Foundation of Classical Electrodynamics
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Lecture - 46
Electrostatic Boundary Value Problem (Contd.,)

Hello students to the foundation of classical electrodynamics course. So, under module 2, we are today having lecture 46 and we would like to continue the electrostatic boundary value problem that we started in the last class.

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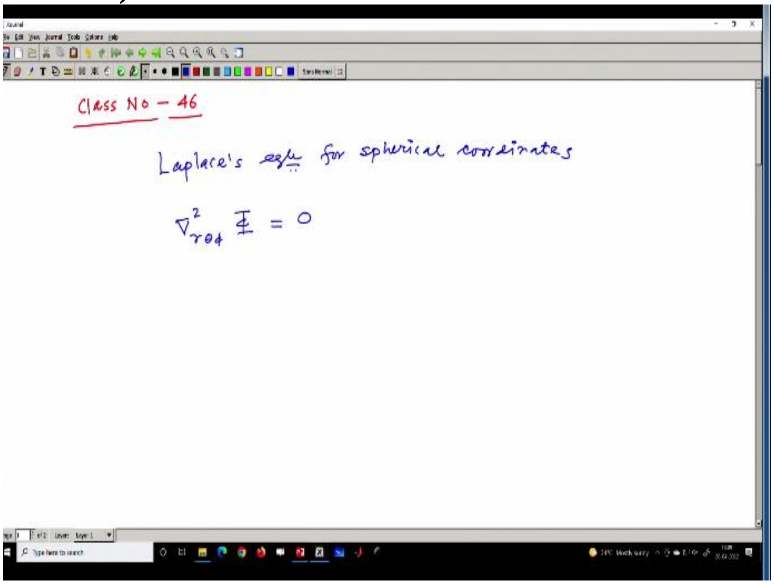

We have class number 46. So, now we are dealing with this set 2 dimensional Laplace equation. So, today we will solve the 2 dimensional Laplace equation for spherical coordinates. So, for spherical coordinates the Laplace equation is simply r, θ, φ and then it will operate over some potential ϕ equals to 0 that is the form.

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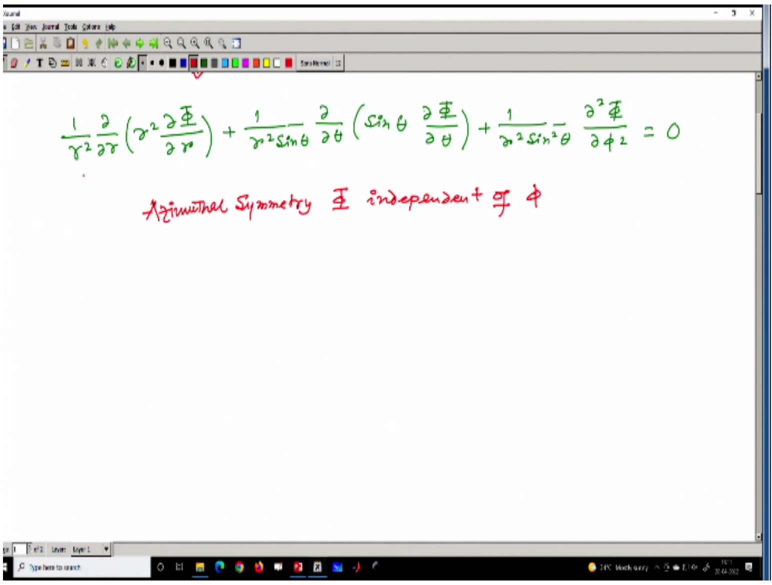

If I now explicitly write this operator it should be $\frac{1}{r^2}\,\frac{\partial}{\partial r}$ and then r^2 $\frac{\partial \phi}{\partial r}$ + $\frac{1}{r^2 \sin\theta}$ and then $\frac{\partial}{\partial \theta}$ and then sin θ $\frac{\partial \phi}{\partial \theta}$ and finally + $\frac{1}{r^2\,sin^2\,\theta}$ $\frac{\partial^2 \phi}{\partial \varphi^2}$ = 0 that is the form we know. Now, this is a 2 dimensional problem. So, let us consider the azimuthal symmetry. So, this is for azimuthal symmetry what we have that ϕ should be independent of a variable φ. So, then this equation whatever the equation will simply contain, so this term will no longer there so this derivative will 0.

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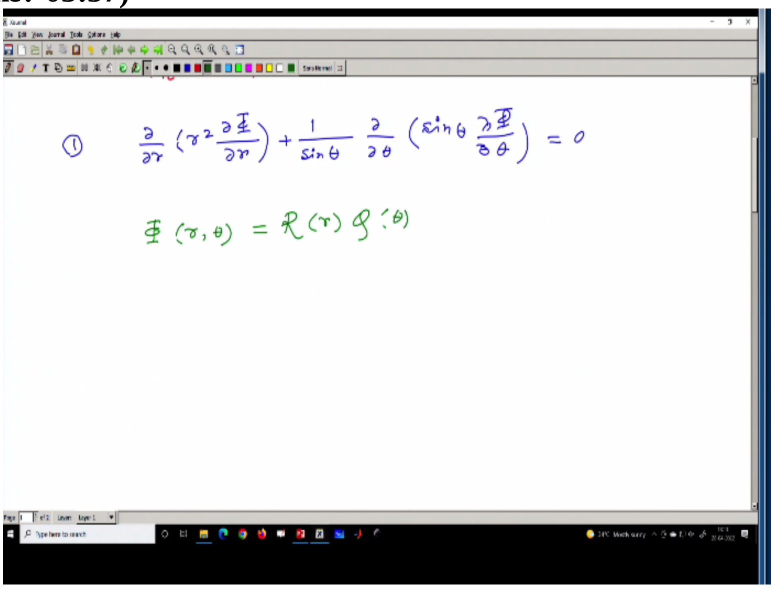

So, eventually we have let us write the equation. The differential equation that we have is $\frac{\partial}{\partial r}$ because $\frac{1}{r}\,\frac{1}{r}$ will cancel out. So, we have $\frac{\partial}{\partial r}$ (r^2 $\frac{\partial \phi}{\partial r}$) + $\frac{1}{\sin\theta}\,\frac{\partial}{\partial \theta}$ and then I should have sin θ $\frac{\partial \phi}{\partial \theta}$ and that is equal to 0 that should be my differential equation. Now, again like the previous problem that we did in the last class, my total potential ϕ, which should be function of r, and θ can be written as a separation of variable like R function of r and Q function of θ. So, I just use the separation of variable and if we put that in this equation.

