# Foundation of Classical Electrodynamics
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## Lecture – 35
## Monopole and Dipole Moment

Hello students to the foundation of classical electrodynamics course. So, under module 2, today, we have lecture number 35, where we will be going to discuss about the monopole and dipole moment.

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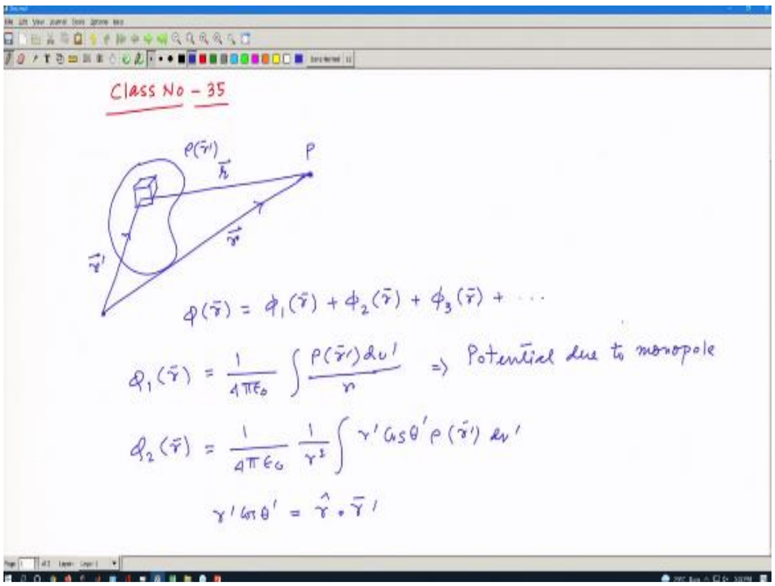

So, today we have class number 35. So, let us remind what we have done in the last class. So, if we have a distribution of the charge and take a small element here from that distribution with some origin and this point was say $\vec{r}\,'$ and we had another point $\vec{r}$ and try to find out the potential at that point. So, this point was P. So, try to find out the potential at this point P and that charge distribution here is defined by this charge density ρ, which is a function of $\vec{r}\,'$.

And when we elaborate that this was our we find that our potential at the point $\vec{r}$ was simply comes out to be the summation of the contribution of the monopole, contribution of the dipole potential due to the dipole, potential due to the quadrupole we will discuss this again and so on. Where this $\phi_1$ as I mentioned simply $\frac{1}{4\pi\epsilon_0}$ then $\int \frac{\rho(\vec{r}\,')dv'}{r}$, this is potential due to the monopole.

Now, similarly, we can write the potential $\phi_2$, the contribution due to the potential the contribution due to the dipole term and this term was this and then $\frac{1}{r^2}$ let us put this r outside integration then r and then ρ $\hat{r}$ not $\hat{r}$ there is a cos term. So, let me write down that first it was r then cos θ' and $\rho(\vec{r}')dv'$. Now, $\vec{r}'$ cos θ' we wrote it $\hat{r} \cdot \vec{r}'$.

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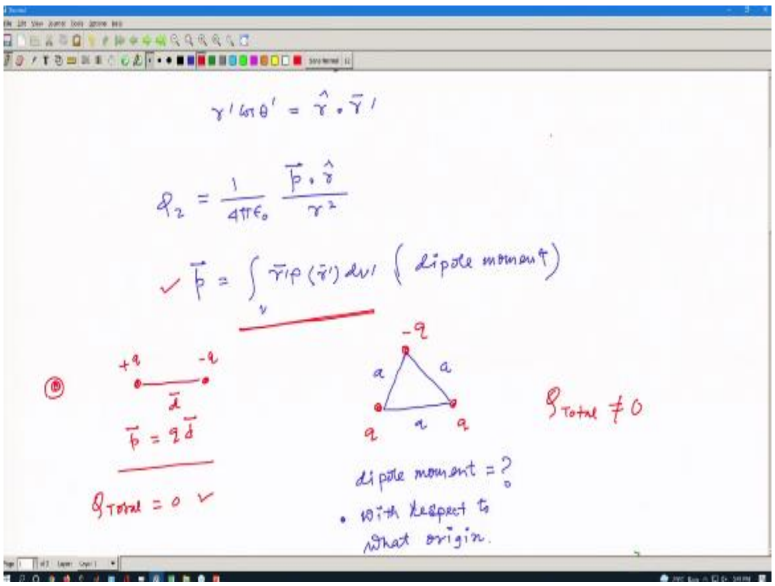


That leads to $\phi_2$ as $\frac{1}{4\pi\epsilon_0}$ and then I can write $\frac{\vec{p} \cdot \hat{r}}{r^2}$ where, $\vec{p}$ was the dipole moment defined as $\int \vec{r}'\rho(\vec{r}')dv'$ over this volume. So, that was the dipole moment term, so, that was last day's work. So, last day's task. So, now, today we will be going to elaborate the concept of dipole and try to by using few important examples. So, normally we know the dipole is generally defined as 2 equal and opposite charges with a separation say d.

