Foundations of Classical Electrodynamics
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Lecture - 04
Coordinate System, Orthogonal Transformation

So, hello students for the course of foundation of classical electrodynamics. So, we are having class 4 today, lecture number 4. And in these particular module mathematical preliminaries, we started the coordinate system last day. Today, we will be going to learn more about the coordinate system and then try to understand what is the meaning of orthogonal transformation.

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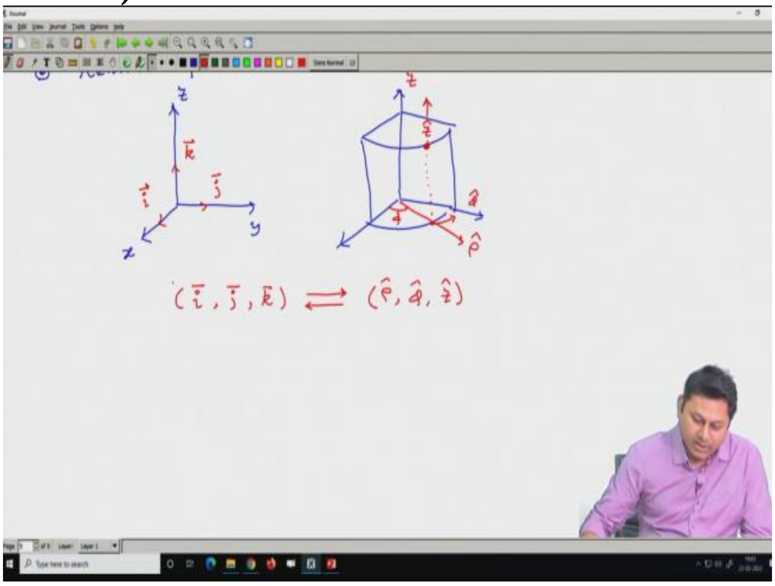

So, today we have class number 4. So, in the previous class if you remember we started the coordinate system and I compared 2 coordinate systems one is Cartesian coordinate system and another is the cylindrical coordinate system and try to understand that how the x, y, z and ρ, φ, z how they are related. So, today we will try to find out the relationship between the unit vector last time I mentioned that.

So, unit vectors because in the Cartesian coordinate system we have, say this is my Cartesian coordinate system and in the same basis if I draw so, I am having a cylindrical coordinate system like this. So, this is x, y, z and this system is basically ρ, φ, z, where this is ρ, this angle is φ and if I move say this I am talking about this point so, this is along these directions z.

So, this is z as usual, but ρ and φ is slightly different in slightly different way. Now, the unit vector along this direction is the ρ unit vector. The φ is this direction, φ unit vector along the tangential direction and along this direction I have z unit vector along this direction. So, the direction of the ρ, φ, z is defined here. In a similar way, here we have in the Cartesian coordinate system, I have unit vector here, I have unit vector along this direction j and I have the unit vector this direction, which is k. That is also well-known and very easy to understand. Now, we will try to you know understand the relationship between the i, j what is the k and how it is related to the unit vector ρ, φ and z. Suppose, I want to define the i unit vector with the unit vector of ρ, φ, z, how do I write this?

So, you can see from the figure that now, if I draw another figure, so, just make a projection of this portion and if I do the projection of this figure, it will look like this.

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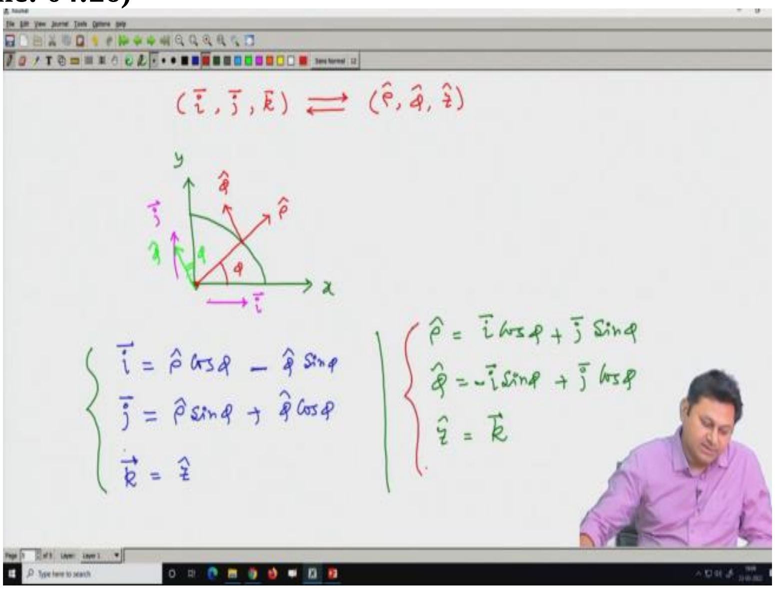

