Foundation of Classical Electrodynamics
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology - Kharagpur

Lecture – 37
Dipole and Quadrupole Moment (Contd.,)

Hello students to the foundation of classical electrodynamics course, under module 2. Today, we will be going to discuss the dipole and quadrupole moment last day also we discussed but today we will be going to do some problem related to this.

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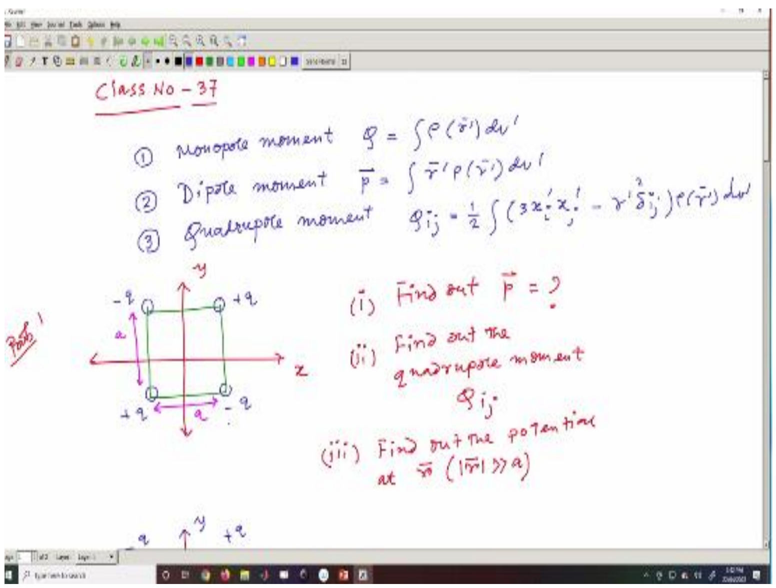

So, today we have class number 37 and we will go to discuss few problems related to quadrupole moment. So, before that let me remind what we have done. So, the definition wise the monopole moment is Q = $\int \rho(\vec{r}')\, dv'$ it is nothing but the total charge. Next the dipole moment it is $\vec{p}$ defined by $\vec{p}$ by definition it is this quantity and quadruple moment, which is defined by Q_{ij} this is a tensor component and we write it as $\frac{1}{2}\int(3x_i'x_j' - r'^2\delta_{ij})\rho(\vec{r}')\, dv'$.

This is the way we defined the quadrupole moment. So, today let us discuss few problems. So, let me first write down the problem so, this problem 1 so, I have a coordinate system like this suppose this is my x and this is my y and we have 4 charged particle or 4 point charge are distributed like this, one is sitting here, another is sitting here so, this charge is say +q, this is –q, this is +q and this is –q, where the separation from here to here this is a and here to here this is a, this is the geometry we have.

And the problem is 1 find out the dipole moment $\vec{p}$, second find out the quadrupole moment that means I need to find out all the components of Q_{ij} and finally find out the potential at some vector $\vec{r}$ such that the $|\vec{r}|$ is very greater than a. Set at long distance I want to find out the potential. So, this is the problem. So, I need to find out first the dipole moment and then I find the quadrupole moment tensors.

And then finally, find out the potential the standard problem last day also we calculated a similar kind of problem, but, we calculate only the dipole moment here we are going to calculate the quadrupole moment as well.

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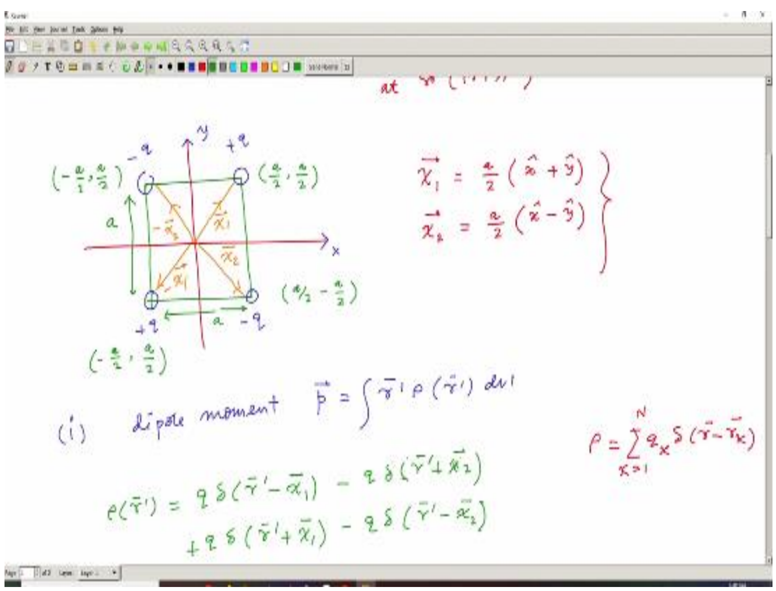

Well, so, let me draw once again so, the solution is like that, so, let me draw this once again and this is my x axis, this is say y and this is +q, -q, +q, -q. So, let us first because in order to calculate the dipole moment or the quadrupole moment I need to know the location the position vector for all these point charges, so, that I first let us figure out. So, from here to here so, this is the position vector for the; charge +q and let us see this position vector is χ_1 in a similar way from here to here, this is the position vector and say this is χ_2.