And the dipole moment what is the dipole moment? The dipole moment is $\vec{p}$ = q $\vec{d}$ that is the very standard result and normally that is the way when you were in school the dipole was introduced. Now, I also mentioned that dipole can also be considered with the fact that this a system of charge having total charge 0 but this quantity, which is written here this integration $\vec{r}$ dot ρ dv this is non 0 then that can also be considered as a dipole.

So, in that case it is not necessarily that 2 charges will be there, there will be more charges and still we can find out the dipole. One example is suppose we have a charge distribution this is also a discrete charge distribution like this. So, one charge is -q and this is say q and this is another q. So, this is a system where the charges are distributed like this where all these are a. Now, if the question is asked what is the dipole moment of the system? Then the answer will not be that simple because it depends on the origin.

So, we need to ask that under I mean with respect to what origin that is the question one can ask what with respect to say with respect to what origin later we will see a very important thing that when the total charge is 0 the value of the dipole moment does not depend on the coordinate system. But if the dipole if the total charge is non0 for here, the q total charge q is non0 I have +q, -q and +q. So, that gives me a total +q, which is not equal to 0.

In that case in that kind of distribution, if you want to find out the dipole moment, it depends on the coordinate system. So, this is independent of the coordinate system because here the total q total was 0. So, independent of the coordinate system but this is not we will be going to prove that. So, let us do one problem and after having the idea about the dipole that it does not contain only the 2 charge but a system of charge. The thing is that you need to calculate this quantity and check what should be the dipole moment.

And another important thing that we understand here that if the total charge is not equal to 0 for example, here we have 3 charges that means the total charge is not equal to 0 and then the question is asked that what should be the dipole moment? The answer is not very straightforward, because in that case you need to know that with respect to what origin I need to calculate that because it depends on the coordinate system. Now, it depends on the coordinate system because the total charge is non0. Let us do one problem then it may be the concept with this problem maybe the concept will be clear.

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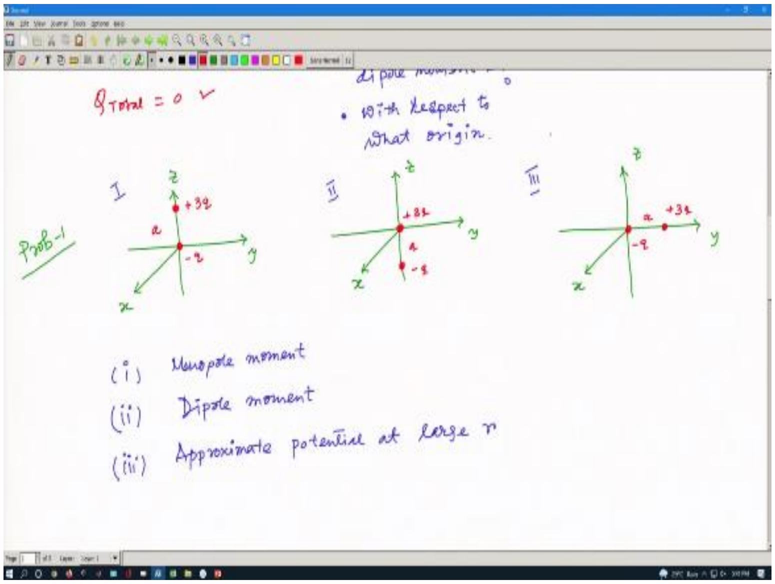


So, problem 1 so, I can have a coordinate system like this, this is the coordinate system that is given and 3 distribution of the charge is there. So, in first case, case 1 I have a charge sitting in the origin and another charge sitting here and the distance between these 2 charges is *a* where the charge, which is at the origin is -q and the charge that is here is +3q again you can see that the total charge is non0. So, that means, if you calculate the dipole moment, so, definitely the coordinate coming to the picture so, this is case 1.

Case 2 I have another similar kind of coordinate system let me define the coordinate say this is x axis, this is y axis and this is z axis so, this is x, this is y and this is z in the next case the charged particle the 2 charges are sitting like this. So, I have 1 charge here and another charge say here this is +3q and this is -q and this length from here to here is *a* like before. So, and finally, I have another distribution 3 let me first draw that coordinate this is x, this is y and this is z.