So, then it will be easier to understand. In Cartesian coordinate system, this is along y and this is along x so, I am making the projection of this part. So, this is my x. This is my y. I am now making a projection, and in that case, I have the base like this. ρ if you look carefully is defined from here to here, along this direction, it is ρ is along this direction, this unit vector. φ is calculated from x to ρ.

So, the φ should be from here to here, this is my φ and the unit vector of the φ should be tangential to this point. So, this is my unit vector φ, and z is perpendicular to the plane. I should not bother about the z because z both the cases the z direction is same. So, if I now along this direction is my i, you should appreciate this fact along this direction is my i and along this direction is my j.

So, I need to just decompose everything and then I can write the vector i in terms of ρ, φ and z. So, you can see that φ is here so, I can replace this φ unit vector here to make it more convenient. So, my φ I just replace it at this point. And if this angle is φ, so, this angle has to be φ, because if you change this, this tangential point will be going to change and if it is 0 then this will be along this direction.

So, that is why this angle φ and this is φ unit vector. Now, if I decompose i along these 2 directions, one I can write along these directions. So, what should be the value? It should be ρ unit vector and the cos component of that. What should be the other direction? Other direction I can say this is i another direction I can have here you can see that ρ and φ they are perpendicular to each other.

So, other direction I can decompose and it should be simply the φ unit vector because it is along the φ direction I am doing and then sin of the angle. I am decomposing this i unit vector along this direction and this direction. Now, if this is φ, then the cos component will go along this direction, but the sin component go along this direction, which is opposite that is why I need to put a minus sign here.

What about the decomposition of the j component? This is straightforward. j is along this direction I can decompose it along ρ. So, if I do that, then it should be simply ρ and this angle is φ. So, that means, it is sinφ plus if I decompose along φ direction it should be φ and this component is simply cosφ. What is z? Both the cases z is same. So, I should not write it z I should write it k.

So, my k vector will be simply z unit vector I should write like k vector because, if I write i, j, k it automatically means these are the unit vectors. So, this is the way I can have i, j, k in terms of ρ, φ unit vectors. Similarly, you can define you can decompose this ρ, which is even simpler to i, j, k component in terms of i, j, k component and if you do then the ρ unit vector, which is sitting here can be decomposed here along x direction and y direction, and if I do it should be simply i cosφ + j sinφ. Similarly, φ unit vector if I want to decompose along i, j φ is in this direction so, φ is in this direction. So, now from this figure if I want to decompose along j direction it should be plus j cosφ and if I decompose along this i directions it should be a negative sign and then i sin of that component.

What about the z? z should be simply along k direction. So, here I decompose i, j, k in terms of ρ, φ, z, here I decompose ρ, φ, z in terms of i, j, k. So, I can have a relationship between these 2 and I can write these in a matrix form that is interesting this is a transformation this is called the transformation of unit vectors.

