Foundation of Classical Electrodynamics
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Lecture - 40
Boundary Condition

Hello students to the foundation of classical electrodynamics course, under module 2, today we have lecture 40. So, in today's lecture, we will go to discuss the boundary condition or the matching condition of the electric field when it goes from one medium to another medium with separated by a boundary.

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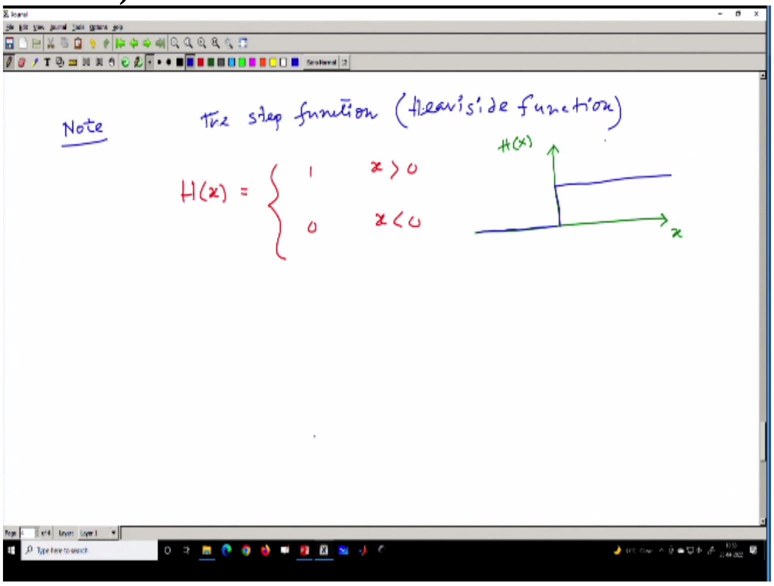

So, we have today class number 40 and our topic today is matching condition or boundary condition so, before finding the matching condition I should like to make a small note here because we will go to use these things and the note is this, the step function or in other words we call it the Heaviside function. So, what this function suggests? So, let us define this function as H function H for Heaviside as a function of x and this function is defined in this way it should be equal to 1 when x is positive and 0 when x is negative.

So, if I plot this function here this is my axis this is x positive and along this direction I am plotting this Heaviside function H(x) the plot will be like this. So, it is 0 here there is a jump here and it is 1 after that this is 1, this is 0. So, this is the form of the Heaviside function or the step function.

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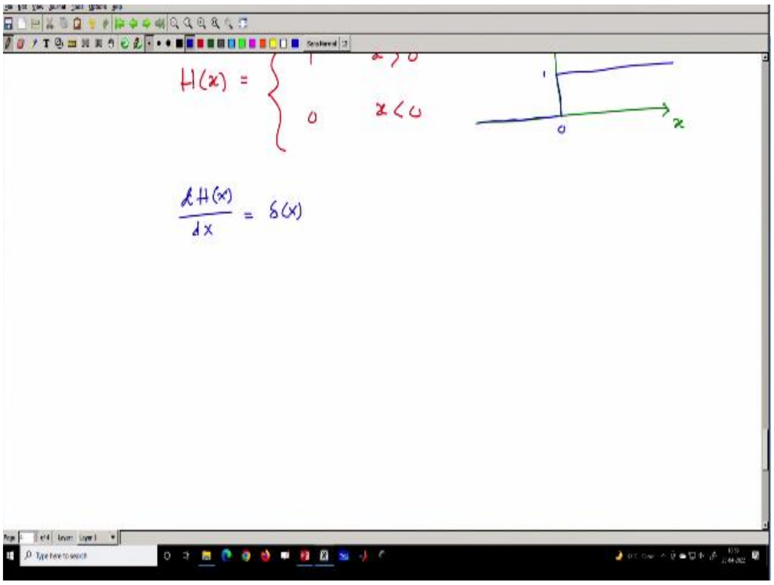

Another important thing that one should note that if I make a derivative of the Heaviside function with respect to say x then I should get a delta function out of that. The derivative of the Heaviside function gives delta function.