And the charge now is distributed like here one charge and over x axis I have another charge. So, this charge is -q and this is +3q as usual the distance between these 2 is *a*. So, these are the 3 system that is given to us and the question is we need to find (i) the monopole moment, (ii) dipole moment and (iii) the approximate potential at large r at far distance. So, now, when we calculate the potential at large r we will be going to consider only the contribution of the monopole and dipole because here you see that when we distribute the $\phi$.

So, that was the contribution of the monopole, that was the contribution of the dipole and so, on. Here for this system, we just try to find out the potential at large r under the consideration that only the monopole and dipole contribution are there.

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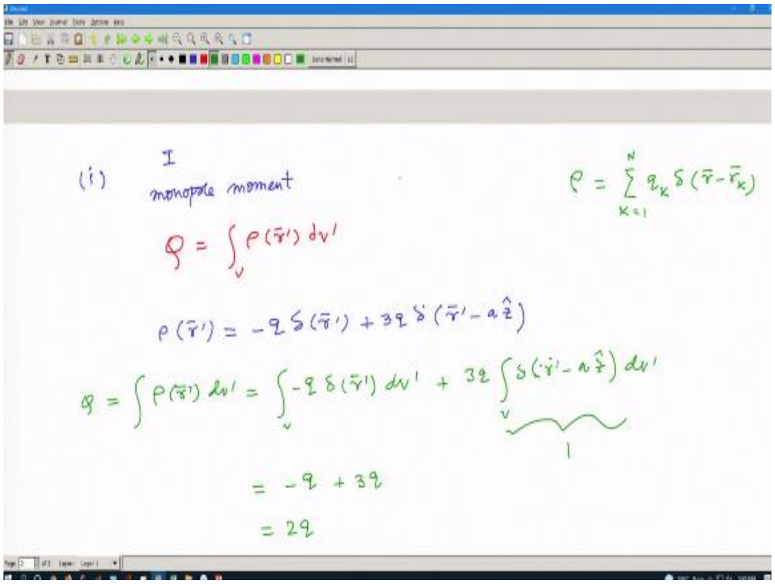


So, first let us calculate for case 1 the monopole moment. So, this is for case 1 and I try to find out the monopole moment. So, monopole moment in general is nothing but the total charge. So, Q is the monopole moment and it defined like $\rho(\vec{r}')dv'$ this is the way we defined. So, this is nothing but the total charge. So, very easily you can see that in all the cases the total charge if you calculate here it should be 2q and here also it should be 2q because +3q -q, it should be 2q and all the cases these values are same.

But we will be going to exploit this expression because we know that for point charge how the ρ's are how the charge density are defined for the point charge, so that we are going to exploit here. So, the ρ that means the ρ for the first case we know because there are a district charge, so, -q charge is sitting at the origin, so, $-q\ \delta(\vec{r}')$ that will be the first term and another term is +3q and it is sitting over the z axis at the distances $a$. So, I simply write $\delta(\vec{r}' - a\ \hat{z})$.

So, that should be the location of this thing and if you add these 2, so, that should be our you know the charge density for the discrete charge if you forget, so, let me write it here once again. So, the charge density of the discrete charge was something like summation of $q_k$ k 1 to say N number of charges are there and $\delta(\vec{r} - \vec{r}_k)$ that was the formula that we derived. So, that is the charge density for the district charge I am just going to use this one the summation sign take care this plus. So, now, simply my Q is integration of $\rho(\vec{r}')dv'$ and ρ is this one.

So, it should be integration of -q $\delta(\vec{r}')dv'$ + 3q I can take this 3q outside integration and integration of $\delta(\vec{r}' - a\,\hat{z})$ dv', but I integrate over entire volume. So, this quantity both the cases this quantity should be 1 and here if I take q outside and this quantity will be 1. So, simply I have the same old thing just q + 3q then it should be 2q this is for case 1 and I believe for case 2 and case 3 you can calculate and find out what is the monopole moment I am not going to do that.

Because this is just a repetition of the same thing you just need to find out the delta function and you can do that if you use the monopole definition this. Otherwise monopole is nothing but the total charge you can easily calculate. So, the value should be 2q for all the cases.