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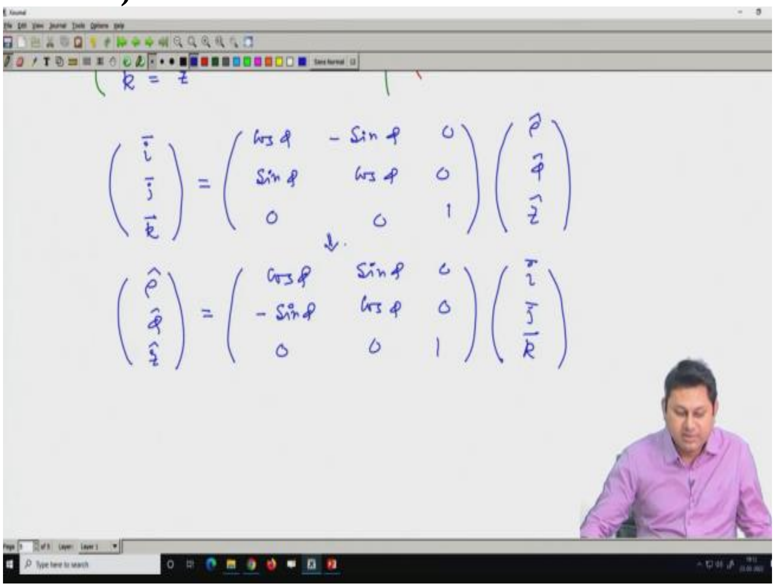

So, if I write this transformation in a matrix form, I simply write i, j and k here. This side if I write in matrix form it should be simply cosφ - sinφ and 0. Then I have sinφ and then cosφ again 0 and then I have 0, 0, 1 and I have ρ unit vector, φ unit vector and z unit vector. This is the transformation from ρ to i if ρ is and φ are given then I can find out what is i, j, k. In the similar way, I can also have another transformation, which is the opposite one that ρ, φ, z is equal to if I make a transpose of these things you will see if I simply make the transpose of these things then I will get the matrix. These are the 2 transformations I am talking about, but interesting thing is this matrix whatever the matrix I formed. So, from the scratch we understand that how i, j, k and ρ, φ, z are related.

And I just decompose these vectors and find that i, j, k can be decomposed in terms of ρ, φ, z in this way. ρ, φ, z can be decomposed in terms of i, j, k in this way and then write a matrix form but interestingly there is a relationship between. So, if I know that this is i, j, k and this is ρ, φ, z and they are related with the matrix M this if I make an inverse of these things.

Then if I find the inverse of this matrix M then ρ, φ, z can be written in terms of i, j, k with this inverse matrix, but here we can see that these things are simply the transpose of that. That means, the inverse and transpose that same matrix so, that means, there is a relationship between these transformations.

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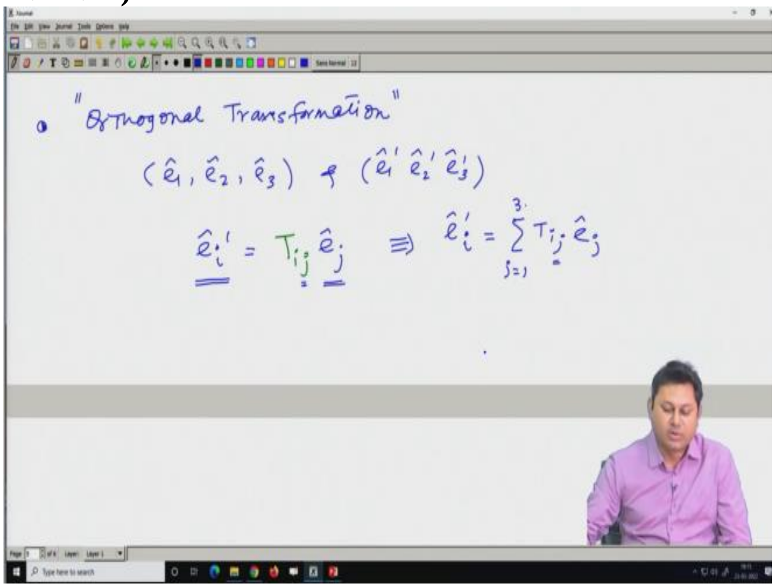

So, these things we will be going to learn next and that is called the orthogonal transformation. So, these kinds of transformations are orthogonal and when we have an orthogonal transformation this transformation matrix follows certain, you know certain unique rule and that thing we are going to understand quickly. So, what is an orthogonal transformation? We have one basis and now we are going to another basis the thing is that 2 bases are forming 2 different orthogonal systems.