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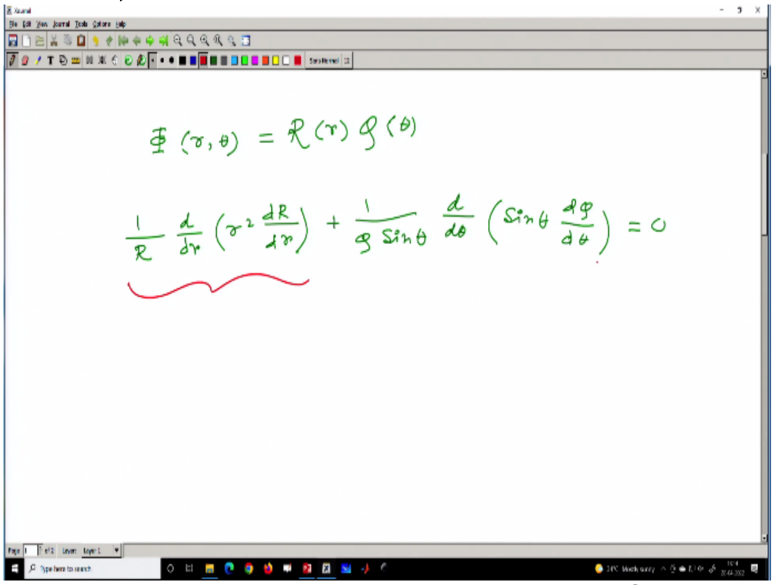

And then I after putting this in equation 1 one can simply have $\frac{1}{R}\frac{d}{dr}$ and then $r^2\frac{dR}{dr}$ because Q is a function of θ. So, this I can take it outside and then I divide everything to $\frac{1}{RQ}$ and then it should be $+\frac{1}{Q\sin\theta}$ and $\frac{d}{d\theta}$ and sin θ $\frac{dQ}{d\theta}$ = 0. Now, this portion is simply function of R and this portion is simple function of Q or simply function of θ.

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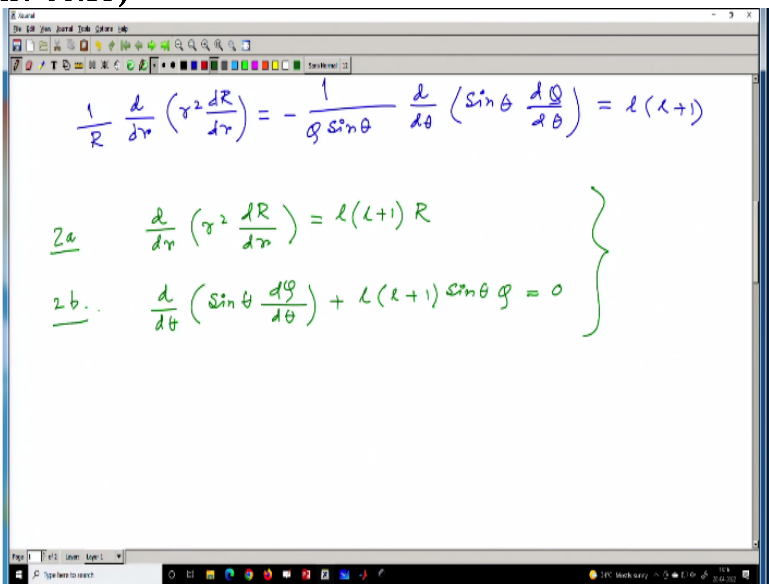

Then I can so, this is a standard procedure, so, I can write it as $\frac{1}{R}\frac{d}{dr}$ and then $r^2\frac{dR}{dr}$ that is equal to $-\frac{1}{Q\sin\theta}$ and then $\frac{d}{d\theta}$ then sin θ $\frac{dQ}{d\theta}$ this has to be some constant and this constant I write like l(l + 1) why I am writing this we will see later, because, if you write this constant that l(l + 1), then we have a well-known differential equation in our hand whose solution is well-known.

So, then I can have from this I can have the equation say 1 equation 2a like $\frac{d}{dr}$ (r^2 $\frac{dR}{dr}$) = l(l + 1) R and equation 2b is $\frac{d}{d\theta}$ and then sin θ and then $\frac{dQ}{d\theta}$ that portion plus I can make a minus sign here and then put it this side plus l(l + 1) and then sin θ and then Q is equal to 0 these 2 equation we have 1 equation is only as a function of R and another equation is only function of θ. So, that is I mean technique we use for separation of variable problems.

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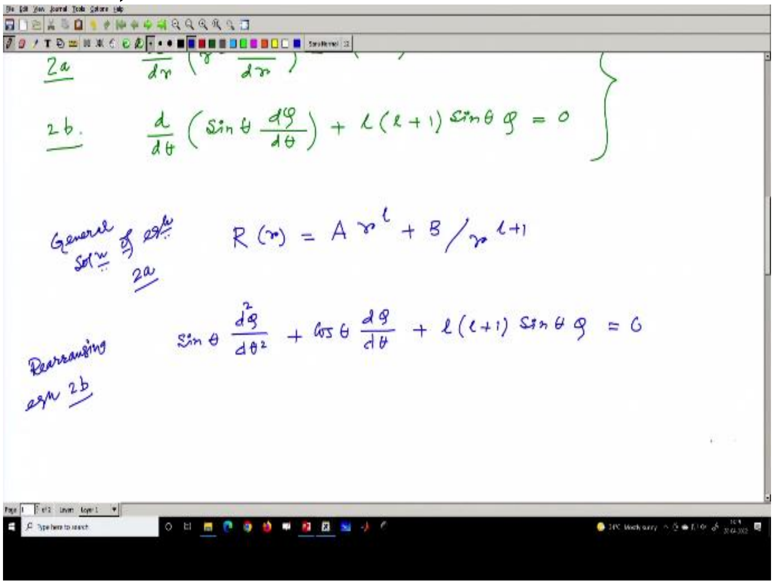

Now, the general solution of the 2a one can find the solution this is you can check it. The general solution of equation 2a is this R as a function of r is equal to constant A r^l + $\frac{B}{r^{l+1}}$. So, one can check it quickly by just putting the solution R to the equation 2a and check that left-hand side and right-hand side are matching or not just put these values here and check left-hand side and right-hand side is matching or not and it has to match because this is a solution.

So, I give you students I give you this as a homework problem, please check that whether whatever is written here in the blue colour the solution it is satisfying equation 2a or not. So, what about 2b? So, I can rearrange these things slide by doing all these derivatives whatever is there and 2b rearranging equation 2b what I get is this sin θ and $\frac{d^2Q}{d\theta^2}$ then + cos θ $\frac{dQ}{d\theta}$ + l(l + 1) sin θ Q = 0 just to rearrange this term, I just make the derivative whatever the derivative we are having here $\frac{d}{d\theta}$ and then make it the derivative and I am going to get this.