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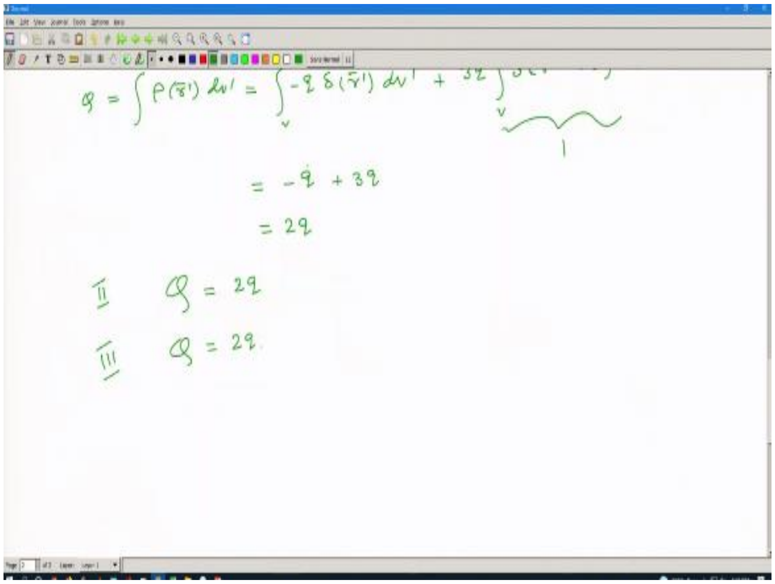

So, for case 2, the monopole moment again comes out to be 2q, I am not calculating that because calculation is exactly same and for 3, Q should be again 2q. But the interesting thing is to calculate the dipole moment, which is important and then this kind of definition will be very much useful.

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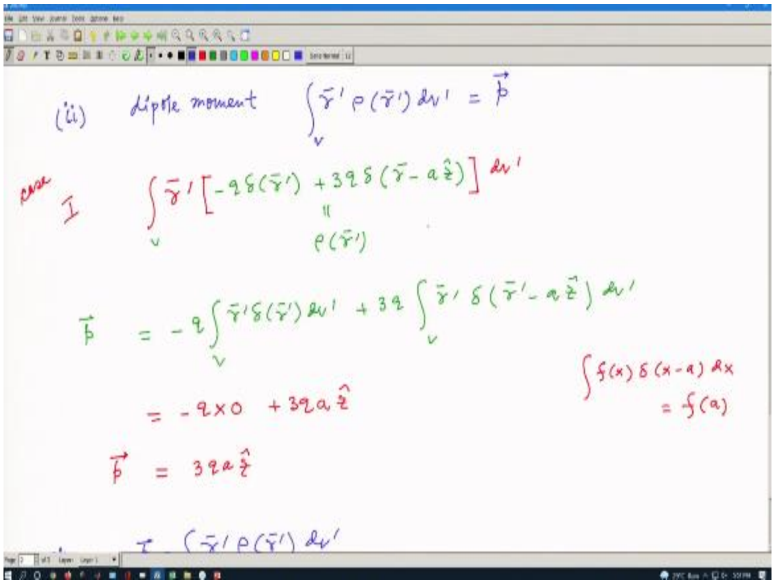


So, now, the problem (ii) that calculation of this is problem (ii), where we calculate the dipole moment. So, dipole moment is simply integration $\vec{r}'$ and then ρ ($\vec{r}'$) and dv' this is the definition of the dipole moment over entire volume. So, in first case what happened in first case, because I find out the ρ in the first case. So, for case 1 because case 1 means for this distribution here, this is 1 for this distribution, I should have my expression like this. So, it should be integration $\vec{r}'$ and then ρ I figured out so, I just write this the value of the ρ.

It was -q $\delta(\vec{r}')$ + 3q $\delta(\vec{r}' - a\,\hat{z})$ the location of the charge, so, that value I put and then finally, I have dv', so, this quantity is ρ($\vec{r}'$), now, I execute this part over entire volume. So, my $\vec{p}$ in this case is simply I can take -q outside and $\int \vec{r}'$ $\delta(\vec{r}')$ dv' and another integration is +3q $\int \vec{r}'\, \delta(\vec{r}' - a\,\hat{z})$ dv' over v. So, now, you can see that this quantity should be 0 because I need to put $\vec{r}'$ 0 as per the rule of the delta function.

I am going to use this rule if you forget again let me write it here $\int f(x)\delta(x-a)dx = f(a)$ very famous identity, so, that I am going to use here So, the first term is simply -q into 0 and second term is +3q in place of r, I just simply put $az$ so, it should be $a\,\hat{z}$. So, my answer for case 1 is simply 3q$a\,\hat{z}$ that is the dipole moment I have for this distribution, now, the distribution has changed and total charge is not equal to 0.

So, as I mentioned the value of the dipole moment very much depends on the coordinate system. So, for this coordinate system I have 1 dipole moment and this case even though the charge looks same, but the location is changed obviously, the dipole moment should have some different values. So, let us find out the dipole moment for case 2.