And so, the question is how I can manage to find this matrix, which can go from one basis to another basis and you know what is the property of these matrix? So, suppose in general I have a coordinate system having unit vector e_1, e_2 and e_3, and another set I am having like e_1 unit vector with prime, e_2 with prime and e_3 with prime. So, the way we have in the previous case if I generalise.

So, e_1' can be figured out from e_1 set of this, but I need to have a matrix here this is j. So, I can find out my e_1' from the value of e_1, e_2, e_3 that is why j is here with the matrix form here i, j and you can see that j are twice here, that means we have a submission here. So, basically, I am having e_i, so these things are equivalent. I am using the Einstein notation. So, j is over summation this value is 1, 2 and 3 if I am dealing with 3 coordinates. So, I just removed the summation sign because I am using this Einstein notation.

So, it is simply e_i' is T_{ij} e_j. Now, I find that both the coordinate here for example, the Cartesian coordinate and this cylindrical coordinate they are orthogonal to each other, I mean i, j, k they are forming orthonormal bases ρ, φ, z also forming orthonormal bases. So, there both the coordinate systems are forming orthonormal bases.

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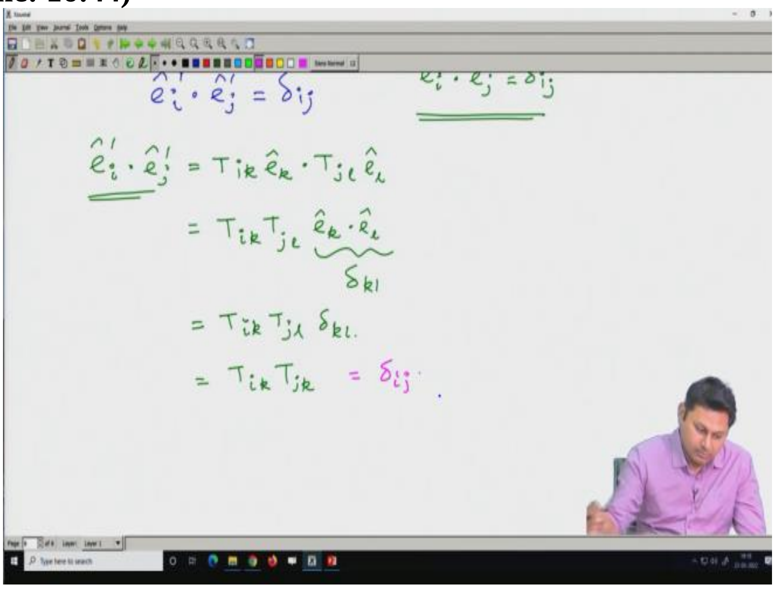

So, that means I can have an so, I should write that e'dot e_j', they are related with delta function because they are forming orthonormal basis. Not only that, the other transformation non-prime frame or non-prime it is also δ_{ij} that we know. So, let us now and now we know the relationship, so, I just put this value here. So, what is the value I am having? e_i' dot e_j' that is T_i say k e_k because I am writing and then dot T_j say l, e_l because this transformation I am using whatever the transformation is written here, so, prime with the non-prime when I write, I write in terms of this matrix T_{ij}. The goal is to find out the properties of these T matrix. I can now write ik T_{jl} and e_k dot e_l this. Now, this quantity is simply δ_{kl} because they are forming again an orthonormal basis here.

So, that thing if I use I simply have T_{ik}, T_{jl} these are 2 matrix element multiplied by this delta. So, it will be 0 if k and l are different. So, when k and l are same, then only I can have this. So, I can simply write T_{ik} and T_{jk}. That thing mind it I start with this quantity, so, that thing is already equivalent to the δ_{ij}.