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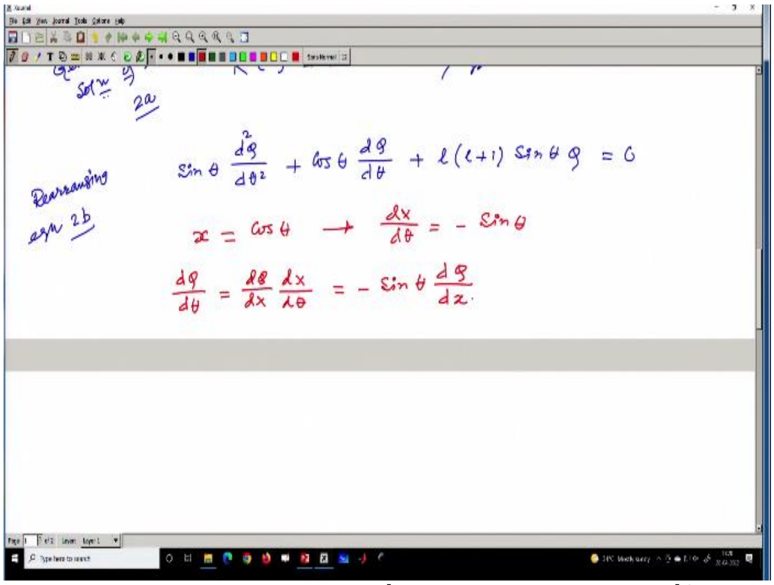

Now, let us take x = cos θ then that leads to $\frac{dx}{d\theta}$ minus of sin θ and $\frac{dQ}{d\theta}$ by using the chain rule will simply becomes $\frac{dQ}{dx}$ and $\frac{dx}{d\theta}$ is - sin θ. So, - sin θ $\frac{dQ}{dx}$.

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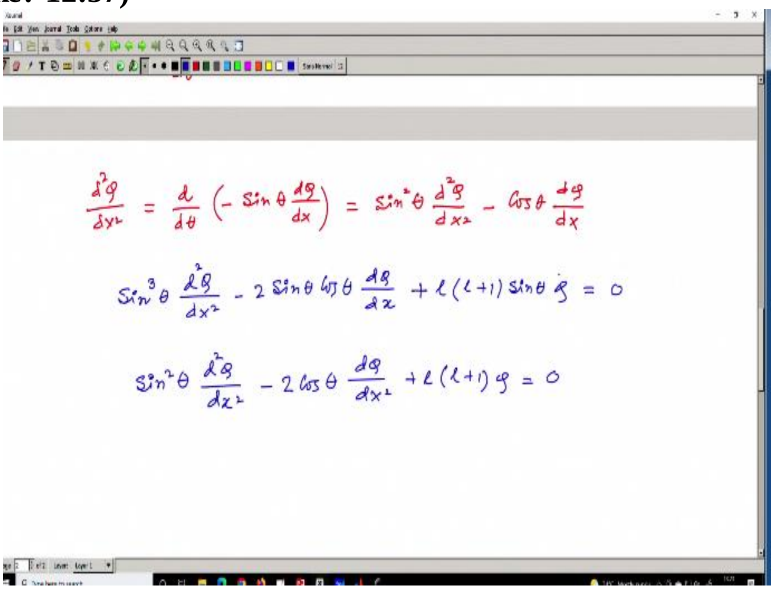

So, $\frac{d^2Q}{dx^2}$ is there so $\frac{d^2Q}{dx^2}$ is simply $\frac{d}{d\theta}$ and -sin θ $\frac{dQ}{dx}$ that is simply if I do this, so, it simply comes out to be sin^2 θ $\frac{d^2Q}{dx^2}$ - cos θ $\frac{dQ}{dx}$ why, because I am making a derivative here. So, the sin θ become cos θ so, -cos θ $\frac{dQ}{dx}$ is one term and another case it is - sin θ and $\frac{d}{d\theta}\frac{dQ}{dx}$ again I can make this $\frac{d}{dx}$ and $\frac{dx}{d\theta}$. So, $\frac{dx}{d\theta}$ I can replace here this value that -sin θ. So, -sin θ multiplied by -sin θ becomes sin^2 θ.

So, I put everything here in this equation and when I replace what I see is this it gives me like $\sin^3\theta \frac{d^2Q}{dx^2}$ and then $-2 \sin\theta \cos\theta \frac{dQ}{dx}$ + l(l + 1) and then $\sin\theta$ Q = 0. So, one $\sin\theta$ will be going to cancel out. So, I have $\sin^2\theta \frac{d^2Q}{dx^2}$ - 2 $\cos\theta$ then $\frac{dQ}{dx}$ and then + l(l + 1) then Q = 0.

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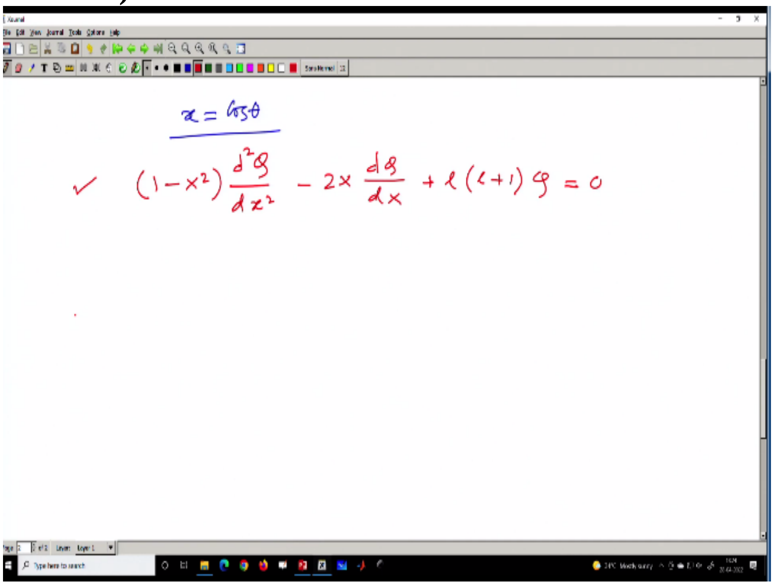

Now, if I put you know this $\cos\theta$ already, so as $x = \cos\theta$, so, that we already took here so, if I just replace here in terms of x, it simply becomes a very interesting equation, which is $(1 - x^2)$ $\frac{d^2Q}{dx^2}$ - 2x $\frac{dQ}{dx}$ + l(l + 1) and then Q = 0. Now, this equation is not new, we already discussed this equation, because this is the differential equation whose solutions are given as a Legendre polynomial and these things we discuss when we are discussing the multipole expansion.

