculate the ϕ now for the system it should be the contribution of the 2 charges so, $\phi_1 + \phi_2$ the superposition. So, that value is simply $\frac{1}{4\pi\epsilon_0}$ and then it should be $\frac{q}{r_1}$ - $\frac{q}{r_2}$.

What is r_1 and what is r_2? The r_1 r_2 is, r_1 is $[x^2 + y^2 + (z - d)^2]^{1/2}$.

(Refer Slide Time: 38:27)

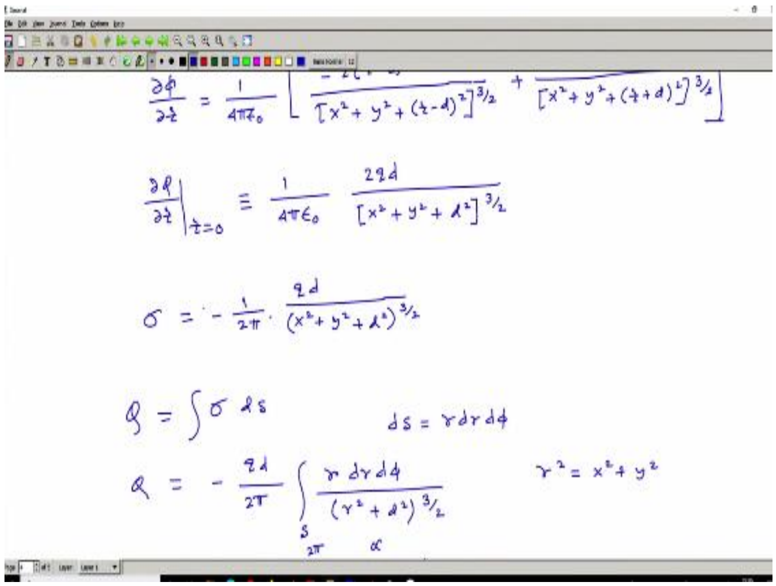


And r_2 from the -q charge the distance is $[x^2 + y^2 + (z + d)^2]^{1/2}$ mind it the coordinate of the +q is (0 0 d) and coordinate of the -q is (0 0 –d). Now, the boundary condition 1 and 2 satisfy by the potential if you look carefully that when z = 0 that is if I put z = 0 then r_1 and r_2 are same, the ϕ will be 0 and also if r tends to infinity, then the ϕ is infinity. So, whatever the boundary condition is there, it is satisfied by this equivalence system. So, that is the important part.

So, the induced now, let us find out what is the induced surface charge because if you put a point charge here q and if you have a conducting plane, then some induced charge will be there and that induced charge we like to find out first. So, the induced surface charge is σ equal to $-\epsilon_0 \frac{\partial \phi}{\partial z}$ evaluated at z = 0 standard formula that we are also using in the previous problem and also the problems now.

$\frac{\partial \phi}{\partial z}$ because ϕ is known, so, $\frac{\partial \phi}{\partial z}$ it is simply $\frac{1}{4\pi\epsilon_0}$ and then $\frac{\partial \phi}{\partial z}$ will be – q(z – d) should go up and then the denominator I should have $[x^2 + y^2 + (z - d)^2]^{3/2}$. And then here I have +q(z + d) and then I have $[x^2 + y^2 + (z + d)^2]^{3/2}$ and that is my derivative.

(Refer Slide Time: 41:29)

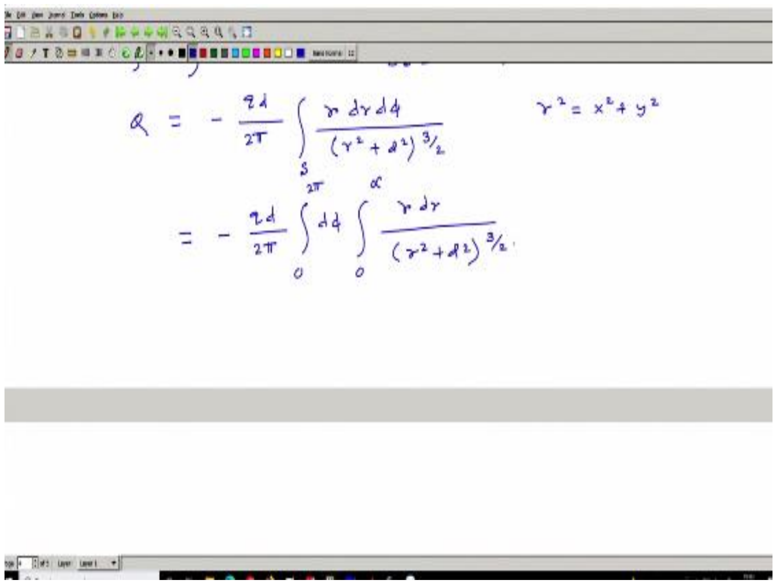


Now, this derivative will go to execute at the point z = 0, so, that simply gives me $\frac{1}{4\pi\epsilon_0}$ and then z = 0. So, this quantity so, I have a 2 here so, it is simply $\frac{2qd}{[x^2+y^2+d^2]^{3/2}}$, know my σ is how much $-\frac{1}{2\pi}$ and then $\frac{qd}{[x^2+y^2+d^2]^{3/2}}$ just use the formula σ is equal to minus of I just multiply $-\epsilon_0$.