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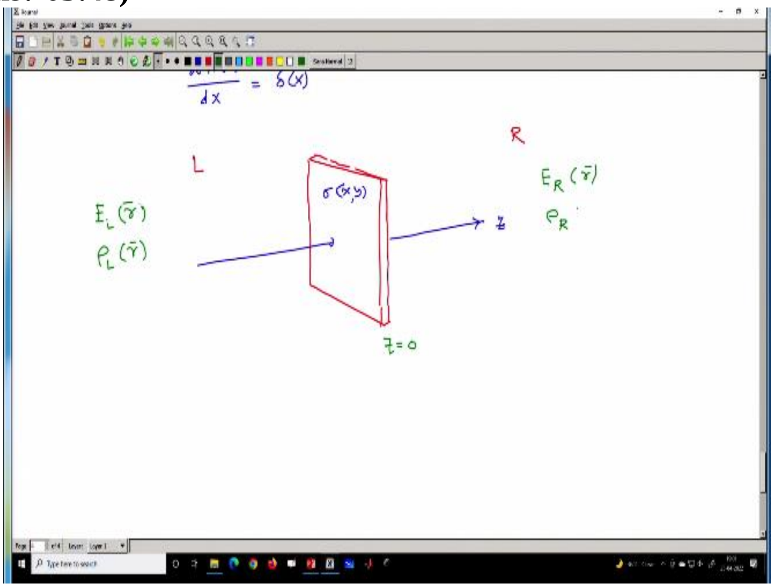

Now, I consider a boundary like this, this is suppose a boundary. So, I am having a boundary like this where some charge distributions are there suppose, it is distributed over the entire surface x, y surface and electric field is going from this direction to this direction like this, this is my z direction. So, this is the left-hand side I write it as L and this is my right-hand side I write it R. For left-hand side the electric fields are represented like E_L as a function of $\vec{r}$ and the charge density I write ρ_L as a function of $\vec{r}$.

This is electric field and this is charge density, this is z = 0 point. In the right-hand side similarly, I have E_R as a function of $\vec{r}$ and also ρ_R as a function of $\vec{r}$ these are electric fields and my charge density.

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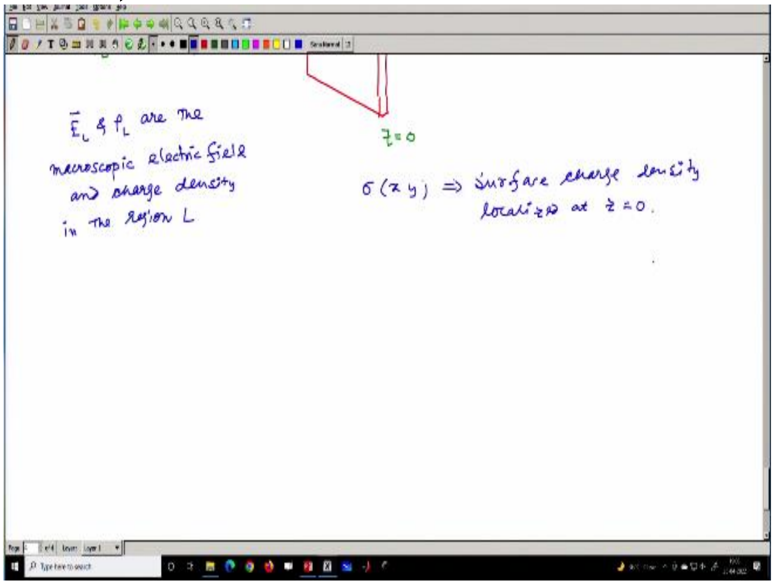

So, let me write clearly what I try to mean here. So, here E_L and ρ_L are the macroscopic electric field and charge density in the region L that means the left side and similarly in the right side we have and ρ I already mentioned that ρ is a surface charge density. So, ρ(x, y) is a surface charge density localized at z = 0 point this is surface charge density localized at z = 0 point now, let us try to understand the electric field at every point in the space how one can write it?

