Foundations of Classical Electrodynamics
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Lecture - 36
Quadrupole Moment

Hello students to the foundation of classical electrodynamics course, under module 2, today, we will be going to have the lecture on quadrupole moment in the last class we discussed in detail about the calculation of the monopole moment and dipole moment. Now, in today's class we will discuss about the quadrupole moment.

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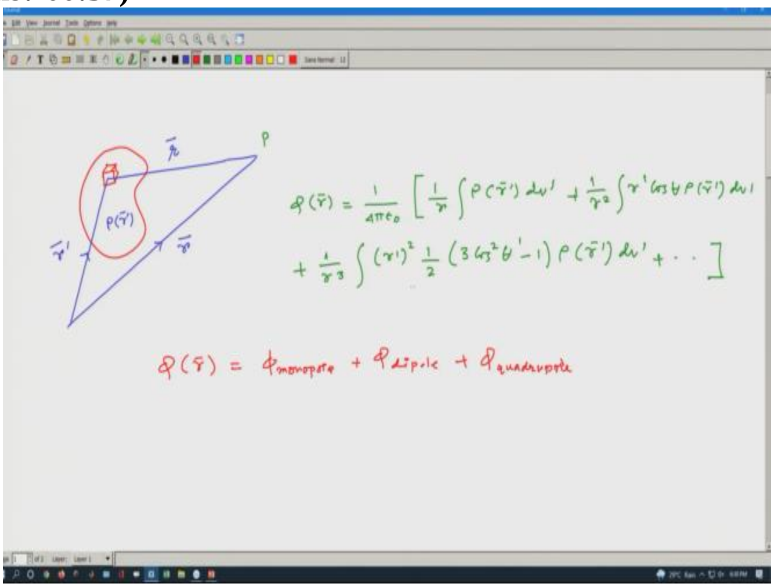

So, we have class number 36 today and today's discussion is on the quadrupole moment. So, let us go back to the calculation we started a few class ago that if I have a charge distribution and this is some reference coordinate system this is $\vec{r}$ this was $\vec{r}\,'$ and this was $\vec{\mathscr{r}}$ this is a system where the charge distribution is defined as $\rho(\vec{r}\,')$ and when we write the potential if you remember the potential at point say this is P point the potential.

And this is at $\vec{r}$ the potential at $\vec{r}$ was this. The first few term if I write, if I just simply write the first few term it should be like $\frac{1}{4\pi\epsilon_0}$ and then I had the first term $\frac{1}{r}$ and $\int \rho(\vec{r}') \, dv'$ that was the first term. The second term was $\frac{1}{r^2} \int \vec{r}' \cos\theta \, \rho(\vec{r}') \, dv'$ and then I had $\frac{1}{r^3}$ integration of r '2 and with the $\frac{1}{2}$ term.

And then (3 $\cos^2\theta'$ – 1) $\rho(\vec{r}\,')$ dv' and so on. So, I have already discussed these first 2 case this if I write, so, I should write at $\phi(\vec{r})$ is the contribution the first term is the contribution of monopole, second term is the contribution of dipole and third term should be the contribution of the quadrupole. So, now, these terms I am going to calculate and try to understand what is the meaning of these terms.

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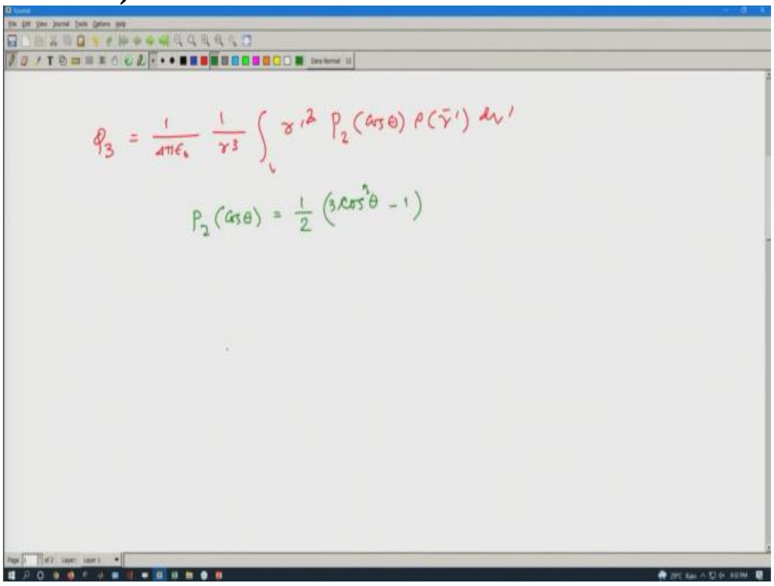

So, ϕ_3 or $\phi_{quadrupole}$ so, if this is my ϕ_1 ϕ_2 according to our old notation ϕ_3. So, the quadrupole contribution ϕ_3 was let me write it once again $\frac{1}{4\pi\epsilon_0}$, $\frac{1}{r^3}$ then integration of r'^2 and then I should write in terms of Legendre polynomial it was P_2 (cos θ) and then $\rho(\vec{r}\,')$ dv' that was the entire term over the volume integral v what was P_2?