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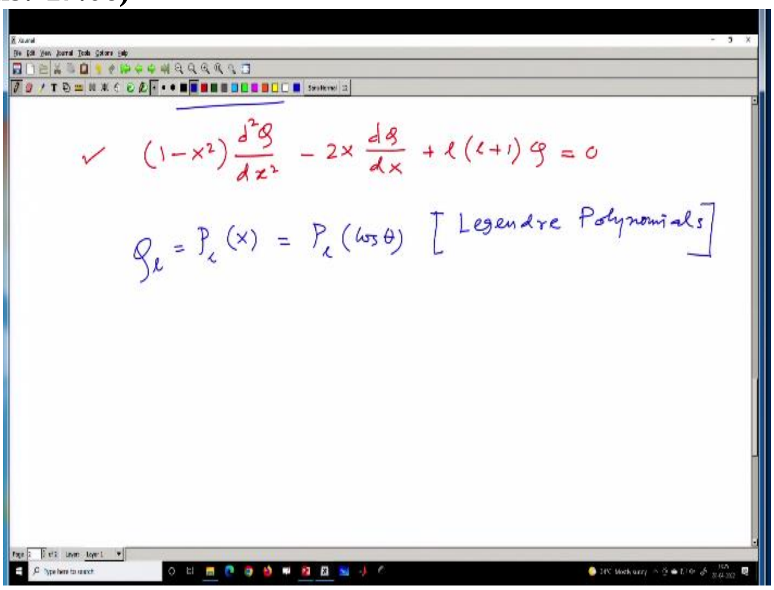

So, the general solution is them known to us and it is Q_l is equal to the Legendre polynomial of x or I should say P_l of $\cos\theta$ this is the Legendre polynomial.

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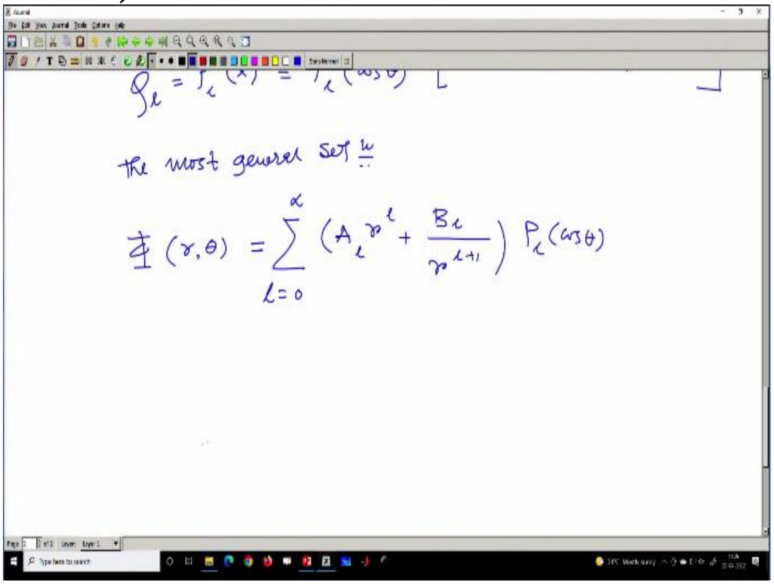

So, the most general solution then becomes simply ϕ, which is a function of r and θ because, both the solutions is now in our hand. So, it should be summation over l tends to 0 to infinity A_l r^l the solution we already figured out $\frac{B_l}{r^{l+1}}$ we check it multiplied by the Legendre polynomial that is the solution for Q so, P_l cos θ. So, this is the way the potential is going to vary for 2 dimensional case when the system is having you know the spherical symmetry.

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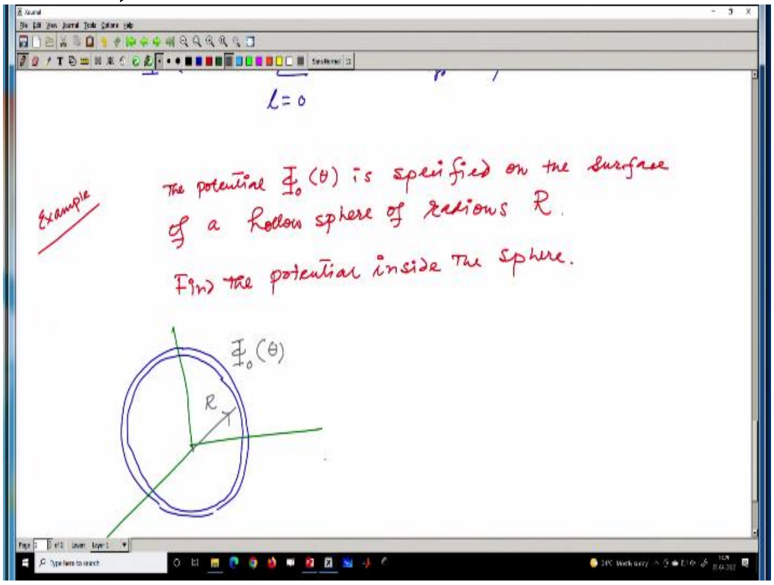

So, let us do 1 example say so, example and the potential so, the example is the potential ϕ_0, which is a function of θ is specified on the surface of a hollow sphere of radius R. So, over a hollow sphere of radius R the potential is defined. Find the potential inside the sphere. So, that is my sphere say this is a hollow sphere. So, I can say this is like a shell in some coordinate system and this is with radius R and the potential here is defined as ϕ_0 and that is a function of θ.

So, when you move the θ, there is a φ symmetry, but if you move that should be a function of θ. Now, the question is given at the you know on the surface what should be the potential inside that means in this region when r is less than R than what should be the potential. Now, let us write the general solution because the general potential for this system is already derived. And that is this.

This is the general form of the potential for any spherically symmetric system. And this solution I will be going to use directly and not going to I mean that is why we did it so, the solution if I directly use here for this problem.