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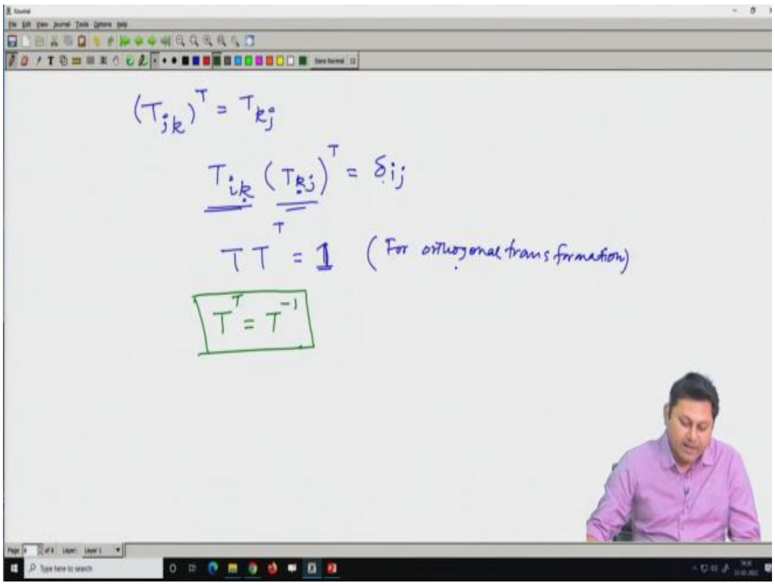

Now, we can because these are the matrix element I can manipulate that and this manipulation is T this jk if I make a transpose of that, then it should become T_{kj}. So, here I can write it in this way T_{ik} and this jk I just write T_{kj} and then transpose of that, that will give you value δ_{ij}. Now, this is a matrix element and this is a matrix element, k sitting here.

So, I can write that simply T matrix T transpose of that thing these are the transformation matrix. Transpose of these things should be equal to unity. That means, the matrix T for orthogonal transformation these should be for orthogonal transformation this is true this has to happen. So, that means, if you now look back to the matrix, so, here this matrix is eventually my T matrix, this matrix and I am making an orthogonal transformation here.

So, whatever the matrix I have here T^T and that should follow that T multiplied by T^T is unity. In other way, $T^T = T^{-1}$ that is the property it always hold. And you can check by yourself that the matrix we figured out the matrix we figured out $\cos\varphi$ - $\sin\varphi$ 0, $\sin\varphi$ $\cos\varphi$ 0, 0 0 1 you make a transpose of this matrix and then multiply with the original matrix whether you are going to get a unity or not.

And you will be going to get unity because this transformation is the orthogonal transformation. I am not going to discuss in detail because that is not the case it is just to make you understand that how from one coordinate system you can transfer to another coordinate system.

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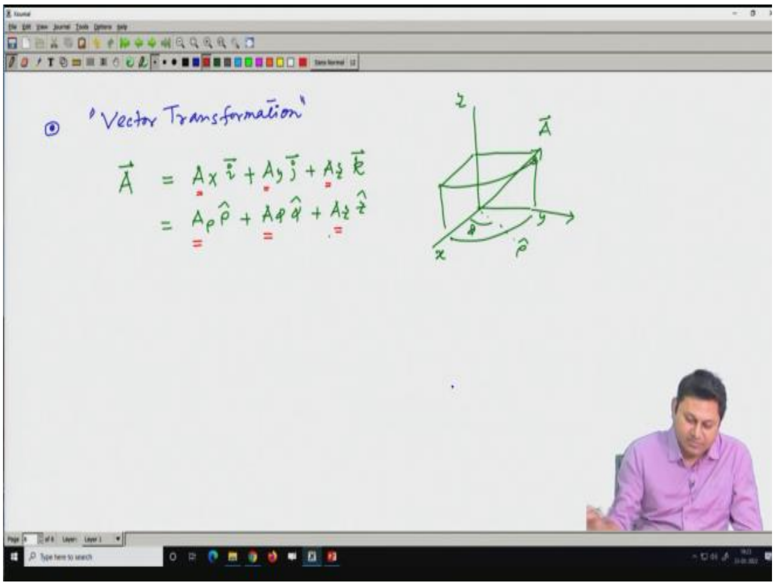

So, let us continue with that now, we will try to understand how the vector is transforming from one coordinate system to another coordinate system taking one coordinate system as you know usual Cartesian coordinate system and other is this cylindrical coordinate system. So, I will be going to have vector transformation. So, what should be the vector transformation means what?