P_2 is the Legendre polynomial of order 2 cos θ this term is $\frac{1}{2}$ (3 $\cos^2\theta$ – 1). So, from these 2 I can define I mean let me simplify a few things.

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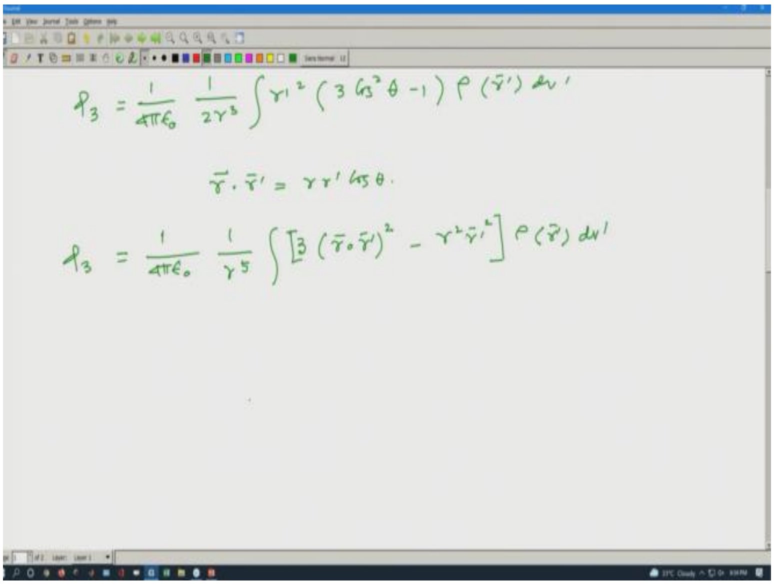

So, my ϕ_3 is $\frac{1}{4\pi\epsilon_0}$ and then $\frac{1}{2r^3}$ integration of r'^2 (3 $\cos^2$ θ- 1) I am writing the same term once again, but now, I will deal with this $\cos^2$ θ term because we know that the $\vec{r} \cdot \vec{r}\,'$ is r r' cos θ so, cos θ I can remove in this way that it is nothing but the square of r r dot and if I do then it should be like this.

So, ϕ_3 I rearrange the term like this $\frac{1}{4\pi\epsilon_0}$ the constant term 1 divided by so, in order to introduce the cos θ I need to multiply the r^2 here and then I divide the r^2 so, it should be r^5 then integration then these cos r now I multiply r^2. So, this term will be 3 and then multiplication of $(\vec{r} \cdot \vec{r}\,')^2$ so, one r^2 is here so, which is taken care of this r^5.

And another term should here also and that is r^2 and then r'^2 and I have ρ usual term and dv. So, now, I can define my quadrupole term here whatever I had, but before that let us try to understand what is $\vec{r} \cdot \vec{r}\,'$.

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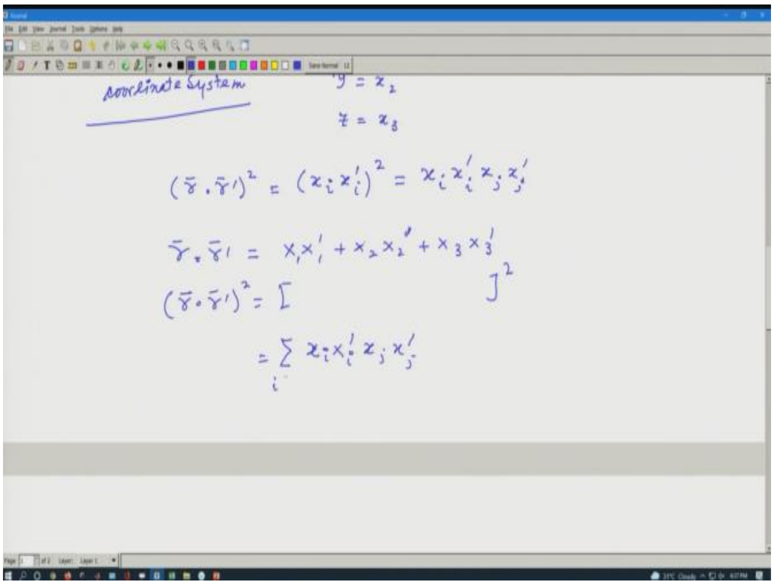

So, if I consider the Cartesian coordinate system in Cartesian coordinate system. So, x component now on I should write as x_1 y I should write as x_2 and z I write at x_3 so, what I should write for $\vec{r} \bullet \vec{r}\,'$ in terms of x y z and that value if I look carefully $\vec{r} \bullet \vec{r}\,'$ is simply $x_i\ x_i'$ because dot this is the way we write and square of that and if I expand this, then it should be $x_i\ x_i'$ multiplication of $x_j\ x_j'$.