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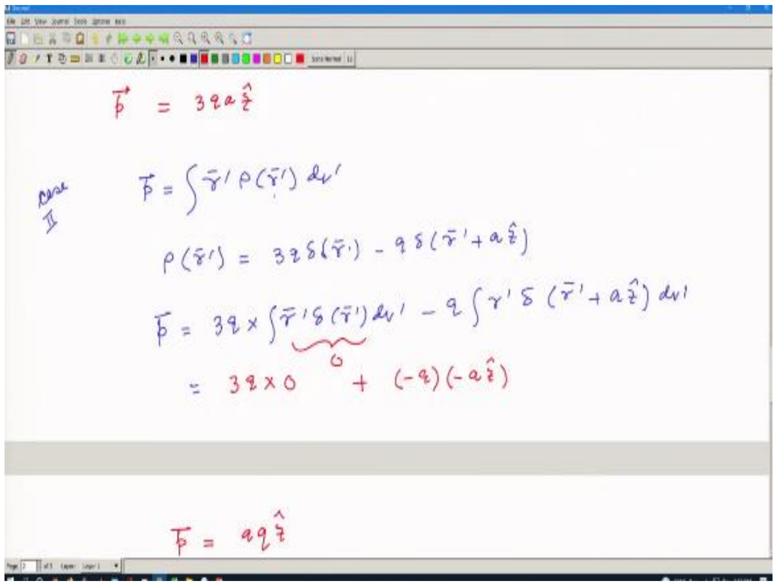

So, for case 2 again I will go to use the same notation same formula that $\vec{r}'$ you know ρ($\vec{r}'$) dv'. So, I need to find out ρ($\vec{r}'$) for this case. So, ρ($\vec{r}'$) for this case is simply let us see what happened here. So, 3q is sitting in the origin and -3q is sitting in the minus direction of z and that means, I simply have ρ = 3q δ($\vec{r}'$) and -q δ($\vec{r}'$ +$a\,\hat{z}$) that is my ρ. And if I put this ρ here in this equation like before, I simply get my $\vec{p}$ to be 3q because 3q is associated to the delta.

So, 3q multiplied by $\int \vec{r}'$ δ($\vec{r}'$) dv'. So, this quantity simply gives me 0 another term is -q $\int \vec{r}'\,\delta(\vec{r}' + a\,\hat{z})$ dv'. So, I can simply have here this quantity is 0, so, I have 3q multiplied by 0 plus this quantity is -q multiplied by - $a\,\hat{z}$ so, that means, for case 2, my the value of the dipole moment is simply $aq\,\hat{z}$. So, previously the value was how much $3aq\,\hat{z}$ now, it is $aq\,\hat{z}$.

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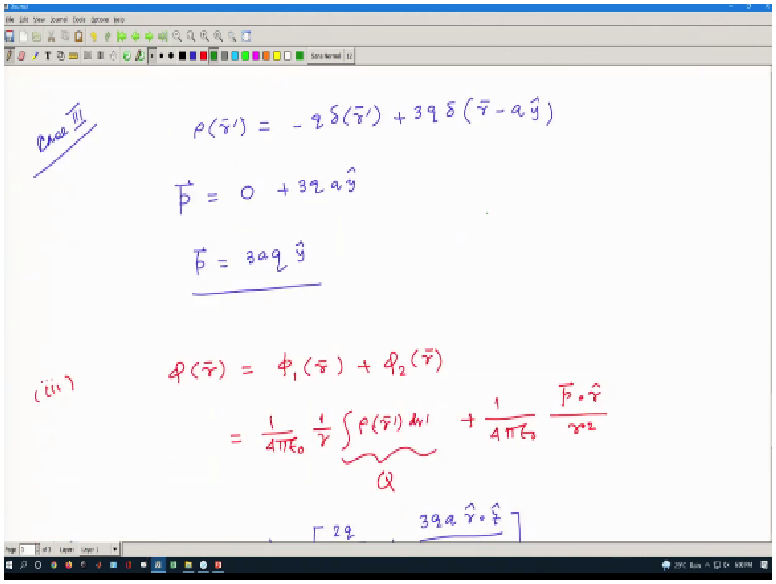


Now, we are almost there. So, let us do the case condition, for case 3, I have ρ so, let us first define the ρ($\vec{r}'$) for case 3 is if I go back so, if I go back here, so, this is a sitting plus minus q sitting in the origin and this is sitting over a y axis. So, I simply have -q δ($\vec{r}'$) because it is sitting at origin and 3q $\delta(\vec{r} \;\; - a\,\hat{y})$. So, from here I can definitely I can simply write my result and my result is -q it will be delta function.