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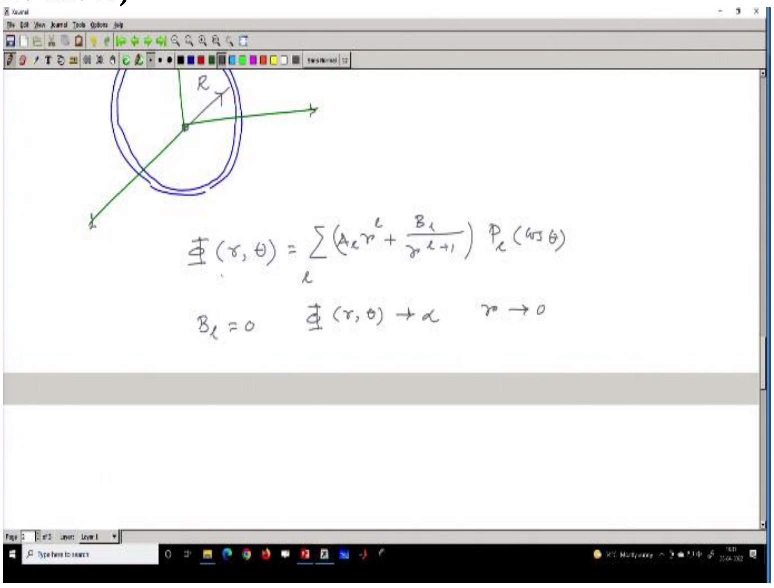

So, the general solution ϕ let me write it ϕ function of r and θ that is summation over l A_l r^l + $\frac{B_l}{r^{l+1}}$ multiplied by the Legendre polynomials this. Now, you can see that when r tends to 0 inside, then what happens these terminal will be going to blow up. So, that means, that is non physical to do you know when the potential is blowing up at r tends to 0.

So, that eventually tells us that B_l = 0 why if it is not 0, then at this quantity tends to infinity for r tends to 0, which is not acceptable. So, that is why from the very beginning I can say that B_l has to be 0.

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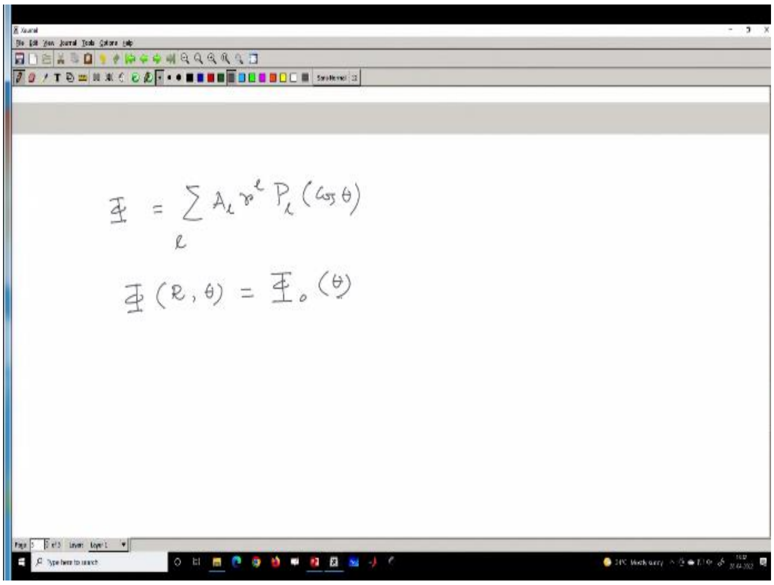

So, then what should be the form of the potential? The form of the potential now, simply ϕ is equal to sum over l A_l r^l and the Legendre polynomial P_l cos θ this is the solution also another boundary condition that is given and this boundary condition is saying that over the surface that when R = R and θ is there, that value is θ ϕ_0, which is a function of θ something like that.

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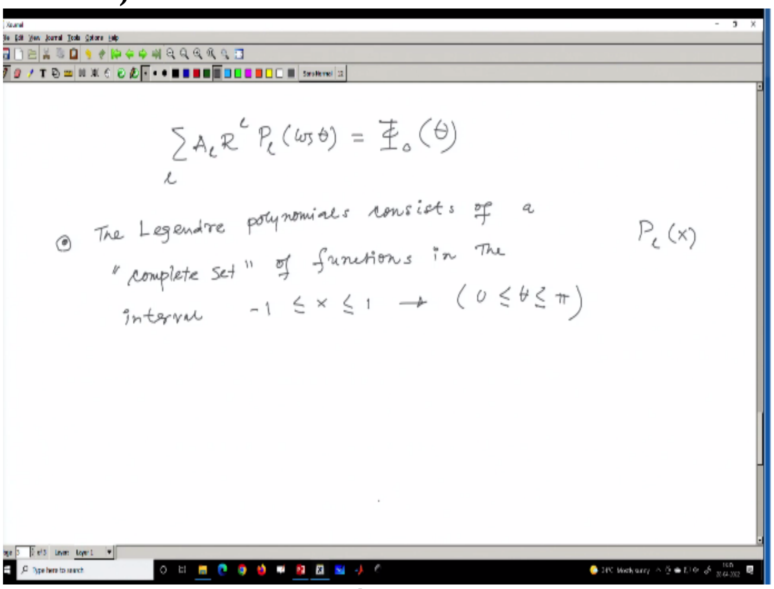

So, I can write here that the summation l A_l R^l I just need to put R because it is at R = R P_l cos θ is simply ϕ_0 function of θ that is the condition. Now we need to find out what is the value of A_l and if you remember the previous day we had a problem were we use the Fourier trick, a similar kind of trick we will be going to use, because, we have again a very nice relationship with all this Legendre polynomials because they are forming the complete set in function space.

So, in function space they are forming the complete set. So, that is why they are having a relationship so, that we will be going to exploit. So, the Legendre polynomials now consists of have a complete set of function in the interval -1 x 1 that basically because this is for x that is I am talking about P_l (x) now, if x is cos θ here so that gives me that it should be 0 θ and π because when it is cos θ is -1 then so when 0 then it is 1 and when π this is -1 so it goes like this and this limit is like this.

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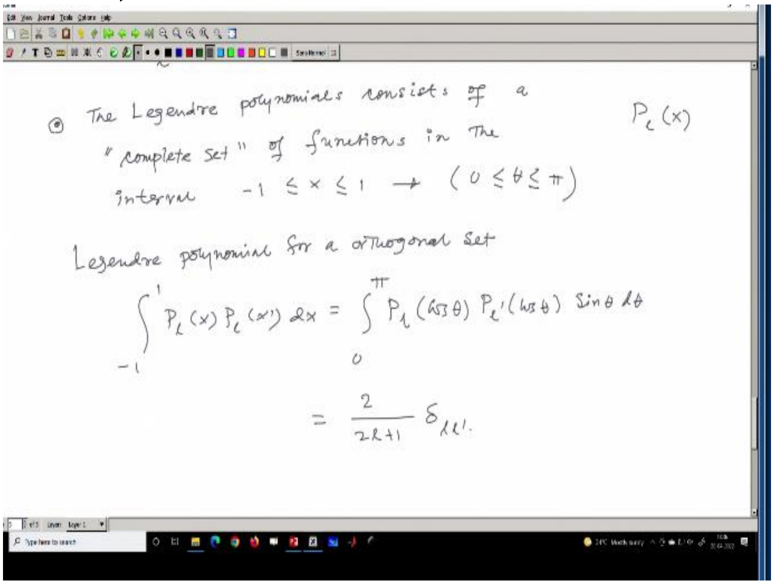

So, I should have for orthogonal set so for Legendre polynomial for orthogonal set so, if I integrate -1 to 1, 1 is Legendre polynomial with x and another with x' dx, which in terms of cos θ it is P_l (cos θ) and $P_{l'}$ (cos θ) sin θ dθ because x is cos θ, so, dx will be sin θ dθ then that value is $\frac{2}{2l+1}$ again it should be $\delta_{ll'}$ like in the previous problem this is the case.