Because when you have a multiplication the cross term will be there, mind it, this is a repetitive index, so, 1 summation sign is there, so, that you should remember always, so, otherwise you can just check it like this way. So, $\vec{r} \bullet \vec{r}\,'$ is simply x x, $x_1\ x_1' + x_2\ x_2' + x_3\ x_3'$ now, if you make a square of that then all the other cross term will be there.

So, that is why these i and j need to be here this is the shorthand notation Einstein notation that is why write it, but you can check it now, if you make a square of this quantity in the left-hand side then in the right-hand side whatever you have you make a square and when you make a square you have the cross term or this multiplication the same term you are going to get a cross term.

And that term will be simply summation of you know $x_i\ x_i'\ x_j\ x_j'$ and the summation is over i and j there are double summations are there. So, now, this $(\vec{r} \bullet \vec{r}\,')^2$ is represented in this way. So, what about r^2 and r'^2?

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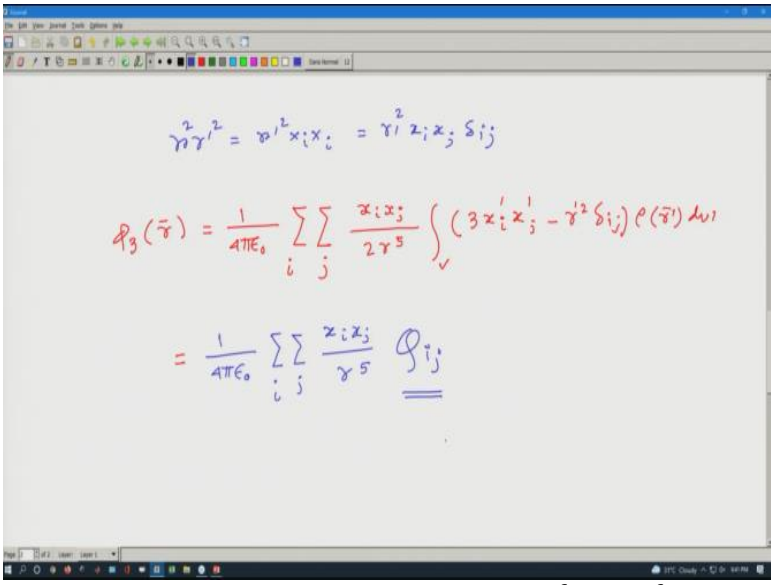

Now, r^2 and r'^2 this term is simply say if I write in terms of r'^2 it is r'^2 and r is x_i r^2 is x_i x_i. So, I can write r'^2 x_i x_j because already x_{ij} is there and δ_{ij} this is the way I can write it. So, my ϕ_3, which is a function of r is simply $\frac{1}{4\pi\epsilon_0}$ and if I write in component way component wise so, let me write in an explicit way i and j.

And then I have $\frac{x_i x_j}{2r^5}$ I take x_i x_j common because here in this term I have one x_i x_j and here in this term I find that I can write in x_i x_j with the delta function and then I take it common. So, this quantity then integration I take x_i x_j out of the integration because this is not if this is not prime and the integration is over prime. So, it should be $3x_i'x_j'$ and $-r'^2$ rather and then δ_{ij} bracket close $\rho(\vec{r}')$ as usual dv' this is over dv.

Now, I am in a position to you know define the quadrupole moment and you can see that there are 2 primes that are associated with that to define the component of the quadrupole moment. So, that means it is not a vector it should be a tensor quantity. So, this is sorry this is p_i so, i summation of j $\frac{x_i x_j}{r^5}$ and rest of the term I write as Q_{ij} this is the quadrupole term, this Q_{ij} and you can see that this comes up to be the tensor quantity.

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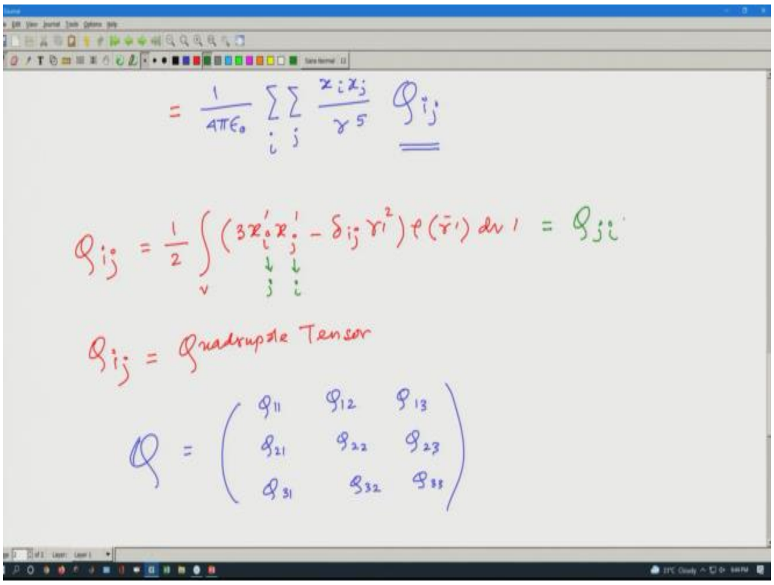