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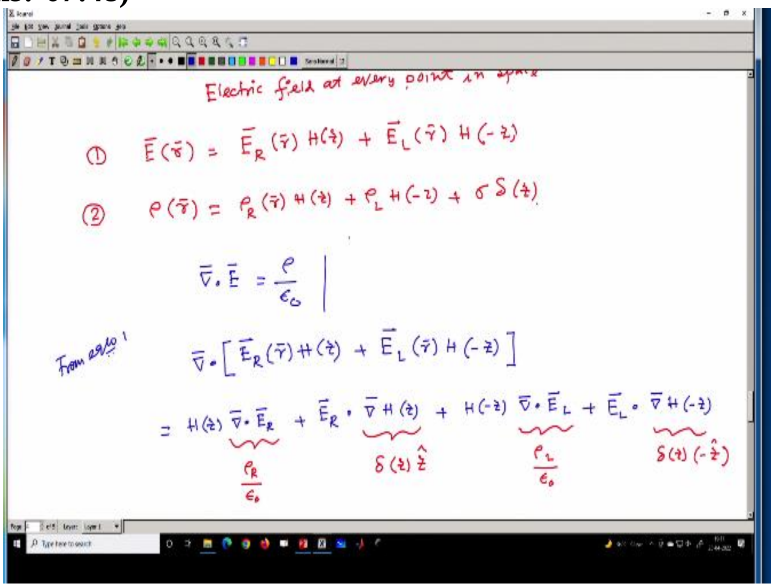

So, the electric field at every point in space; how can I write it? $\vec{E}(\vec{r})$ is the total electric field, total electric field consists of the electric field on the left-hand side electric field on the right-hand side. So, I simply write in terms of you know Heaviside function, I can write it electric field at right-hand side. And then this is a function of $\vec{r}$ multiplied by the Heaviside function, which is the function of z here.

So, there will be no $\vec{E}_R$ in the left-hand side because I am multiplying with Heaviside function plus $\vec{E}_L$. That is the left-hand side function and multiplied by the Heaviside function with -z when I write -z that means the Heaviside function will flip, will flip means we will have opposite so, now I have negative sign 1 and then positive side 0. So, just put x = - x here I am doing the same thing and what is the charge density?

This is the total electric field left-hand side field right-hand side field this is total electric field. What about the total charge density? The total charge density in the similar way $\rho(\vec{r})$ is equal to what is the left-hand side ρ, what is the right-hand side ρ? This is $\rho(\vec{r})$ with the Heaviside function like before what is the left-hand side with the Heaviside function with -z and whatever we have over the surface charge density and that is at z = 0. So, I should use my delta function here so δ(0).

So, that is the total charge density we are having left-hand side this charge density right-hand side this charge density and z = 0, we have a charge distribution here. So, that charge distribution I write in terms of delta function. Now, I have this equation in my hand my fundamental equation that $\vec{\nabla} \bullet \vec{E}$ is $\frac{\rho}{\epsilon_0}$ that is the equation I always I can explore this equation according to my convenience so, here we are having this.

So, now total $\vec{E}$ I know, so, I will just simply write it. So, from equation 1 what I write that total $\vec{E}$ I already evaluated this is $\vec{E}$ at right-hand side vector sin function of $\vec{r}$ with Heaviside function at z plus $\vec{E}_L$ the left-hand side with Heaviside function –z and that is all. But here, you can see that this operator we are going to operate both because this is a Heaviside function is a function so, this quantity is this.

So, first I write a Heaviside function at z and then this is going to operate over $\vec{E}_R$ then I have $\vec{E}_R$ dot then I have the gradient of the Heaviside function with z plus this is straightforward vector rule I am using plus Heaviside function and then $\vec{\nabla} \bullet \vec{E}$ left and plus $\vec{E}$ left • gradient of Heaviside function with –z. Now, this quantity I already mentioned that the derivative of the

Heaviside function you know give us the delta function. Here Heaviside function is a function of z only.

So, when we make this quantity divergence. So, it eventually gives us the derivative and that gives us the delta function at z with the unit vector this. This quantity is also in the similar way give us delta function z but in the opposite direction -z unit vector because 1 negative sign I have. So, with that also I can have $\vec{E} \cdot \vec{r}$ that quantity this one I can also have $\frac{\rho_R}{\epsilon_0}$ this quantity I can write $\frac{\rho_L}{\epsilon_0}$. So, all the 4 quantity I can write in this way.