Now, by symmetry I can say that so, the position vector of these 2 will be simply -$\vec{\chi}_1$ and -$\vec{\chi}_2$ these are the position vectors. So, I can find out the position vector very easily because I know what is the coordinates here. So, coordinates since this is a I already mentioned that from here to here it is a and from here to here this is a they are forming a, a box like that. So, here the coordinate is $\frac{a}{2}, \frac{a}{2}$ the coordinate of this point is $\frac{a}{2}, -\frac{a}{2}$ coordinate of this point is $-\frac{a}{2}, \frac{a}{2}$.

And this is $-\frac{a}{2}, \frac{a}{2}$ these are the locations of these point charges. Once you will know the location then the next thing is very straightforward and that is find out the $\vec{\chi}_1$. So, the $\vec{\chi}_1$ vector, which is the location of this is simply $\frac{a}{2}$ $(\hat{x} + \hat{y})$ and $\vec{\chi}_2$ is $\frac{a}{2}$ $(\hat{x} - \hat{y})$ in a similar way other 2 are negative of that vector. So, I am not writing that for other 2 points you can find out this is one is $-\vec{\chi}_1$ and another $-\vec{\chi}_2$.

So, next let us calculate directly the dipole moment. So, dipole moment of the system is according to our rule, which is $\vec{p}$ this is $\int \vec{r}'\rho(\vec{r}')\,dv'$, now ρ is the density and we know for discrete charge how the density is calculated. So, several time we are using this concept a very important concept, so, let me write it so, the ρ for discrete charge is the density is $q_k\ \delta\ (\vec{r} - \vec{r}_k)$ this is the way we define the charge density of discrete charge if there are N number of charges I can write it in this way.

So, I am going to exploit this expression here and write down r, density of this system charge density of the entire system and that is q and then δ ($\vec{r}\,'$ - the location of that) - q δ (the location of $\vec{r}\,'$ this one will be + $\vec{\chi}_2$ because I am calculating this one. And then +q δ ($\vec{r}\,'$ + $\vec{\chi}_1$) and -q δ ($\vec{r}\,'$ - $\vec{\chi}_2$) this is the density of the system in this discrete charge having this 4 discrete charge whatever we are having here.

So, these are the 4 discrete charges. So, one charge is sitting here and for that it is $\vec{\chi}_1$, one charge is sitting here for that is -$\vec{\chi}_2$ so, that is why I have a plus sign here for this it is charge +q for that it is again - $\vec{\chi}_1$ that is why writing +$\vec{\chi}_1$. Here in the delta function and another is this one for that it is $\vec{\chi}_2$ and I write - $\vec{\chi}_2$.

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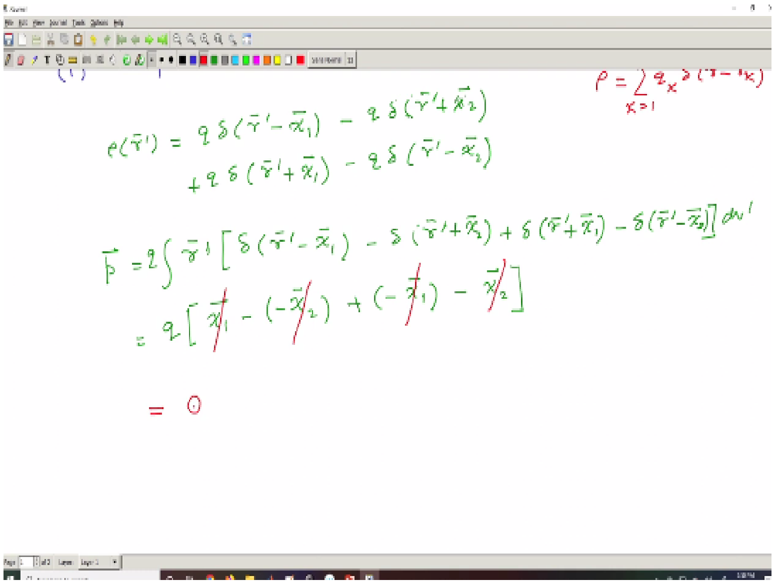

So, now, simply my this will be integration of $\vec{r}$' and q is common to I can take q outside and we left with only the delta functions and that is δ ($\vec{r}$' – this) - δ ($\vec{r}$' + this) + δ ($\vec{r}$' + this) and -δ ($\vec{r}$' - this) over dv'. Now, the delta function once we have the delta function then rest is very straightforward because simply I just need to put. So, in this case I should put the value simply $\vec{r}$'.