So, now, Q_{ij} will now consider it only this term is equal to half of integration over the volume $(3x_i{}'x_j{}' - \delta_{ij}\, r'^2)\, \rho(\vec{r}\,')$ dv′. So, this is the value of the quadrupole tensor and this is the value this the ij components. So, there should be 9 component like this. So, this is Q_{ij} is our quadrupole tensor so if I write the total value.

So, Q_{ij} if I write my total Q that is this one, Q_{11} Q_{12} because i can take 1, 2, 3, j can take 1, 2, 3. So, I should have this value Q_{21} Q_{22} Q_{23} Q_{31} Q_{32} and Q_{33}, but also I have another important information and that is the symmetry, the symmetry is you can see that $Q_{ij} = Q_{ji}$ that means, if you just replace i to j here if I just replace i to j and j to i then there will be no change actually, this will be the same like the before so, Q_{ij} and Q_{ji} should be same so, that quantity is Q_{ji} as well.

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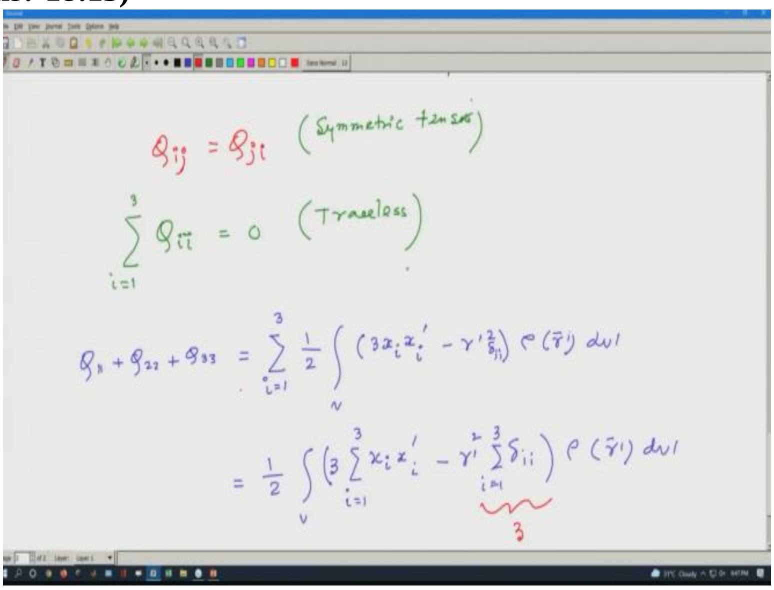

Not only that, this is one important thing, so, this is a symmetric tensor. Another important thing if I now check the trace of this matrix whatever the matrix we are having right now, that is Q_{ii} sum over i 1, 2, 3 then that value is 0 so, this is traceless. So, I can prove that because I mean let me quickly show it. So, $Q_{11} + Q_{22} + Q_{33}$ these are the diagonal terms and if I add all these diagonal terms, then that quantity is called trace and this value has to be 0 here.

So, if I add over all i, so, I only take the value when i = j and then sum over i. So, i and I take this quantity, which is you know $3x_i$ and I take x_i because i and j are same here. So, this is prime – r'^2 it will remain like this and δ_{ii}. So, that is 1 and then I have $\rho(\vec{r}\,')$ dv'. So, now I make a summation over i 1, 2, 3. So, when I make a summation i 1, 2, 3 what you will normally expect is let us take half outside and let us put this integration inside.

So, then you will have 3 and this integration is over only this term. So, it should be $x_i\ x_i'$, i goes to 1, 2, 3 and then minus let us put δ_{ii} here for convenience and then r'^2 and the sum over δ_{ii} again i can go to 1, 2, 3 and over $\rho(\vec{r}\,')$ dv'. So, this quantity again we calculated earlier this should be 3 and what is this quantity?