So, that is the important thing the same vector I mentioned earlier the same vector A in Cartesian coordinate system I write A_x i + A_y j + A_z k. The same vector I can write also in you know this cylindrical coordinate system and then it should be $A_\rho\ \hat{\rho}$+ $A_\varphi\ \hat{\varphi}$ + $A_z\ \hat{z}$. Mind it I am defining the same vector in 2 coordinate system if this is my coordinate system, this is the vector I am talking about A vector if it is x y, z.

I can decompose it according to x, y, z components, but if I am using this cylindrical coordinate system still it is possible to define this vector. So, this is my cylindrical coordinate system. In cylindrical coordinate system, the same vector I can define, but in that case the component will be going to differ. Now, the question is if I know the component the point is if I know the component of A_ρ, A_φ, A_z can I get back the component A_x, A_y, A_z or vice versa?

So, that we are going to check. Suppose A_ρ, A_φ, A_z component is given to you. So, you know the vector but A_ρ, A_φ, A_z components are there. Can I able to figure out what is my A_x component, what is my A_y component and what is my A_z component or vice versa? Like A_x, A_y, A_z component is given to you and you are asked to find out what should be the component A_ρ, A_φ, A_z in Cartesian in cylindrical coordinate system so, that I will be going to you know figure out.

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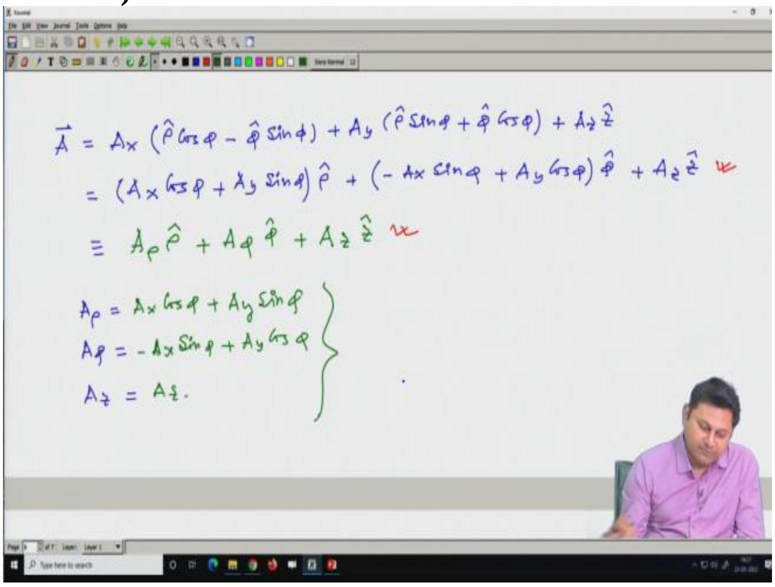

So, let me write down the complete vector A. So, it is A_x and now I know what is the relationship between i with the ρ, φ, z because that I just figured out i what is the relationship between the ρ, φ, z I know, so that I going to use here. A_x and then I write ρ unit vector cosφ in place of i I am just writing this. φ then sinφ + A_y. what was j? It was ρ unit vector then sinφ + φ unit vector cosφ and what is z? z is simply z unit vector that was.

So, now, I just simply write take all the ρ component one, all the φ component φ unit vector component in 1 bracket and z component in 1 bracket if you do I am going to get this I just rearranged that A_x then cosφ + A_y then sinφ that should be with unit vector ρ plus minus of you know A_x sinφ and then plus of A_y cosφ that should be in under unit vector and A_z will be simply A_z z unit vector.

Now, this vector is equivalent to whatever I am having is equivalent to $A_\rho\,\hat{\rho}$+ $A_\varphi\,\hat{\varphi}$ + $A_z\,\hat{z}$. So, these 2 vectors whatever I have here this one and this one, are same. Unit vectors are there so, obviously the component wise we are having the same value. So, I can now have a relationship with the component and that is this. So, from here I can write my $A_\rho = A_x\cos\varphi + A_y\sin\varphi$.