So, I just replace $\vec{r}$' to $\vec{\chi}_1$ in this case I just replace –(- $\vec{\chi}_2$), in this case, + (-$\vec{\chi}_1$) and here minus of we have $\vec{\chi}_2$, so - $\vec{\chi}_2$. So, here you can see that $\vec{\chi}_1$ and this $\vec{\chi}_1$ will cancel out and + $\vec{\chi}_2$ and - $\vec{\chi}_2$ will cancel out. So, eventually what I am getting is 0. So, the dipole moment is 0 for this system after doing all this calculation, we find that the dipole moment is 0. Now, the next thing that what should be the quadrupole moment for it.

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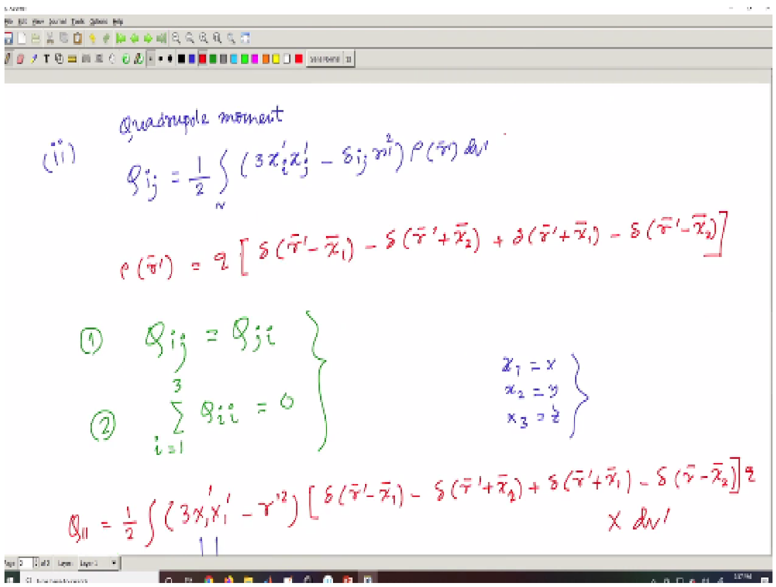

So, next we calculate the second part of this problem that this is 2 and we want to calculate the quadrupole moment and that we define definition is very important you need to remember this definition and it is $3x_i' \, x_j'$ and then $-\delta_{ij} \, r'^2$ and then $\rho(r') \, dv'$. Now, I have already figured out the ρ and that will be going to help me because in this problem, I need to put this ρ value and rest of the thing is identical.

So, let me write down the ρ once again the $\rho(r')$ for this system was simply taking q common it was I am writing the same thing multiple times, but anyway then $\delta \, (\vec{r}' + \vec{\chi}_2)$ and then plus no q I take common I should not write q anymore so, it should be simply $\delta \, (\vec{r}' + \vec{\chi}_1)$ and then $-\delta \, (\vec{r}' - \vec{\chi}_2)$ this is my density. So, that density I will put here one by one. So, first let us calculate, because there should be 9 terms Q_{11} Q_{12} Q_{13} and then Q_{21} Q_{22} Q_{23} and so on.

But, we know so, let me write down we know that there are few properties that we discussed last day and that is $Q_{ij} = Q_{ji}$ the tensor is symmetric that is property number 1 and the property number 2 for this case is $\sum Q_{ii} = 0$ it is a traceless I can go to 1 to 3. So, if it is not that I need to record all the nine components, but I can simply calculate few components and my total value total tensor can be calculated. So, let us start with the calculation of Q_{11}.

So, Q_{11} is simply $\frac{1}{2}$ of integration and then I have $(3x_1'x_1' - r'^2)$ and then I have the entire delta function and the entire delta function if I right so, let me write it, it should be $\delta \, (\vec{r}' - \vec{\chi}_1)$ then $-\delta$

$(\vec{r}' + \vec{\chi}_2)$ and then +δ $(\vec{r}' + \vec{\chi}_1)$ and -δ $(\vec{r}' - \vec{\chi}_2)$ how the $\vec{\chi}_{1,}$ $\vec{\chi}_2$ is defined already we mentioned and we take q term sitting here multiplication over dv'.

So, now, we have only the delta function and then here we have the function, but in this function here I am having x x only the x component because I according to our notation x_1 = x, x_2 = y and x_3 = z so, only the x component of the $\vec{\chi}_1$ $\vec{\chi}_2$ these x components will put and in place of r'^{2} I just put whatever we have in the argument. That is $|\vec{\chi}_1|^2$.