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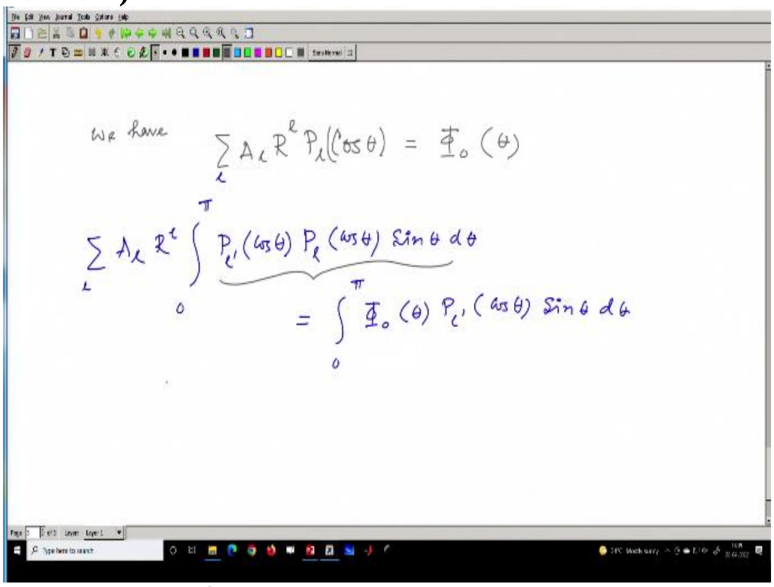

So, we have some over A_l then R^l P_l (cos θ) = ϕ_0 the function of θ so, now we are going to use this trick because we know that it is related to delta function. So, we will go to use this trick and it will be like this. So, sum over A_l, which is already there, this is over l, and then R^l, I can put it outside because it has nothing to do with cos θ and then I am going to integrate 0 to say π.

And then $P_{l'}$ and then cos θ then I multiplied this P_l (cos θ) to P_l (cos θ), which is already there and then sin θ will be there sin θ dθ and in the right-hand side, whatever the function we are having, because I do not know function of θ, which is given which has to be supplied to find explicit form it should be $P_{l'}$ (cos θ) and then sin θ dθ. So, this quantity is forming a complete set, which we discuss this quantity is forming a complete set so, I just mentioned here.

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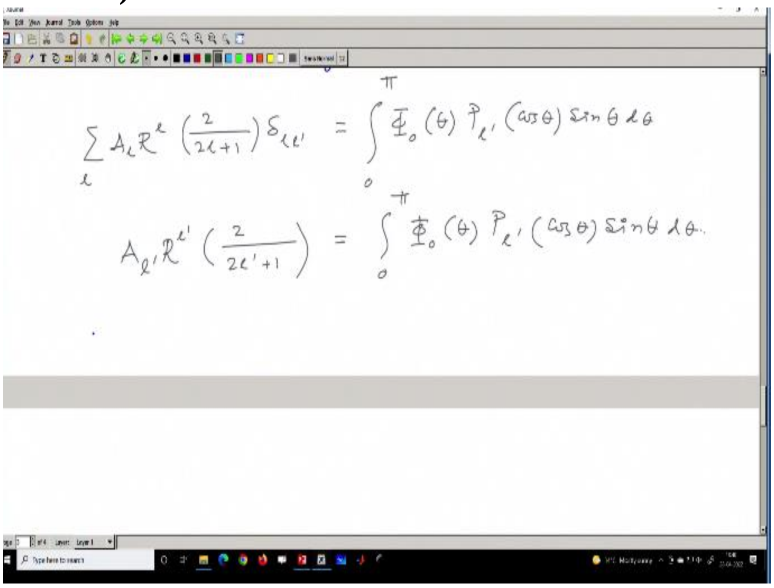

So, that thing I am going to use, so, it will be l A_l R^l and this part $\frac{2}{2l+1}$ and then delta function l l' that is again equal to integration of 0 to π and whatever is there, I just simply write ϕ_0 (θ) and it is $P_{l'}$ (cos θ) sin θ dθ. So this is related to the delta function. So, what I do that I just after integration, we can remove this delta function and I just write $A_{l'}$ $R^{l'}$ when l = l' then that is meaningful.

And I have $\frac{2}{2l'+1}$ and that side, I have simply 0 to π because we cannot do anything right now, because ϕ_0 (θ) is not given this is a functional form. So, it will remain like this.

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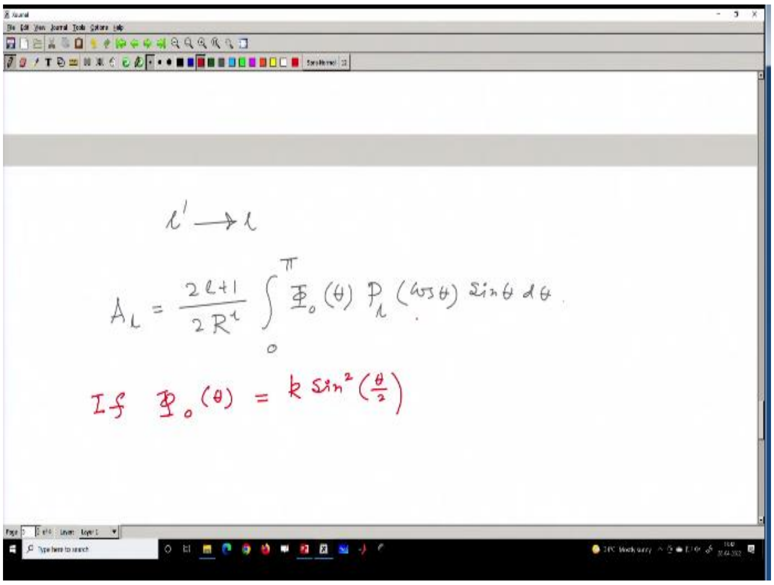

So, I can simply have so now I can replace because this is the dummy index so, I can replace I am just replacing l' to l. So, I can have A_l as $\frac{2l+1}{2R^l}$ integration of 0 to π and then ϕ_0 functional form, which is still unknown P_l (cos θ) and then sin θ dθ. Now, as I mentioned that this is not given, but now I am saying if ϕ_0 is given, then I can execute the value of A_0 suppose ϕ_0 (θ) is given and this value is say k $\sin^2$ $(\frac{\theta}{2})$.