My A_φ is this quantity, which is $-A_x\sin\varphi + A_y\cos\varphi$ and finally, my $A_z = A_z$. So, this relationship I figured out and from this relationship I can make everything in matrix form.

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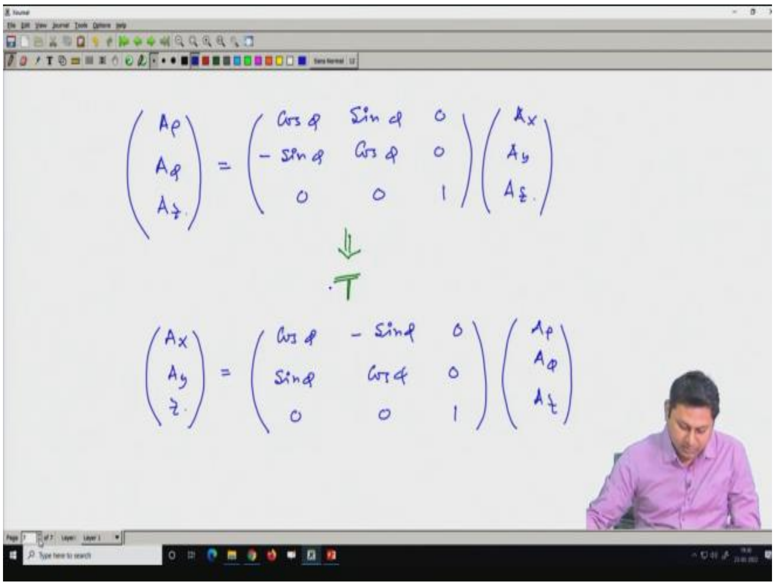

So, A_ρ A_φ A_z is there so, I can write it in this matrix form, which is this. You can see that you are getting in same matrix that we figured out last time. So, this is the transformation matrix. So, if I want to find out so, if A_x A_y A_z components are given then you can find out A_ρ, A_φ, A_z component with this by using this expression and in the similar way if I can figure out these things by making a transpose of this so, I will be going to get this. What is there?

I have $\cos\varphi$ here, $\sin\varphi$ here, 0 here, $-\sin\varphi$, $\cos\varphi$ and 0, and then 0, 0, 1 and that here I just write A_ρ, A_φ, A_z. So, this is the way we can transform from one coordinate system to another coordinate system. Finally, I will introduce another coordinate system, which is very important and that is the spherical coordinate system.

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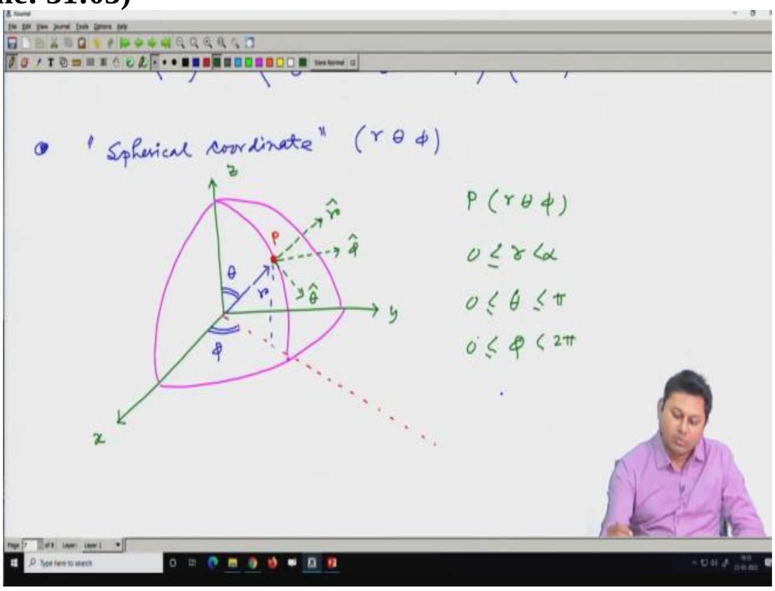

So, we know we now deal with the Cartesian coordinate system, so far, we are dealing with Cartesian coordinate system and then the corresponding transformation etcetera. So, now, we will move to another very, very important coordinate system in electromagnetic theory, this is very, very important that is the spherical coordinate system, let us define first and then we will discuss about how the vector is defined etcetera.