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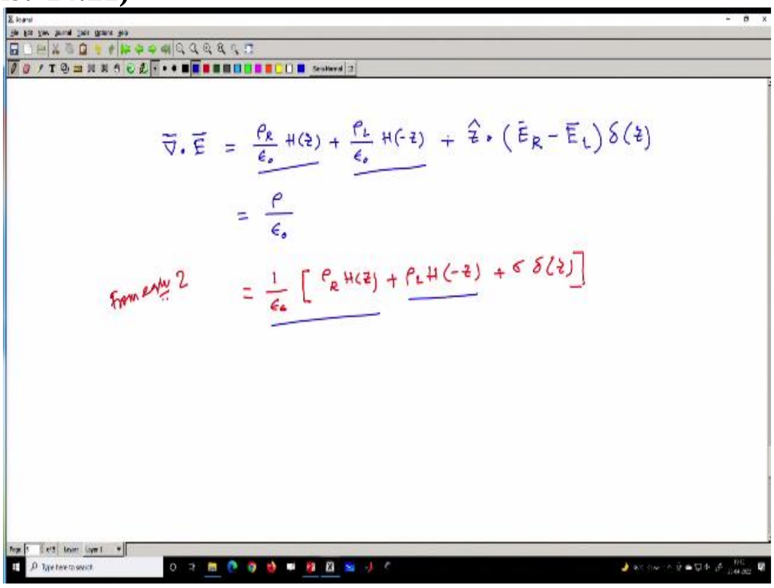

So, finally, what we get? We get $\vec{\nabla} \cdot \vec{E}$ that is my left-hand side is equal to that we are getting $\frac{\rho_R}{\epsilon_0}$ multiplied by the Heaviside function that is plus $\frac{\rho_L}{\epsilon_0}$ multiplied by Heaviside function with negative z these 2 plus what I get? I get these 2 term $\vec{E}_R$ $\vec{E}_L$ and dot these things. So, I simply have $\hat{z} \cdot (\vec{E}_R - \vec{E}_L)$ with the delta function. Now $\vec{E} \cdot \vec{E}$ is right-hand side I still have $\frac{\rho}{\epsilon_0}$ so, this quantity is simply $\frac{\rho}{\epsilon_0}$.

Now, I have also another equation if I write equation I am just exploiting equation 1, but equation 2 is suggesting that my ρ is ρ_R H(z) ρ_L and σ. So, I can write it here is equal to $\frac{1}{\epsilon_0}$ from equation 2 this red part I can write and ρ I write ρ_R with Heaviside function z plus ρ_L with Heaviside function -z plus σ delta function of z that use. So, left-hand side and right-hand side you can see that ρ this quantity will cancel out. So, this quantity will cancel out here this quantity will cancel out here.

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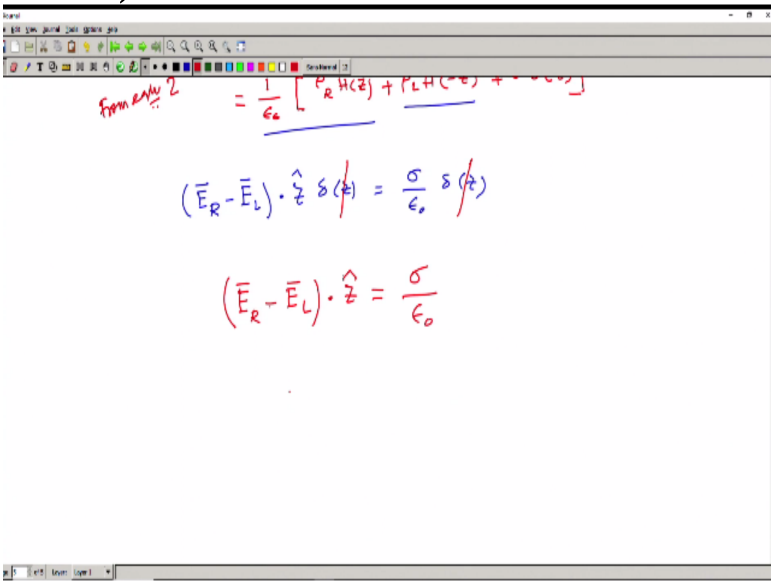

And eventually what I get is this one $(\vec{E}_R - \vec{E}_L) \bullet \hat{z}\ \delta(z) = \frac{\sigma}{\epsilon_0}$ and δ(z), δ(z) again going to cancel out both the site and I have my first matching condition and that is $(\vec{E}_R - \vec{E}_L) \bullet \hat{z} = \frac{\sigma}{\epsilon_0}$. So, these components are the parallel components.