So, it should be 0 contribution and it should be +3q and whatever the value I have here that is $a\hat{y}$. So, my final value is simply $3aq\;\hat{y}$ so, this is the value of the dipole moment for case 3 that way. Now, the next the third part of the problem that find out the approximate potential at large r so, I can write my potential ϕ($\vec{r}$) as a contribution of ϕ monopole and ϕ($\vec{r}'$) and ϕ dipole this is for monopole and this is for dipole I should not take all the higher order terms other higher order terms just.

So, it is simply $\frac{1}{4\pi\epsilon_0}$ and then $\frac{1}{r}$ then ∫ ρ($\vec{r}'$) dv' this is my monopole moment, which is total charge Q and plus $\frac{1}{4\pi\,\epsilon_0}$ and then I have $\frac{\vec{p}\bullet\hat{r}}{r^2}$.

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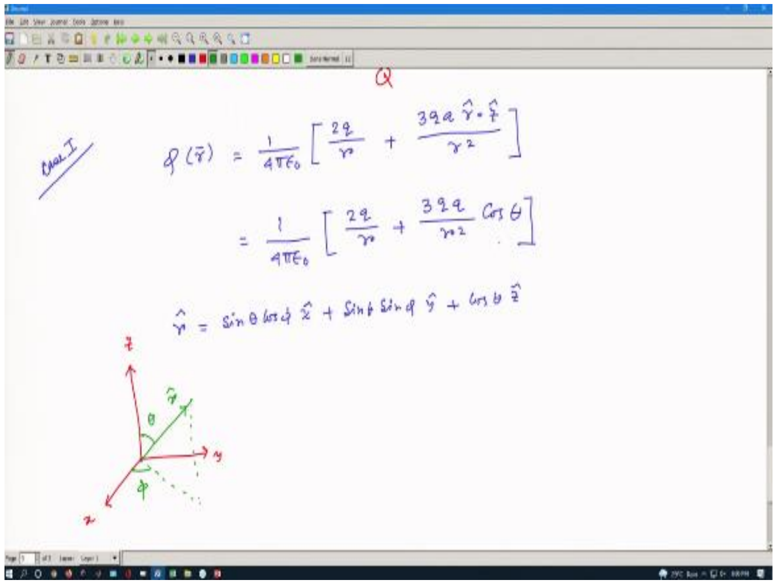

So, simply I can write this for case 1 I know what is a monopole and I know what is q and what is p so I simply have the result. So, my results should be $\phi(\vec{r})$ for case 1. It is I can take $\frac{1}{4\pi\epsilon_0}$ common and the total charge was $\frac{2q}{r}$ and plus the monopole the dipole moment I calculated and it was 3qa and then it should be $\frac{\hat{r} \bullet \hat{z}}{r^2}$. So, that is the contribution I should have now, $\hat{r} \bullet \hat{z}$ you can simply write $\hat{r} \bullet \hat{z}$.

You can simply write a cos θ if the angle between these 2 is cos θ the θ then $\frac{1}{4\pi\epsilon_0}$ [ $\frac{2q}{r}$ + $\frac{3qa}{r^2}$ and then we have cos θ] that should be the value of the potential now, again I am not going to do the calculation for case 2 because in case 2 it will be similar, but you should be careful enough in calculating because the dipole the direction of the I mean you should be careful to you know, find out this r for example, let me write down the r here.

So, the $\hat{r}$ is simply sin θ cos φ $\hat{x}$ + sin θ sin φ $\hat{y}$ + cos θ $\hat{z}$ why it is? Because you need to consider the coordinate system because the coordinate system it is there actually. So, you can see that this value is x, this value is y and this value is z in this coordinate system, if I draw a $\vec{r}$ like this, this is my $\vec{r}$ and the direction of the unit vector is along this direction and if I make a projection here, so, this angle is my φ and this angle is my θ.

So, according to this coordinate system this is my $\vec{r}$. So, that is why when we have $\vec{r} \cdot \hat{z}$, I have a cos θ here, because $\hat{z}$ is associated with the cos θ.

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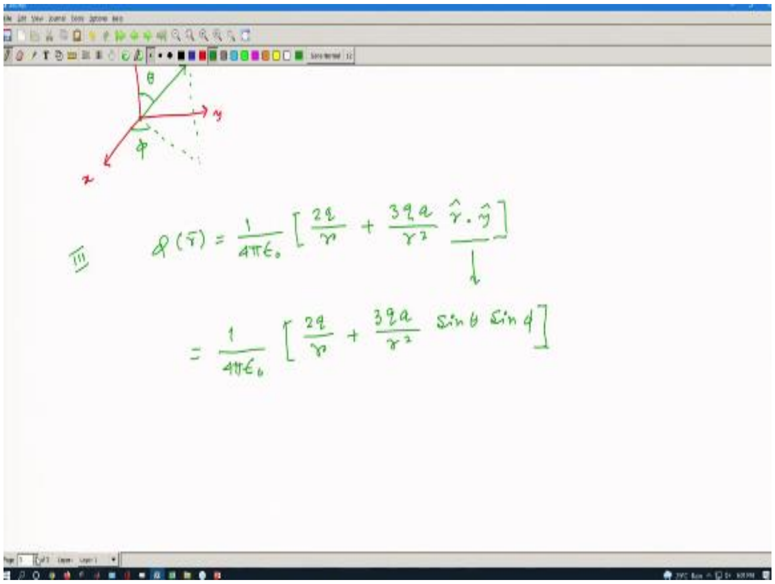