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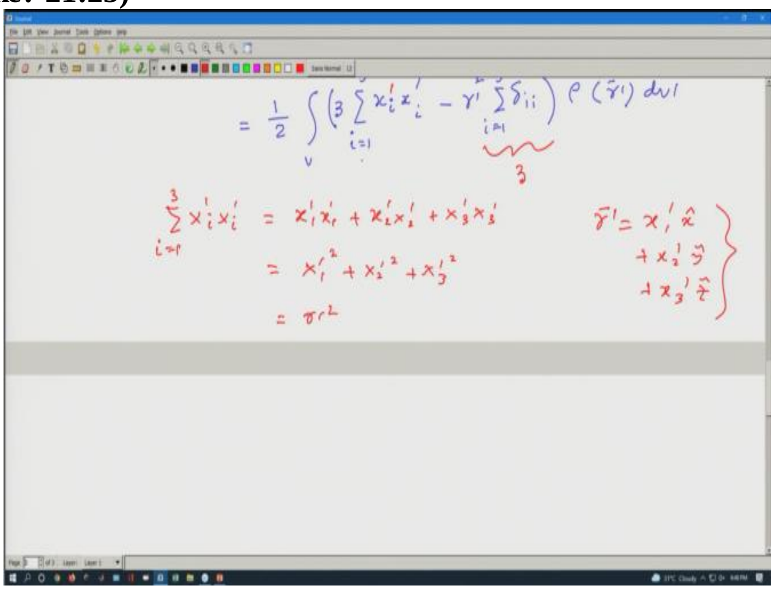

Summation of $x_i\ x_i'$ sorry it is both the cases it is prime I missed this prime here. So, x_i' multiplied by x_i' is 1, 2, 3 is simply $x_1\ x_1' + x_2\ x_2' + x_3\ x_3'$ this is nothing but $x_1'^2 + x_2'^2 + x_3'^2$, which is nothing but r'^2 because r' prime is $x_1'\hat{x} + x_2'\hat{y} + x_3'\hat{z}$.

So, after having that we are almost done because this quantity comes up to be $3r^2$ and already 3r is here.

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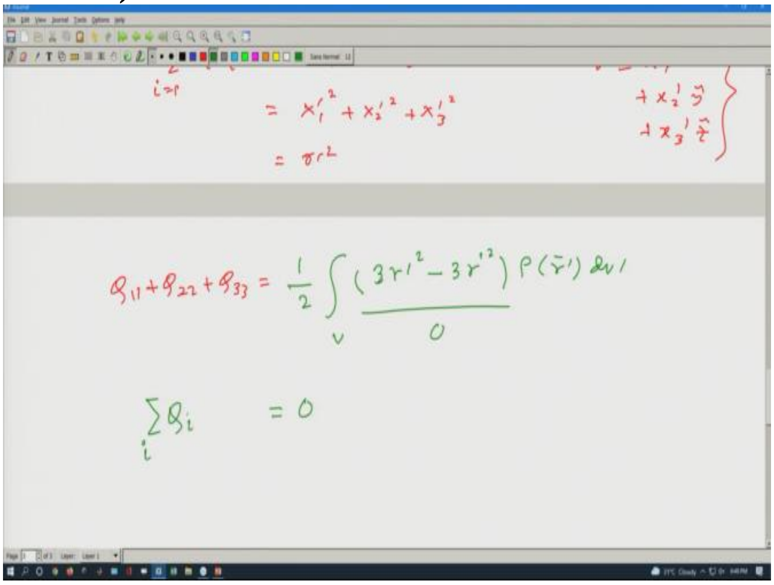

So, eventually what I get is Q_{11} + Q_{22} + Q_{33} is equal to right-hand side half integration ($3r'^2$ -3 r'^2) it is a very important minus sign so, I should not mess up here and then bracket close $\rho(\vec{r}\,')$ dv' and this quantity is simply 0. So, that tells us that the summation over i Q_{ii} is 0 that means the tensor is simply traceless whatever the quadruple tensor that you have.

So, whenever you calculate the quadrupole moment this tensor should have this matrix should have a it is a traceless matrix the trace should not be a non 0 it is 0 and also it is symmetric. So, there are many relation between the Q_{11} Q_{12} Q_{13} once you find 1 value then they are not independent. So, there are the relation 2 very important relation we are having that they are symmetric and traceless.

When you calculate then this this information, will going to help us. So, let us jot down what we have so far, because this is important. So, let us jot down what we have so far in last few classes.