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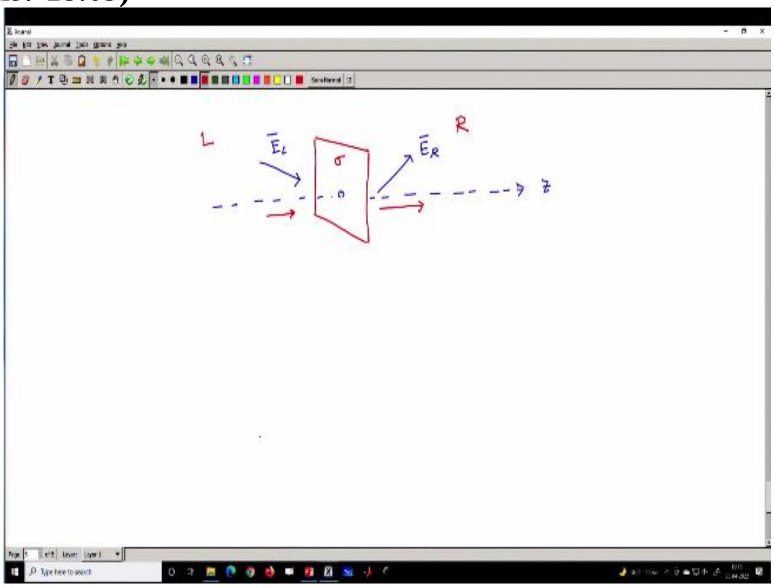

So, if I look carefully that I have a boundary here and whatever the $\vec{E}_R$ I am having suppose, this is my left side and this is my right side so, here suppose I am the right side this is my electric field $\vec{E}_R$ so, z is along this direction this is my z direction. So, and also $\vec{E}_L$ is say along this direction is my $\vec{E}_L$. So, here what I am doing is that $\vec{E}_R \bullet \hat{z}$ means I am taking the z component here along this direction and I am also taking the z component here and this z component I just subtract.

And if I subtract and find that value is equivalent to whatever the σ I am having now, if σ is 0, we can say that this z component of E and L are same. So, that means we have a matching condition. So, that is the z component of E remain conserved in these 2 sections L and R. In a similar way, we can find another because I have another equation I just so far I exploit the divergence I exploit this equation. But another equation is still in my hand and that is the $\vec{\nabla} \times \vec{E}$.

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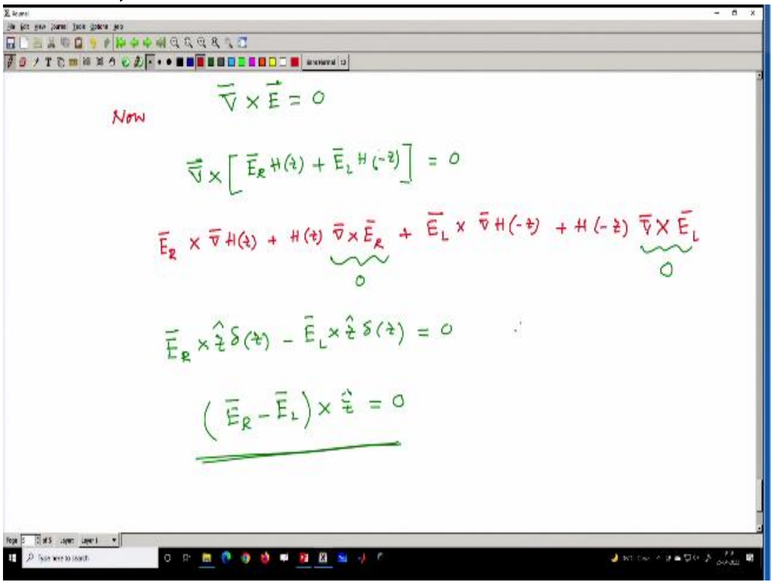