But for the case 3 let me write here for the case 3 the potential that you calculate is $\frac{1}{4\pi\epsilon_0}$ monopole contribution is simple, but the dipole contribution should be 3q$a$ that we figured out divided by $r^2$ and then I should have $\vec{r} \cdot \hat{y}$ because it is along y direction if you remember for case 3, if I check it is along y direction this one. So, now, $\vec{r} \cdot \hat{y}$ when you calculate, so, it should not be cos θ everything will be same.

So, my results so, let me write down the result. So, $\frac{1}{4\pi\epsilon_0}$ and it should be $\frac{2q}{r}$ and then $+\frac{3qa}{r^2}$ and this quantity should be sin θ sin φ. So, that should be roughly the potential. So, the next problem that I like to do quickly is this.

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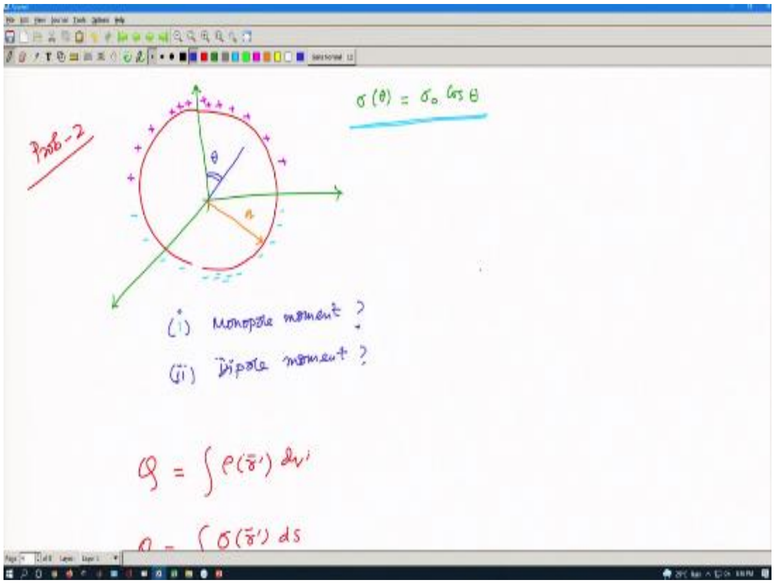


So, another important problem 2 so in problem 2 a spherical shell I have a spherical shell whereas put a coordinate system here and it has a charge distribution σ but that is a function of θ and that is $\sigma_0$ cos θ. This is the way the charge is distributed so, that means, if I have a value r here over this sphere shell then this is the angle θ. So, these are when the θ is 0, I should have the maximum value here and that here we have all $\rho_0$ that is the maximum value and the charges distributed like here.

But when θ is $\pi$ that is here, the value of the cos θ is -1. So, we have all the minus charge. So, that means, the charge is distributed in this way very, very unique kind of way. So, the positive charge let us put the positive charge here is distributed mostly here and gradually it is reducing and no charge here over this axis because when θ is 0 there is no charge over the axis I should not put so, this is a charge free region. On the other hand, the negative charge let us put this color is concentrated here and gradually it is reducing.

And no negative charge is gradually it is reducing and no negative charge here. So, obviously, you can see there is a charge distribution here because as a surface charge is distributed in this form and the question is the question is find out the dipole moment of this system. So, find out so, first so, there are 2 parts of the question. So, the first question is; find out the monopole moment and second part is dipole moment. So, how you calculate the monopole moment?

The monopole moment is total charge simply and that is $\rho(\vec{r}')$ dv' that is my formula. So, here for this case, I should not have this because this is a spherical shell.