So, that is the value of the potential you know over these surface and it is a function of θ. So, if you change the value of the θ the potential will change. So, from here you can see that when θ is 0 the potential is 0, but when θ is say π, the potential is k and so on. So, there is a variation of the potential of the surface then what is my ϕ?

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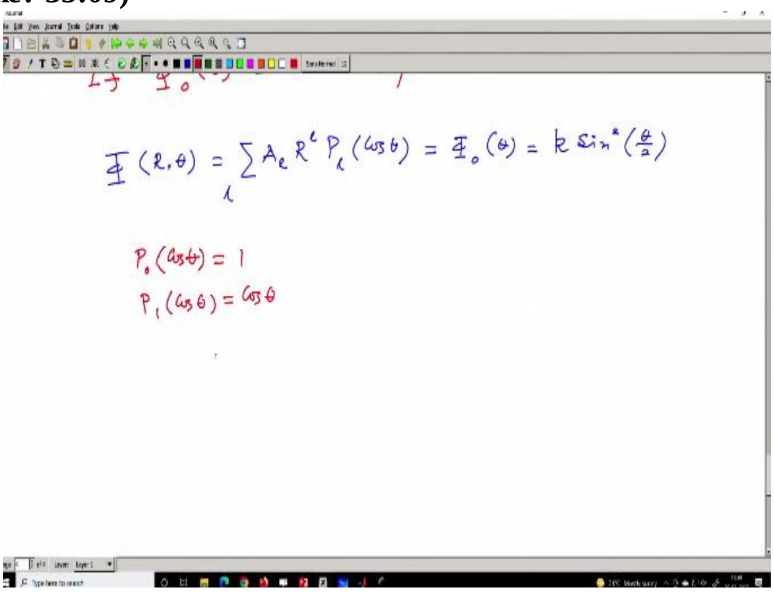

The general solution is there so, $\phi(R, \theta)$ that is l, which goes to 0 to infinity then A_l R^l P_l (cos θ) that value is now, we know that this was before θ 0 θ but now I put the explicit form. So, this is k $\sin^2$ $(\frac{\theta}{2})$ now, it is very interesting treatment because this is a function of θ at right-hand side and left-hand side also I have a polynomial. So, I know the explicit form of this polynomial, so if I have this polynomial in such a way that this $\sin^2$ θ is there.

So, the coefficient of that quantity should match from the both the side and I can explain we can find out the value of A_l let us do that. So, we know that P_0 (cos θ) = 1, P_1 (cos θ) = cos θ and so on. So, let us expand up to these 2 term because after that we have $\cos^2$ θ and so on.

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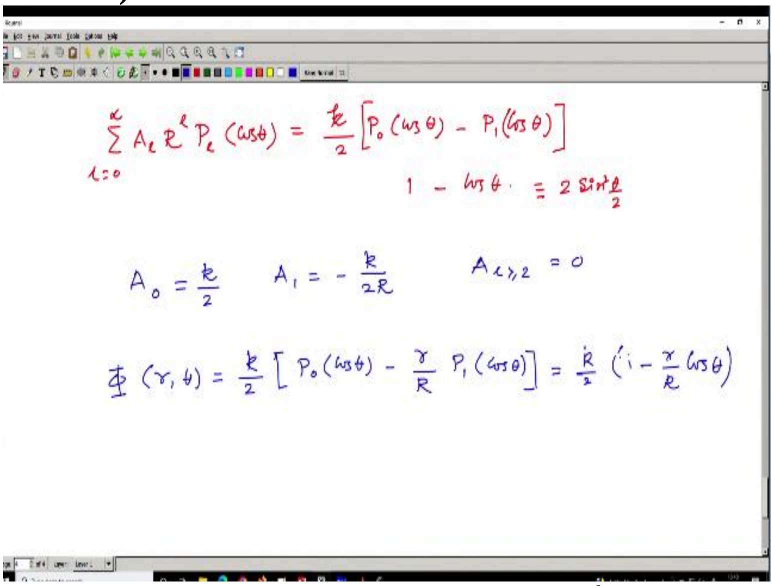

So, let us expand up to this so, summation l = 0 to infinity, A_l R^l P_l (cos θ) that is when I have l 0 so that term I write in the right-hand side it is you know $\sin^2$ θ, so that I write in form of this you know this Legendre polynomials, so this I write like $\frac{k}{2}$ then P_0 (cos θ), which is 1 and then $-P_1$ (cos θ) so, if I bracket it. So, this is 1 and this value is cos θ.

So, that is (1 - cos θ) and so this value is equal to 2 $\sin^2\frac{\theta}{2}$ $\sin^2\frac{\theta}{2}$ + $\cos^2\frac{\theta}{2}$ - $\cos^2\frac{\theta}{2}$+ $\sin^2\frac{\theta}{2}$ + $\sin^2\frac{\theta}{2}$ so, it should be 2 $\sin^2\frac{\theta}{2}$. So, that quantity is this one so that is why half is here. So, the point is I can write this right-hand side in the form of these Legendre polynomials. And here I must say that any function because these Legendre polynomials are forming a complete set in function space.

So, any function can be expanded in terms of these Legendre polynomials. So, here we have $\sin^2 \frac{\theta}{2}$ so, I can suitably adjust my Legendre polynomial only I can use this first two Legendre polynomial and I find that I can regenerate this function. So, now, if I tally from left-hand side and right-hand side then I simply have A_0 is $\frac{k}{2}$ and A_1 is $-\frac{k}{2R}$ because A_1 R = $-\frac{k}{2}$. So, A_1 has to be $-\frac{k}{2R}$ and all the l greater equal to 2 has to be 0.

So, my potential what I get (r, θ) is simply $\frac{k}{2}$ then [P_0 (cos θ) - $\frac{r}{R}$ P_1 (cos θ)] = $\frac{k}{2}$ simply (1 - $\frac{r}{R}$ cos θ) and that is my result here when all these parameters are given I can find it that this is my result. Now, A_l can also be figured out directly because here in this formula I already write that A_l can be written and after doing this tally I just find out what is my A_1 and what is my A_2, A 0, 1 and then 2 etc. So, these I just expand the right-hand side in the term of Legendre polynomial and then fine, but still we can find A_l directly.