So, now we have $\vec{\nabla} \times \vec{E}$ this is also a fundamental equation is equal to 0 if that is the case then I can have $\vec{\nabla} \times \vec{E}$ I know from equation 1 I can write it and that is $\vec{E}_R$ H(z) and then plus $\vec{E}_L$ this that quantity is 0. Now again I can use this curl expression. So, here one important thing is in the right hand side is 0. So, I do not need to use the equation that 1 equation is sufficient the first equation this equation 1 is sufficient.

So, I simply write here that this is nothing but $\vec{E}_R \times \vec{\nabla} H(z)$ + H(z) $\vec{\nabla} \times \vec{E}_R + \vec{E}_L \times \vec{\nabla} H$ (-z) gradient of these things and then H of this things and it should be H(-z) and then curl of that thing and then curl of $\vec{E}_L$. So, this, quantity $\vec{\nabla} \times \vec{E}_R$ that will be 0 and that quantity is also 0 since these are 0. So, what I get is left with this quantity $\vec{E}_R$ and cross $\hat{z}$ δ(z) - $\vec{E}_L \times \hat{z}$ δ(z) that is equal to 0.

So, delta function I can cancel out so, simply what I get is this, so what is another? So, $(\vec{E}_R - \vec{E}_L) \times \hat{z} = 0$. So, that is the parallel component if I look carefully so, this is my second matching condition the first matching condition is saying that E dot so this is my first matching condition is saying that $(\vec{E}_R - \vec{E}_L) \cdot \hat{z}$ is $\frac{\sigma}{\epsilon_0}$ and my second condition is saying that $(\vec{E}_R - \vec{E}_L) \times \hat{z} = 0$ when you make a cross that means, you are now talking the parallel component.

Because when you make a crossing, so, then it should be you know the cross will be perpendicular to the E along z both that means it should be an x, y plane. So, that is nothing but you know this parallel component and this parallel component is always conserved.

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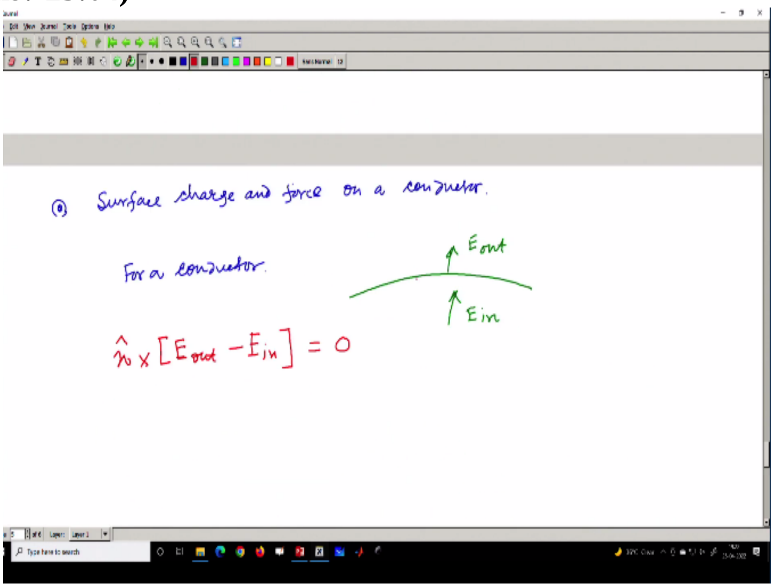

So, now after having this information, we can do few more things. And one of the things is to find out the surface charge and the force on a conductor. So the next thing we do is surface charge and force on a conductor what is the meaning? So, we already for a conductor so, what we have let me write so, this is my E_{out} say and this is my E_{in} so, I already have this equation $\hat{n} \times (E_{out} - E_{in}) = 0$ that is this equation simply considering that is the z direction. So, now I am changing say this is my n direction and this is the boundary.