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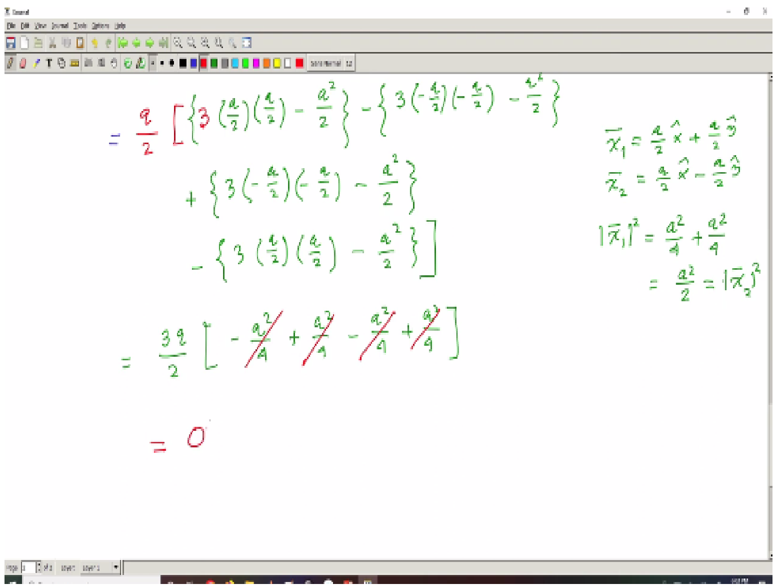

So, this simply gives me if I start putting the value because now I am dealing with the delta function then it is let us take $\frac{q}{2}$ outside because q is constant so, I can take it outside and then I execute the integral. So, the first case I write 3 and for this case this is $\vec{\chi}_1$ and I know what is $\vec{\chi}_1$? It is a_{11} $\frac{a}{2}$ $\frac{a}{2}$ x and y so, x component and y component are having $\frac{a}{2}$ so, x component is important which is $\frac{a}{2}$. So, I simply better I write it here it will be helpful.

So, somewhere so, my $\vec{\chi}_1$ is $\frac{a}{2}\hat{x} + \frac{a}{2}\hat{y}$ and $\vec{\chi}_2$ is $\frac{a}{2}\hat{x}$ - $\frac{a}{2}\hat{y}$. So, here I am going to put this value only the $\vec{\chi}_1$ value. So, in place of x' I just have $\frac{a}{2}$ in place of x_1 again I have $\frac{a}{2}$ and in place of r I should simply have the mod square of these things. So, what is the $|\vec{\chi}_1|^2$? It is simply $\frac{a^2}{4}$ + $\frac{a^2}{4}$. So, it is a square divided by 2 that is the value.

So, that value is simply $\frac{a^2}{2}$ my first this I execute next I need to execute the next one. So, that is minus of 3 and now, this is $-\vec{\chi}_2$ so, $-\vec{\chi}_2$ means I need to put the minus value of $\frac{a}{2}$ but I multiply twice. So, it should be $-\frac{a}{2}$ and $-\frac{a}{2}$ and then $-\frac{a}{2}$ and then I put r^2 so, that is $\frac{a^2}{2}$ that is the second. What about the third? This is plus and in a similar way I have 3 multiplication of a plus $\vec{\chi}_1$.

So, $+\vec{\chi}_1$ means I should have $\vec{\chi}_1$ with negative values $\frac{a}{2}$ again I need to put $-\frac{a}{2}$ here in place of x, $-\frac{a}{2}$ in place of x and then - r^2, which is same, which is $\frac{a^2}{2}$, because mod of all the values is same that is equal to $|\vec{\chi} \quad|^2$ and finally, we have minus of 3 is here and $\vec{\chi}_2$, $\vec{\chi}_2$ is simply $\frac{a}{2}$ $\frac{a}{2}$ and then $-\frac{a^2}{2}$. So, all the 4 values I calculated.

And now what I get let us see that I am getting here let us take 3 outside $\frac{3q}{2}$ and then I get here a term $\frac{a^2}{4}$ $-\frac{a^2}{2}$. So it should be $-\frac{a^2}{2}$, these term I am having the same thing but minus sign, so I should have $+\frac{a^2}{2}$, this term is plus I should have $-\frac{a^2}{2}$ and this term again, I have sorry so, this is 2. So, sorry so it is a^2 not $\frac{a^2}{2}$, I should have $\frac{a^2}{4}$ all the cases.

So, this is 4, this is 4, this is 4 and I should have $+\frac{a^2}{4}$. So, after doing all this calculation, we find that it seems to be cancelling and I am having 0.