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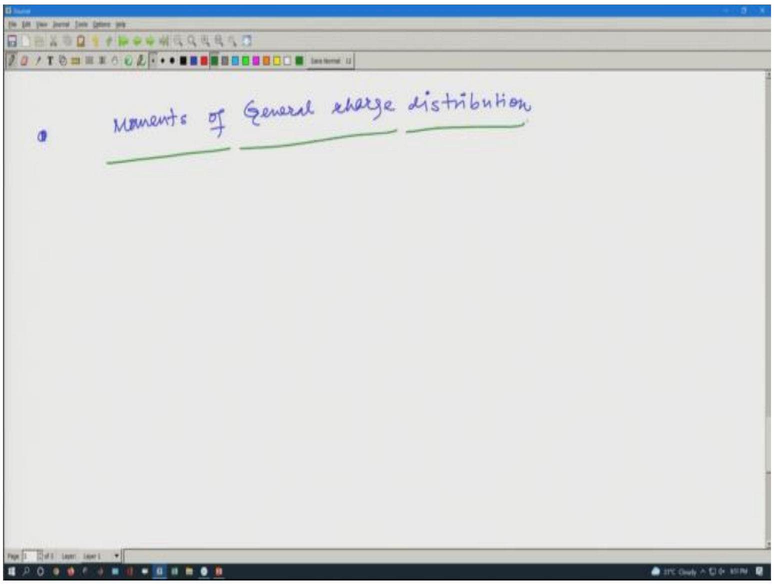

And then we are going to proceed we maybe calculate if time permits today. We will go to calculate the quadrupole moment for a given charge distribution so, moments so, what we find is moments of general charge distribution. So, I have 2 kind of we discuss about 2 kinds of charge distribution one is the for first case.

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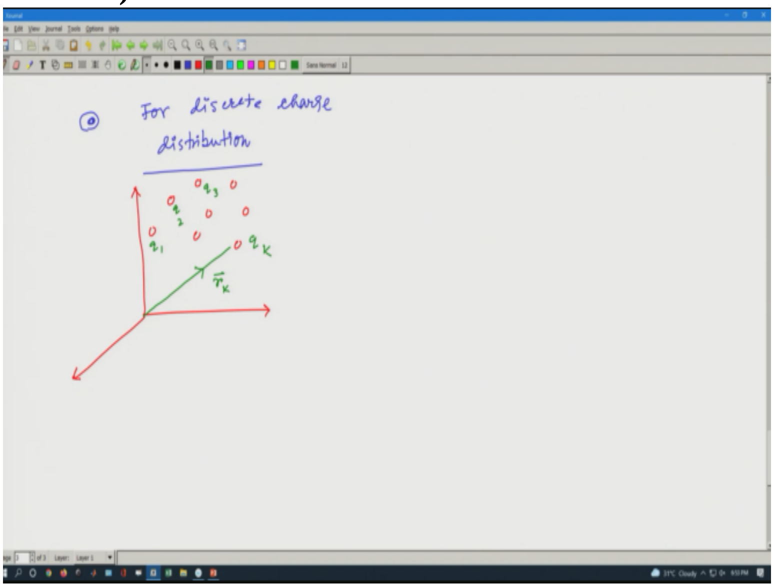

It is say for discrete charge distribution so, for the discrete charge distribution if I visualize I should have a coordinate system like this and I have the charge particle q_1 q_2 q_3 q_4 etcetera q_k like randomly distributed. So, this is my coordinate system and so, this is the q k^{th} charge and the location is r_k similarly, you have q_1 q_2 q_3 and so on but location of this is r_k. So, if that is the geometry of the charge distribution then one by one let us write.

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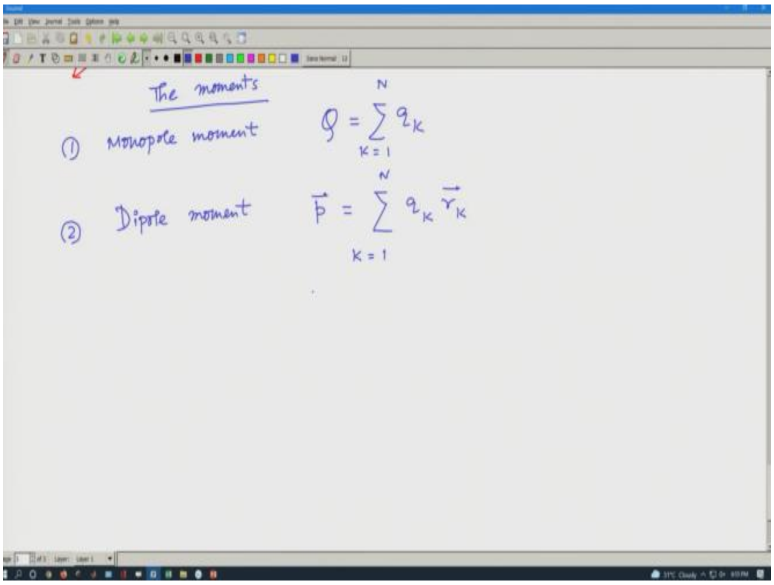

So, the moments so, first I should have the monopole moment so, in monopole moment what do we find that q is equal to sum over q_k, k goes to 1 to N that is nothing but the total charge that is the way we define the monopole moment. Second thing for this discrete charge this is the way we define second thing is dipole moment. The dipole moment p this is the way we write dipole is defined by some over k = 1 to N and q_k r_k this is the way we define the dipole moment.