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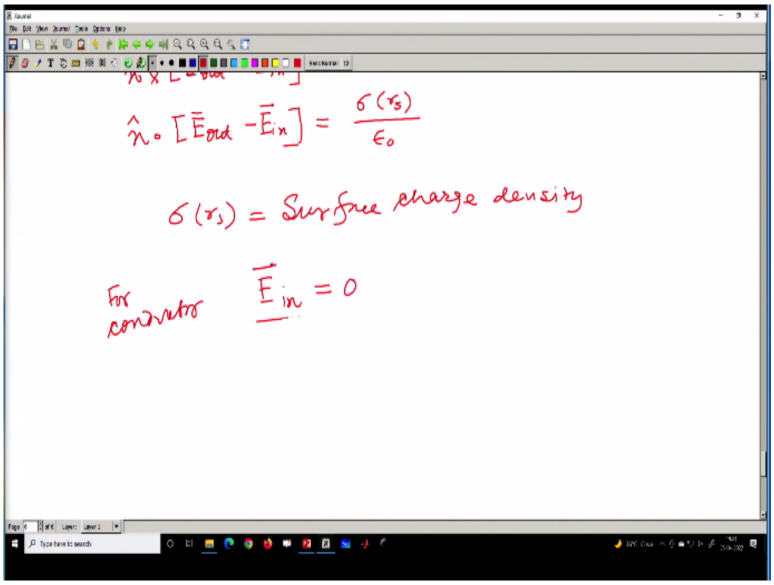

And another equation is $\hat{n} \bullet (\vec{E}_{out} - \vec{E}_{in})$ is the I should put the vector sign here is σ over the surface for conductor over the surface I am having r_s that is over the surface this is r_s and this is ϵ_0 and $\sigma(r_s)$ as I mentioned $\sigma(r_s)$ is the surface charge density and $\hat{n}$ is the unit vector to the conductor of this $\hat{n}$ is a unit vector along this direction perpendicular to the surface. So, now, for conductor what happened that $\vec{E}_{in}$ that is the inside the conductor there should not be any field so, $\vec{E}_{in}$ should be 0.

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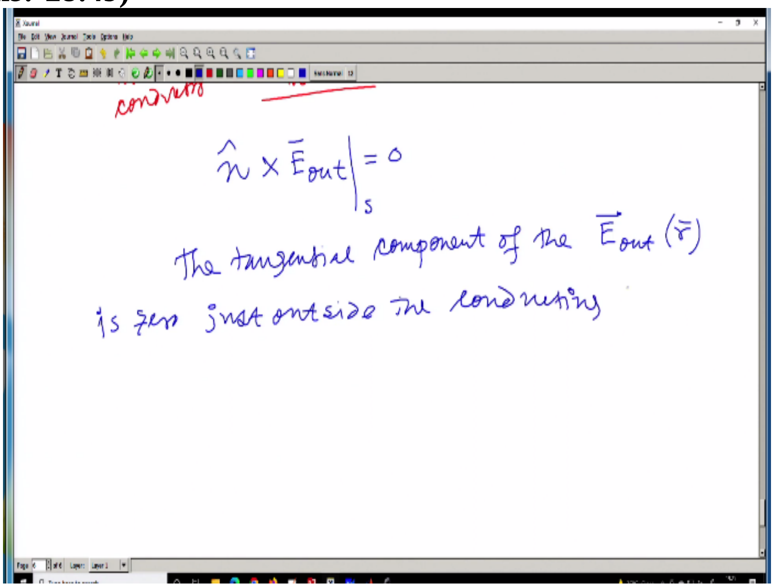

So, I am left with if $\vec{E}_{in}$ is 0 I am left with this equation $\hat{n} \times \vec{E}_{out}$ at over the surface is 0 so, the tangential component to the $\vec{E}_{out}$ is 0 just outside the conducting surface. So, that means the tangential component of the field $\vec{E}_{out}$ r is 0 just outside the conducting surface.