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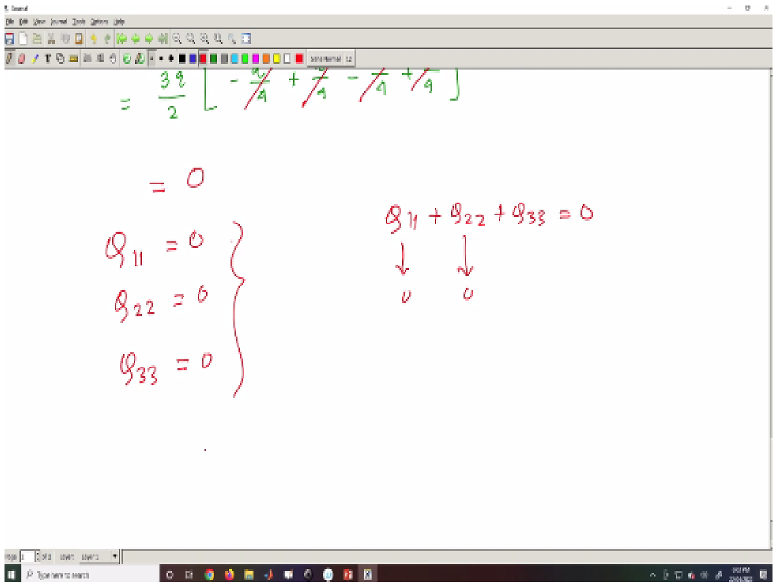

So what I get is $Q_{11} = 0$. In the similar way, you will find $Q_{22} = 0$. Now, I know $Q_{11} + Q_{22} + Q_{33}$ has to be 0. Since this is 0, and this is 0, this has to be 0. So that means Q_{33} is also 0 so, all the diagonal term will be 0. After doing all calculation, we figure out so the next thing that I will be going to calculate is Q_{12}, the calculation seems to be a lengthy. But once you do this calculation by your own, then you are confident enough that how to calculate the quadrupole moment tensor.

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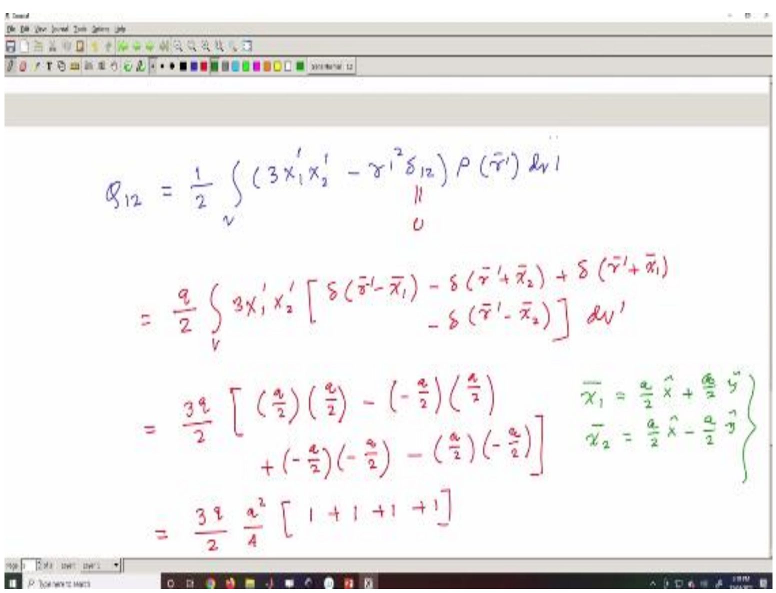

The exam this is nowadays this is a very popular question to calculate the quadrupole moment for a given charge distribution like so, for 1 2, I have an advantage, the advantage is if I write down because a delta function is there, so I can simply have, let me write it first $3x_1' \, x_2' - r'^2$ and I should write δ_{12} and then $\rho(\vec{r}')dv'$, you can see that this quantity is 0. Eventually, I am having only this term r^2 term is no longer for all the off diagonal terms, it is true.

So I simply have ρ is known so I simply have $\frac{q}{2}$ and then I have the integration over volume and I have $3x_1'x_2'$ and the delta function whatever the delta function is there, so let me write it once again delta. Once you do one problem, the rest of the thing is now just a repetition of the same thing plus delta one value you will get r' minus you know let me follow the order otherwise, so, here I should write minus of these things and it is plus and we have $+\delta\ (\vec{r}' + \vec{\chi}_1)$.

And then I have a $-\delta\ (\vec{r}' - \vec{\chi}_2)$ then this is dv not delta this is dv'. So, now, again I need to just put the value and if I put this value I should get $\frac{3q}{2}$ here and let us start putting the value. So, first when I put this one, so, the value is $\frac{a}{2}$ and $\frac{a}{2}$ first value, second value with a negative sign I should put it but it is 1 2 mind it 1 2 means it is x y. So, let me write it because this is important. So, $\vec{\chi}_1$ was $\frac{a}{2}$ $\hat{x}$ + $\frac{a}{2}\hat{y}$ and $\vec{\chi}_2$ was $\frac{a}{2}\hat{x}$ - $\frac{a}{2}\hat{y}$.