Please mind it the dipole moment now defined for a system of charges, this is not 1 charge, this is N number of charges is there and still I can define the dipole moment. So, if you have an impression in your mind the dipole is only consists of 2 plus and minus things. So, this is a one kind of dipole but in general, if you do not have if you have a charge distribution still I can have the dipole moment and you can check it for 2 values whether this general definition is fine or not?

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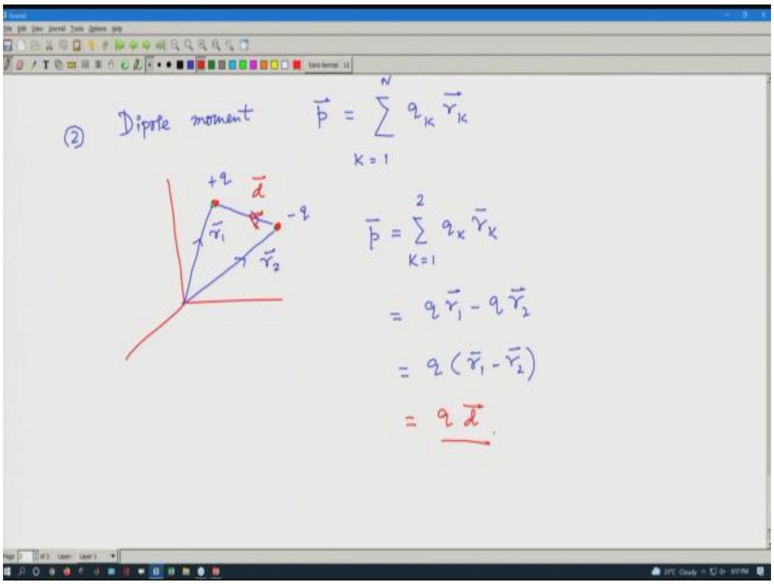

So, let me do this so, that there should not be any confusion. That suppose you have to charge equal and opposite that is the condition you need to put. So, let us have the coordinate say, this is $\vec{r}_2$ and this is $\vec{r}_1$ and this charge is say you know that this charge is say -q and this is +q. So, according to this, I have $\vec{p}$ = summation of k = 1 to 2 because 2 charged particles there and then q_k and then r_k that is so, that means, I have $q\vec{r}_1 - q\vec{r}_2$.

So, this is q ($\vec{r}_1 - \vec{r}_2$) and $\vec{r}_1 - \vec{r}_2$ is this vector. So, it is from opposite direction. So, it is from minus to plus so, this is the direction. So, I should have if this is my $\vec{d}$ then it is simply $q\vec{d}$ that is the general definition we had that the separation between 2 charged particle 2 charge quantity having equal and opposite charge 2 point charge having the equal and opposite value and the distance between these 2 is if it is $\vec{d}$.

Then we have so, from here you can see that even from this definition I can get this and finally, for discrete charge I can have the definition for quadrupole moment.

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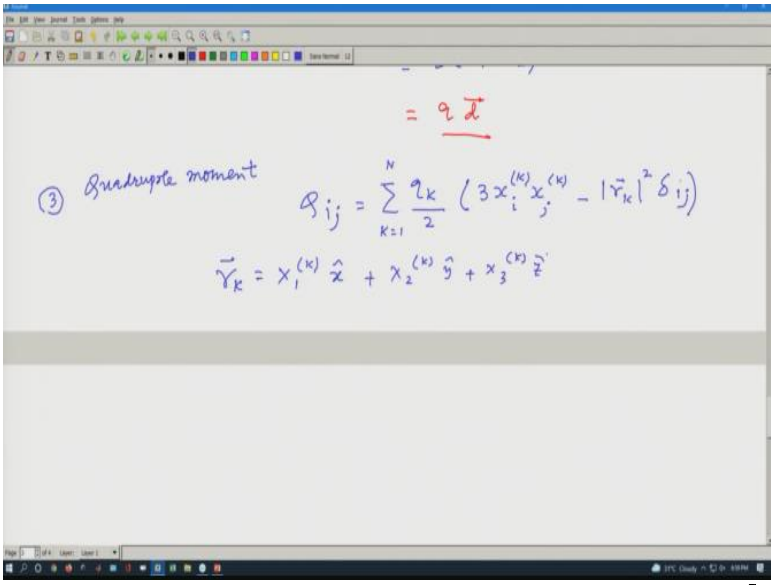

So, quadrupole moment is Q_{ij} for discrete charge it should be summation $\frac{q_k}{2}$ the sum over k, which is say 1 to N particle N charged particles and then I have $3x_i$ this is the k^{th} charged particle x_j this is for k^{th} charged particle and then $-|\vec{r}_k|^2$ and δ_{ij} that was already there. So, $\vec{r}_k$ is defined like x_1 this is the k^{th} r so, that is why I put a $\hat{x}$ + x_2 $^{(k)}$ $\hat{y}$ + x_3 $^{(k)}$ $\hat{z}$ this is the way it is defined. Now, we will write down what happened for continuous charge distribution.