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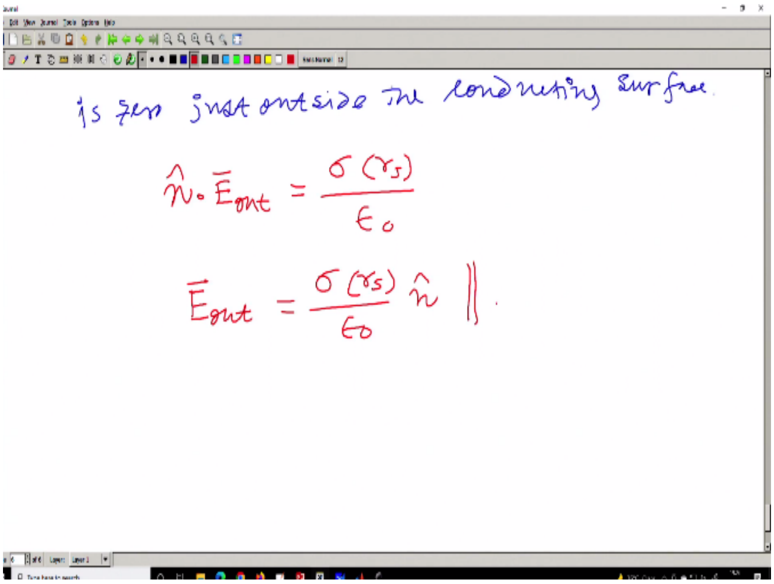

So, but another equation still I am having that is $\hat{n} \cdot \vec{E}_{out}$ is how much it is σ, which is at r_s divided by ϵ_0 as $\vec{E}_{in}$ 0 because $\vec{E}_{in}$ is 0 here. So, whatever $\vec{E}_{in}$ that has to be 0 for conductor, if this is a conductor then this is inside the conductor we should have 0 field. So, from here we have a well-known equation that just out what we are having? We are having the electric field as $\frac{\sigma(r_s)}{\epsilon_0}\hat{n}$. So, that expression we already figured out in the last class, again I am getting using exploiting these boundary condition or boundary matching condition.

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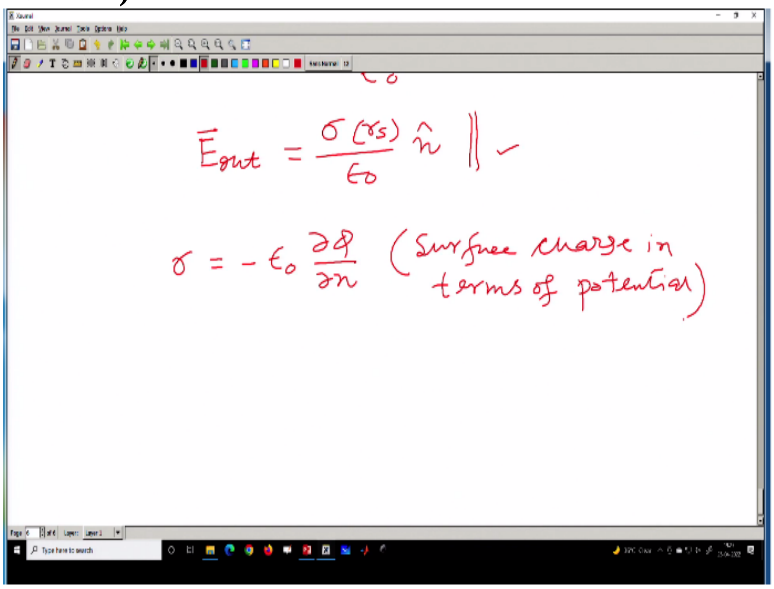

So, this gives us σ equal to you know, we can calculate it with the potential because σ here and that is by simply calculating this quantity. So, this is the surface charge in terms of potential so, the point is using the boundary condition that we discussed this boundary condition is important in many places we will be going to use this boundary condition where the tangential and perpendicular component of the electric field how they are related is given.

So, using that we finally figured out that what is the field exactly the outside of the conductor and that value is already derived using the Gaussian surface last class I think we calculated that and we are getting the same value. So, we are almost there because today I do not have much time to discuss in the next class we will continue that and try to understand that what should be the force on a conductor.

So, I started here if you look that the surface charge and the force on the conductor, but this force thing I will do in the next class and also try to do some problems. Suppose we have 2 conductor placed hemisphere 2 conductor and how much force they exert on each other. That calculation we will try to understand quickly in the next class. So thank you very much for your attention and see you in the next class.