So, that component that x and y value is important here. So, for $\vec{\chi}_1$ I have x value $\frac{a}{2}$, y value $\frac{a}{2}$ I put it, for $-\vec{\chi}_2$ I should have the x value as $-\frac{a}{2}$ and the y value is $+\frac{a}{2}$ because it is $-\vec{\chi}_2$ and then I have plus $-\vec{\chi}_1$, for $-\vec{\chi}_1$, I have x and y both are $-\frac{a}{2}$ and $-\frac{a}{2}$. And finally, for $\vec{\chi}_2$ I have value $\frac{a}{2}$ for x and $-\frac{a}{2}$ for y, so let us see.

So, $\frac{a}{2}$ all the cases it is $\frac{a}{2}$ say I can simply take $\frac{3q}{2}$ and this is $\frac{a^2}{4}$ outside then what I am getting is 1 + 1 + 1 + 1.

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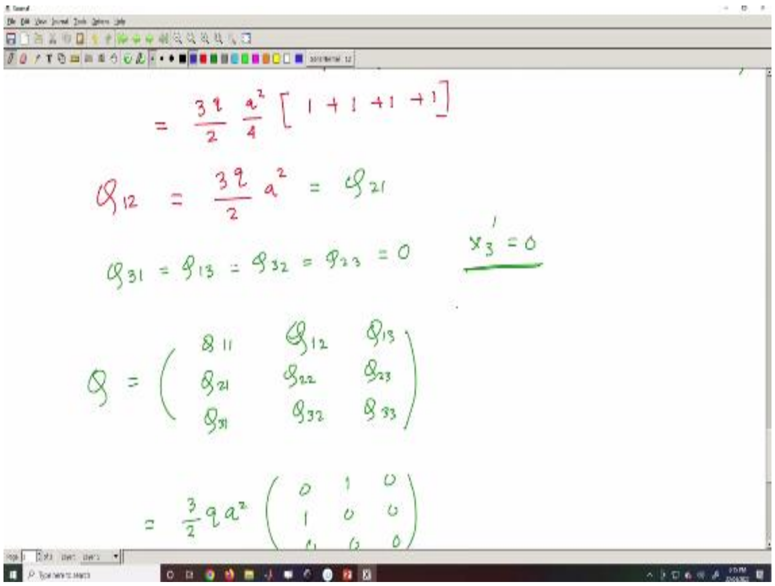

So that value is nonzero mind it and what I am getting is 4. So, $\frac{3q}{2}a^2$ that is the value I am getting. So this is the value of Q_{12}. Now, once you know Q_{12}, then you know that this is equal to Q_{21}. And now if I calculate Q_{31} or Q_{13} or Q_{32} or Q_{23} everything will be 0 why? Because there is for this is 2 dimensional there is no z component sitting here since there is no z component, so, I will be going to get all whenever I have x_3 here in this picture these things will be 0 for all the cases.

So, all this quantity should be 0, because in our case $x_3{}'$ is 0, I mean there is there should not be any $\vec{\chi}_3$ I should not have because this is in plane, no z company. So, I have already figured out almost all the values and also 23 32 All these things as 0. So, 1 2 3 4 5 6 and other 3 7 8 9 so, all the nine components are there. So, if I put these 9 components together, I will have Q = Q_{11} Q_{12} Q_{13} Q_{21} Q_{22} Q_{23} Q_{31} Q_{32} Q_{33} and the value should be this.

This seems to be $\frac{3}{2}qa^2$ and I should have 0, this value is 1, this is 0, this value is 1, this is 0 0 0 0 0, this is the value of the quadrupole moment of the system.

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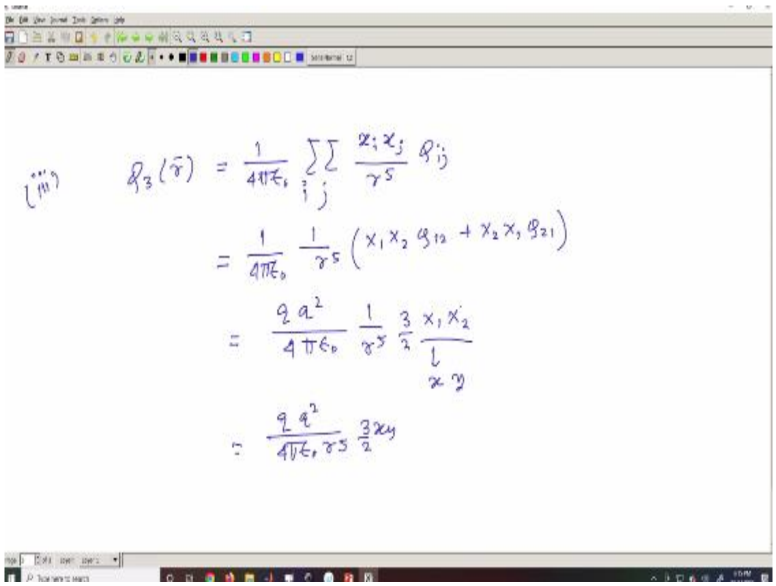

Now, the third part is you know, what is the value of the potential? That was the third problem. So, let me do it here as well. So, ϕ_3 that is the potential contribution of the quadrupole for discrete case it is $\frac{1}{4\pi\epsilon_0}$ then sum over i and j this is $\frac{x_i x_j}{r^5}$ and Q_{ij}. So, that quantity Q_{ij} I already figured out so, that value if I put here it should be $\epsilon_0 \frac{1}{2r^5}$ and this.