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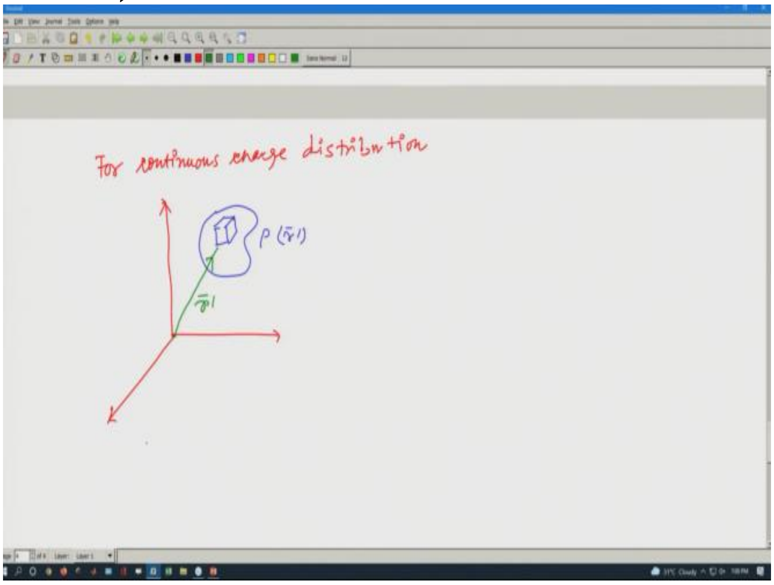

And now for continuous charge distribution so, we have a coordinate like this and here this is the charge distribution we have this is defined by $\rho(\vec{r}\,')$ and this is the location of that small section this $\vec{r}\,'$, so, now the monopole moment, quickly.

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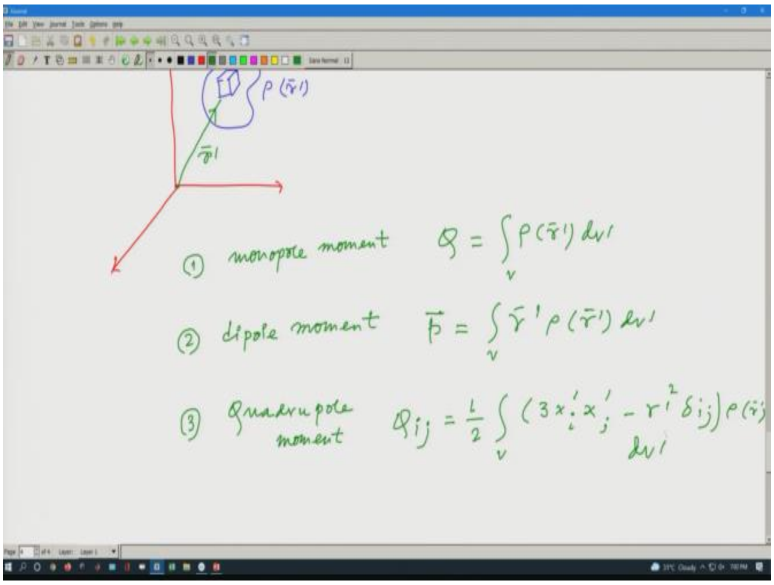

First the monopole moment will be defined by q is equal to integration of $\rho(\vec{r}\,')$ dv', which is nothing but the total charge, second is dipole moment, which is $\vec{p}$ and that we know that is $\vec{r}\,'$ then $\rho(\vec{r}\,')$ dv' and finally, the quadrupole moment, which we discussed today in detail so, quadrupole moment in continuous case, it is Q_{ij} that is half of sometimes in few books you find this half is not there.

But anyway 3 x_i' 3 x_j' – r'^2 δ_{ij} and then I have $\rho(\vec{r}\,')$ dv'. So, this is the overall you know jest of the moments for discrete charge and for continuous charge distribution, so, I do not have much time today to discuss further. So, in the next class maybe I like to discuss about 1 problem, where we calculate the quadrupole moment exploiting this expression, which is now is there in the screen.

And then maybe we discuss more about I mean, once you do the problem, then you can understand that how the quadrupole moment can be calculated. And then we will be going to end our discussion on this moment. But before ending we will also show that how, as I mentioned that when the charge is equal to 0, the dipole moment does not depends on how the dipole moment depends on this coordinate for total charge equal to 0.

So, with that note I would like to conclude today's class. So, hope you can able to solve a few problems. And then based on these formulas, and in the textbooks, you have many problems and please solve these problems. And with that note I would like to conclude today, thank you.