So, in spherical coordinate, we define a point with r, φ with a coordinate r, θ and φ. So, if I draw so, let me first draw carefully. This is x, this is y and this is z and I am having so, along this direction is the ρ. So, I should draw it here and I am having like this. So, now one by one define so, if vector point here red P, how to define this point? So, I can have a line from here to here.

So, this is the point this is the value of say r. This r can be in any I mean over this circle any point everything is r so; I need to introduce another coordinate and that is this how this r is making an angle with this x axis like we had you know in this cylindrical coordinate system. So, this angle should be my φ, but it can be any point over this line. So, another coordinate I need to introduce here and that coordinate here is θ.

So, with these 3 coordinates you can define any point over space using this spherical coordinate system. Now, what should be the unit vector because we are more interested with the unit vectors here. Along this direction it should be the unit vector r. I have a you know a surface here so, over the tangent along this direction because θ is from here to here it should be the unit vector θ, direction of the θ is in this direction.

So, unit vector and φ will be here along this direction this will be the φ this is the tangential direction it will be the φ unit vector so, these are the 3 unit vectors we have here in these points. Along this direction it is r, the tangential direction it is φ and downward tangential direction it is θ, what are the restrictions? I should also mention the restriction, so, the point P so, this is the point P.

So, quickly I write so, this point P is defined by r then θ and φ what are the limits of this? So, $0 \leq$ r $< \alpha$, $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$. So, these are the restrictions we have for r, θ and φ.

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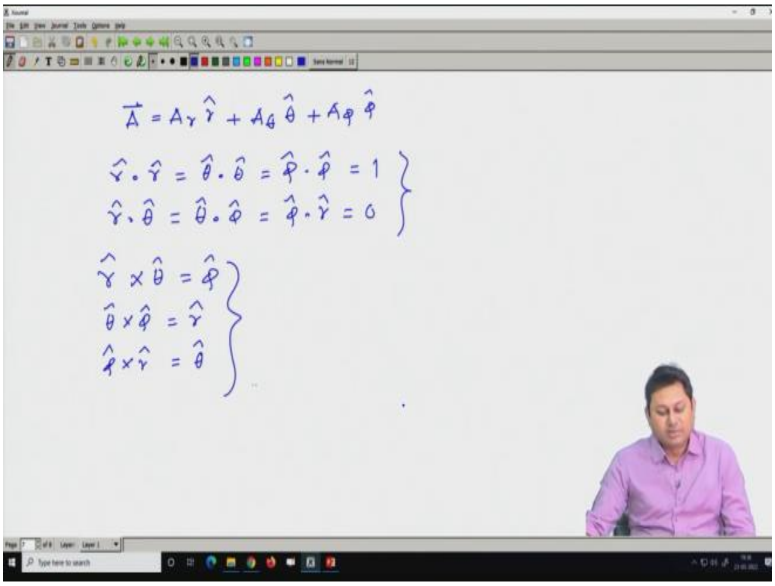

What are the unit vectors that is important and also need to define the unit vectors. So, A is A_r then r unit vector, a vector can be defined in this way θ, θ unit vector and φ, φ unit vector, and r dot r is θ dot θ = φ dot phi unit vector = 1. r dot θ, θ unit vector dot phi unit vector, phi unit vector dot r unit vector, this is 0, that means, they are forming orthonormal basis not only that, these are the right-handed system, so, I can have r cross θ = φ, then θ cross φ = r, and φ cross r = θ. So, they are forming right-handed orthonormal basis. So, I do not have much time today. So, with this note I would like to conclude. In the next class, we will discuss more about the spherical coordinate system how the vectors can be transformed, and how the unit vectors are transformed.

I am not going to do the detailed calculation the way I did in here, too and rather, I like the students to you know, do all this calculation by your own and try to find out you are getting the same result or not. And please take it as an exercise but next class I will do everything. So, with this note, I like to conclude here. Thank you very much for your attention.