So, this is not 2 here so, I just need to put the Q_{ij}. So, what Q_{ij} I get here. So, it is 1 2, so, it is x_1 x_2 because this is meaningful all the values are 0 other values are 0 1 2 plus x_2 x_1 Q_{21} because only x 1 2 and 2 1 is non vanishing other all the values are 0. Now, if I put this it simply comes out to be $\frac{qa^2}{4\pi\epsilon_0}\frac{1}{r^5}$ and 3 x_1 x_2 this is the value. So, x_1 x_2 means it should be x and y in terms of x and y it should be $\frac{qa^2}{4\pi\epsilon_0 r^5}$ and then 3xy this should be the value.

Another 2 terms should be here because it is $\frac{3}{2}$. So, I should put a 2 here and another 2 should be here. So, this should be the potential due to this the structure. So, we are almost finishing our time but quickly I like to show another thing and that is for single charge system what happened.

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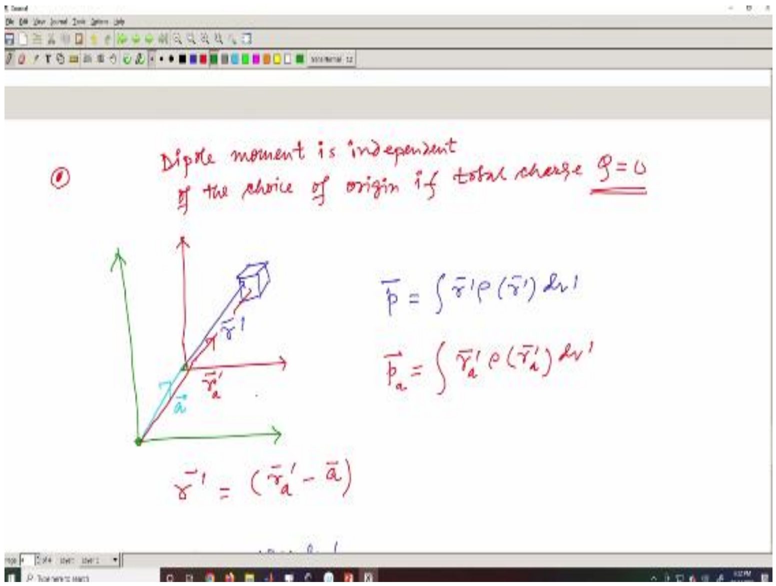

So, for a single charge system, so, let us first calculate that thing that if we mentioned that the dipole moment is independent of the choice of the origin if total charge Q = 0 in the dipole moment, if the total charge Q = 0, then it is independent of the choice of the origin. So, we can prove it very quickly. So, suppose I have a coordinate system like this and a point here charge distribution is here and whose location with this coordinate system is $\vec{r}\,'$.

So, according to my definition, it should be $\vec{r}\,'$ $\rho(\vec{r}\,')$ and dv' now what I do? That I am going to shift this coordinate system some other point say here I shift it such that this is the origin now, I have a new origin here which is shifted by a quantity vector $\vec{a}$ and from these new origin the point the location with this new origin the location is say now $\vec{r}_a\,'$. So, for $\vec{r}_a\,'$ for this origin for this coordinate system.

According to my definition, the dipole moment should be written as $\vec{p}_a$ is equal to integration of in place of $\vec{r}\,'$ I should now write $\vec{r}_a\,'$ because this is the location with this new coordinate and then $\rho(\vec{r}_a\,')$ dv' this is the new value with the new coordinates. Now, you can see that $\vec{r}\,'$ $\vec{r}\,'$ is this quantity and so, $\vec{r}\,'$ this is not $\vec{r}\,'$ is simply $(\vec{r}_a\,' - \vec{a})$ so, if I put that and try to understand the relation between these 2 these 2 dipole moment.