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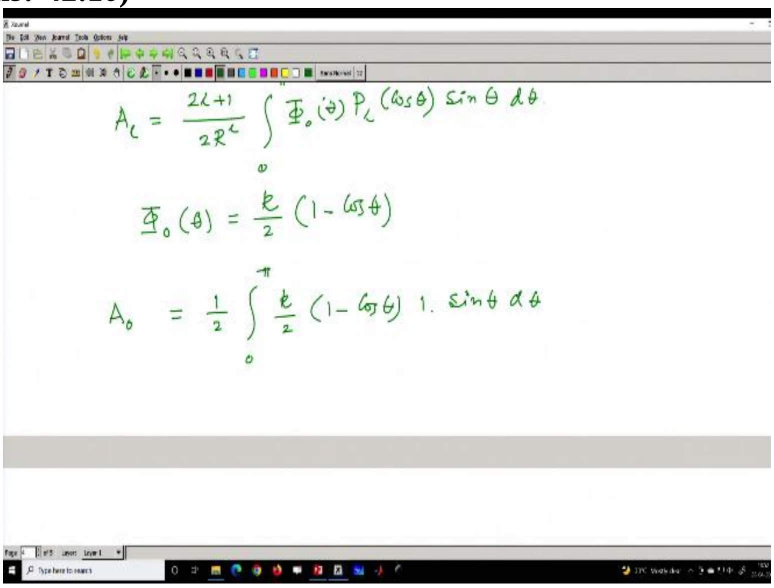

So, if I do that finding A_l directly so, what I have here is A_l = $\frac{2l+1}{2R^l}$ integration 0 to π and then we have ϕ_0 (θ) and then P_l (cos θ) and then sin θ dθ this is what we had here. So, now ϕ_0 (θ) in explicit form is given it is $\frac{k}{2}$ (1 - cos θ). So it is k multiplied by $\sin^2$ θ so, that I write in this form. So, my A_0 is what then so my A_0 when I put l = 0 then it should be half integration 0 to π and then $\frac{k}{2}$ (1 - cos θ) and then this is l, P_0 is 1, so 1 and then sin θ dθ.

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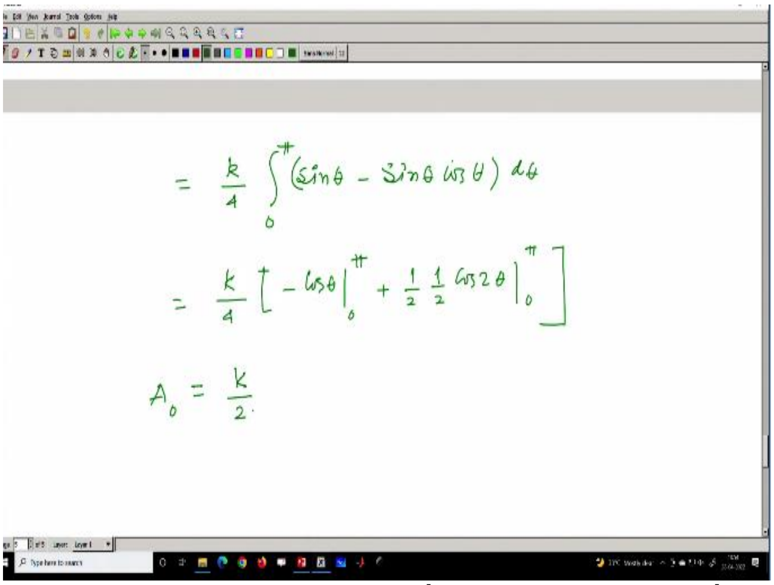

So, these integration if I execute so let us put this $\frac{k}{2}$ outside so, we have $\frac{k}{2}$ then integration 0 to π, sin θ and then -sin θ and cos θ that over dθ this is a very straightforward integration and I can have $\frac{k}{4}$ and then this integration is -cos θ that we need to execute at 0 to π and then $-\frac{1}{2}$ sin so another $\frac{1}{2}$ will be there it will be sin 2θ and then it will be cos 2θ. So, it will be cos 2θ with the negative sign, so it should be plus and 0 to execute at this point.

So, this value when we execute 0 to these things and it simply gives us A_0 as $\frac{k}{2}$ and this is -cos θ, so when you put π, it should be -1 and then -1 so, it should be - 2 + and then when put 0 then 1 and then 2 π 0 is $\frac{1}{4}$. So, it is simply 2 so, I should have it is $\frac{k}{2}$. Now is A_0 if you look here already we figure out with the expansion that it is A_0 is $\frac{k}{2}$.

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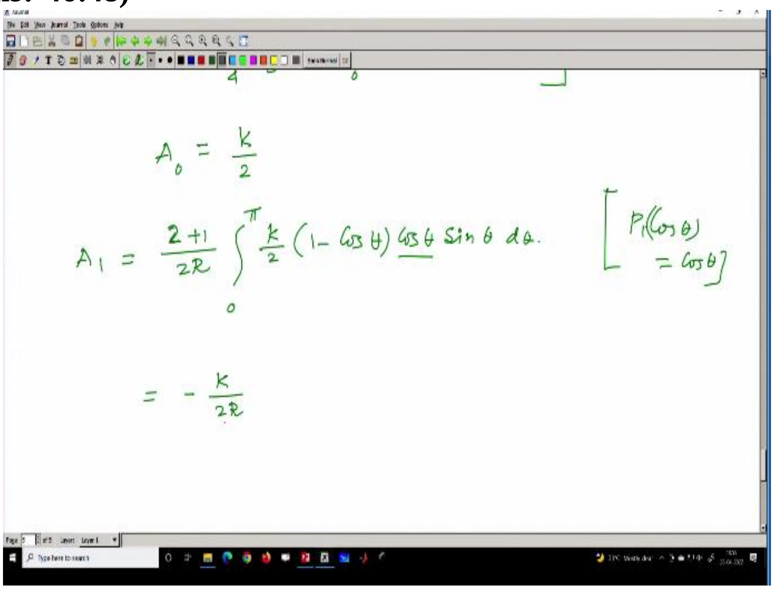

In the similar way, if you calculate A_1, I am not going to do the full calculation, you just it should be $\frac{2+1}{2R}$ and the integration is 0 to π and it is $\frac{k}{2}$ (1 - cos θ) and then cos θ sin θ dθ because we have P_1 (cos θ) = cos θ that we put here. Now, if you execute this integral, then you will get the same value like $-\frac{k}{2R}$, which we derive here this. So, I am not going to do the entire integration I have already shown that from this equation, whatever the equation you get, you can directly find out A_l.

So, there are 2 way we calculate and show that how to calculate this constants A_0 I calculate and A_1 again you can calculate by executing this integral this is not a very big deal to do this integration. So, if you do this integration you will find the result like $\frac{k}{2R}$. Today I would like to conclude here because my time is limited. So, in the next class I will try to do few more problems regarding the boundary value issue and another technique like image method that is important, we will not be going to do very detailed.

But few problems I like to show that how works and these things. And then we will be going to start maybe after next day's class we will start our third module where we will start the magnetostatic. So, thank you for your attention and see you in the next class.