ϵ_0 ϵ_0 will be going to cancel out and then one 4 will be canceled out by these 2. So, these 2 is here. Now, what is total charge that is the thing so, total charge should be integration of σ and ds where ds is the surface element over the xy plane and in. So, ds should be simply r dr you know dφ. So, now I have here σ, so my Q should be $-\frac{qd}{2\pi}$. And I need to integrate now I changed my coordinate because here $x^2 + y^2$ is there and but ds I write in terms of r φ for my convenient I did it.

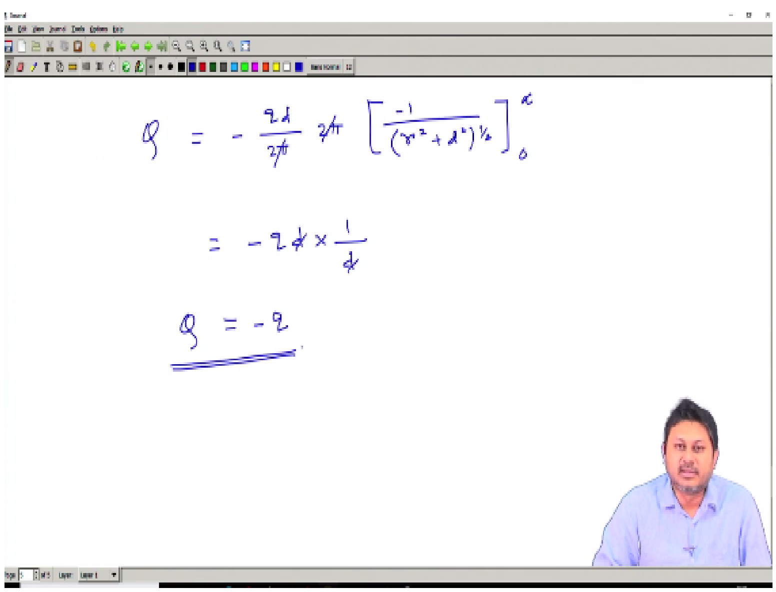
So, I need to write my x and y because $x^2 + y^2 = r^2$, so that I can use here that is $(r^2 + d^2)^{3/2}$ I can write r^2 as $x^2 + y^2$ here. So, now I can, so what is the integration.

(Refer Slide Time: 44:23)



So, now I can write it as minus of this is over surface, this integration is over surface. So I need to put the proper limit. So that is 0 to 2π for φ and 0 to infinity, because it is infinitely extended plane $\frac{rdr}{[r^2+d^2]^{3/2}}$ this integration is very much durable.

(Refer Slide Time: 45:01)



So, I can have $-\frac{qd}{2\pi}$. And then from here I have another 2π, because I am going to integrate dφ here and this integration if you do because if you take ($r^2 + d^2$) to be x than 2 r dr will be here. So then you can be able to find out the value of this integration is $\frac{-1}{[r^2+d^2]^{1/2}}$ with the limit 0 to infinity.

Now, if I put this the limit 0 to infinity, I simply find that these 2π 2π it is going to cancel out it is -qd × $\frac{1}{d}$. So, d d will be going to cancel out.

So, eventually I get -q and that is expected because if I put a q here, so the induced charged here, whatever the induced charge, should be -q and that induce surface charge and that precisely we get by exploiting the potential so how we get this -q? The -q we get by just exploiting the value of the ϕ that we calculated using the superposition principle, so that everything is in right place.

And using that actually, we find that the total potential using this simple image method, we find that the potential at any point can be represented by just simply making equivalent system. Where we have a +q and -q, instead of having an infinite plane, I just replace the entire infinite plane to an image. So that is why it is called image as if I am having a mirror here and a +q is imaged here with a negative value.

And then it satisfy the boundary condition the 2 boundary conditions which are marked here. And then we find the potential and exploiting that potential, we find the induced surface charge is also becoming Q = -q, which is expected. So this is an elegant way to solve again the boundary value problem. So today, I do not have much time to you know extend to show you this process to show another problem, but next class, I will like to show another problem.

And that will be the end of our module 2. After that, we are going to start module 3 where we understand the magnetostatic problem. So with this note let me conclude today's class. Thank you very much and see you in the next class.