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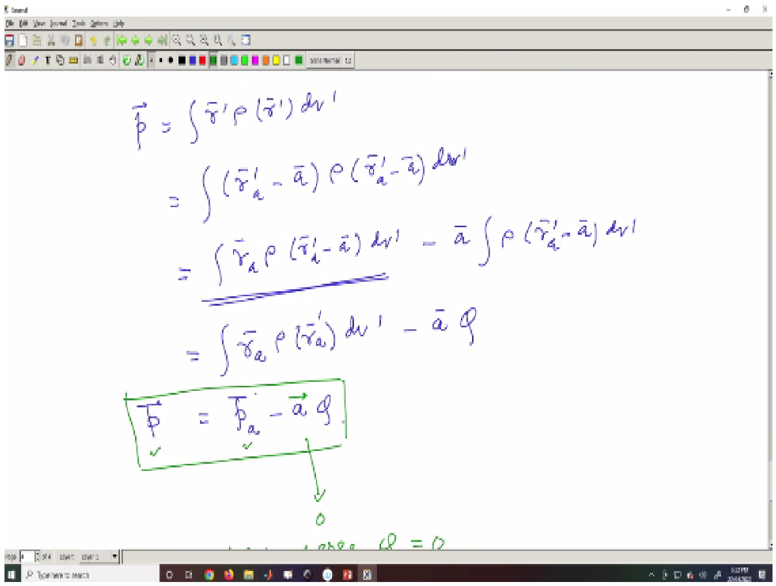

So, $\vec{p}$ = ∫ $\vec{r}$' ρ($\vec{r}$') dv' now, we will go to put the value so, this is ($\vec{r}_a$' – $\vec{a}$) and ρ ($\vec{r}_a$' – $\vec{a}$) and dv'. So, this quantity is integration of $\vec{r}_a$' and then ρ ($\vec{r}_a$' – $\vec{a}$) dv' and $-\vec{a}$ integration of ρ ($\vec{r}_a$' – $\vec{a}$) dv'. So, this quantity is simply represented this you know this because this is even if I put this. So, let me write it here this is simply $\vec{a}$ρ and ($\vec{r}_a$' – $\vec{a}$), whatever the value of the density I will have the same amount of density I should have for $\vec{r}_a$.

Because both the cases it is showing the same point and $-\vec{a}$ this quantity is a total charge ρ this quantity dv' is my total charge $\vec{a}$ into Q. So, this quantity is my $\vec{p}$ is this my $\vec{p}_a$ on the new coordinate $-\vec{a}$ Q. So, there is a relationship between the dipole moment when we have I should have $\vec{a}$ here, when we have a shift of the coordinate system this is for old coordinate system and this is for new coordinate system. Now, you can see that it will be independent of the coordinate system if this quantity is 0.

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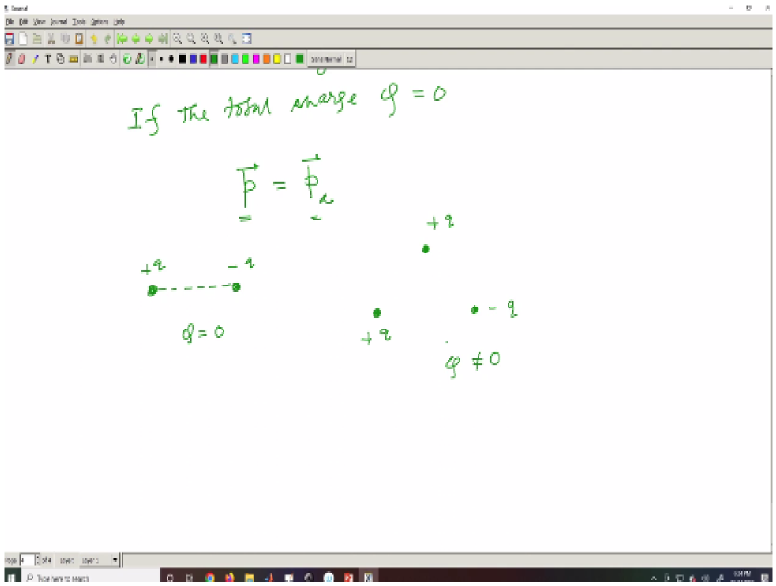

So, that means, if the total charge distribution or the total charge in the distribution if the total charge Q = 0 that means, the total charge is 0, we simply have $\vec{p} = \vec{p}_a$. So, whatever the dipole moment we have in the previous case I can have the same. So, we can say that under the condition that when the total charge is 0 the dipole moment is a conserved quantity. So, dipole moment will depend on the coordinate system only if the total charge is nonzero if the total charge is 0, then it is independent of the choice of the coordinate system.

So, that is why when we have if you remember when we have 2 dipole plus and minus with +q and -q, this is a simplest example of a dipole then the total charge Q was 0 and we find that the dipole moment whatever the coordinate system you use will remain same. On the other hand, if you have a 3 charge system say this is +q, this is -q and this is again +q and if somebody asks what should be the dipole moment of this system, then it the total charge Q is not equal to 0 here.

And we did it one problem actually in the last class and we checked that it entirely depends on that how it depends on the coordinate system it even though the charge is same, but because of the distribution it is different that means the choice of coordinate system matters. So, for this case when there is total charge 0, I should have the dipole moment should not have any preference over the coordinate system. But here we should have a preference when it is nonzero.

So, with this note, I think I should stop here because my time is limited. So, in the next class we will start a new topic related to the potential equipotential surface and conductor and dielectric will be covered in that part. So, thank you very much for your attention and see you in the next class.