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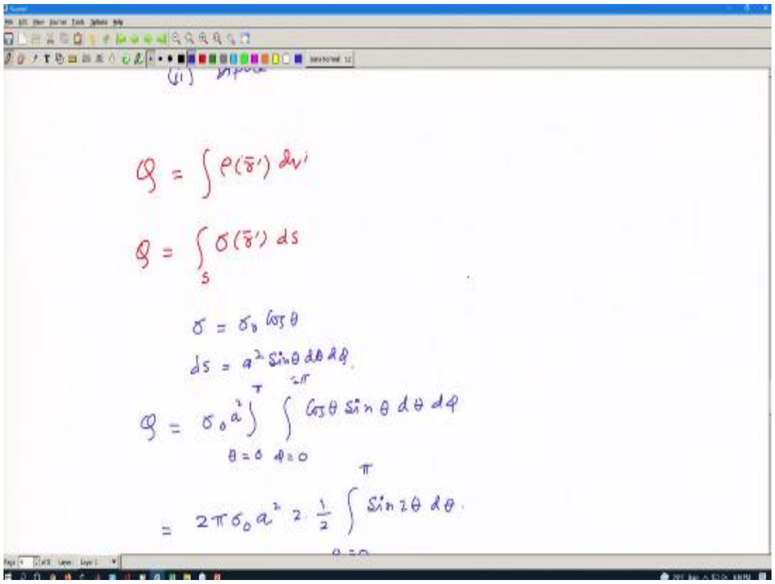


So, for this case my Q should be integration of σ that should be a function of some $\vec{r}'$ and $d\vec{s}$ that should be the not $d\vec{s}$ it is simply $d\vec{s}$ because this is not a vector quantity ds over the surface integral I just need to the surface you to make the surface integral. So, it is simple because ρ is given so, ρ is given as $\rho_0$ cos θ and my ds for spherical system is simply $a^2$ I need to mention the radius of this shell and that is $a$. So, it is simply $a^2$ and then sin θ dθ dφ.

So, now, if I calculate my Q it should be simply $\sigma_0$ integration θ goes to 0 to $\pi$, integration φ goes to because there is φ. So, φ goes to 0 to $2\pi$ and then I have cos θ and then $a^2$ is there. So, I should put $a^2$ outside and then we have sin θ and dθ and dφ. So, I have $2\pi$ because dφ is there $\sigma_0$ $a^2$ out and I need to integrate. So, another 2 I can have 2 multiplied by half and then my integration over θ will be θ say 0 to $\pi$ and this quantity cos and sin become sin 2θ dθ.

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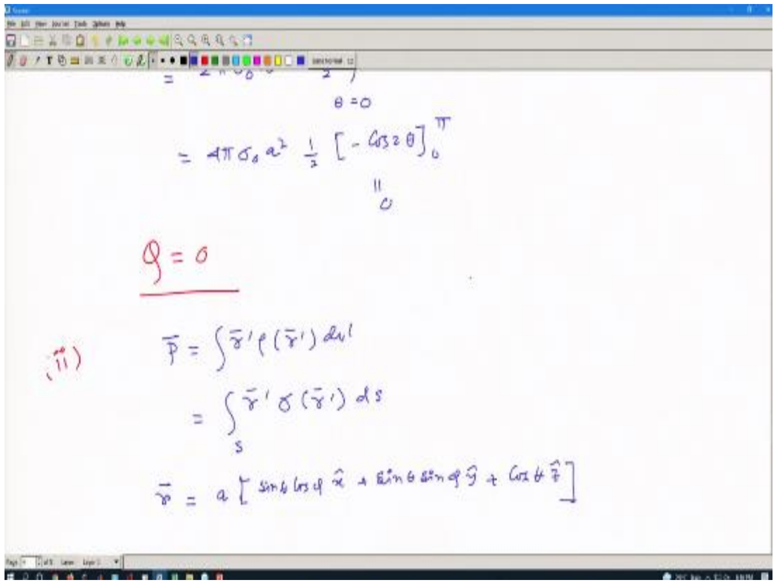


And this is $4\pi\ \sigma_0\ a^2$ and the rest of the term is half. And if I do the integration is - cos 2θ and over 0 to $\pi$ and we know that this integral integration is simply 0. So, I have my total charge here 0, that means monopole moment is 0. And it is expected because the way the charge is distributed by symmetry, you can see that there should not be any effective charge whatever it is positive side, whatever is the upper side of the hemisphere that is the lower side.

Now, the next thing is important that what should be the dipole moment. So, case 2 what should be the dipole moment. So, the dipole moment of the system as per our notation, it should be written like this. Again, since it is sphere, it should be simply $\sigma(\vec{r})$ and ds over the surface integral and $\vec{r}$ now I need to write because it is a spherical coordinate system. So, $\vec{r}$' or simply I write $\vec{r}$ is over the sphere.

It is $a$ sin θ cos φ $\hat{x}$ + sin θ just before I do it, I did it sin θ sin φ $\hat{y}$ + cos θ $\hat{z}$ that is the value of the $\vec{r}$. So, now, I will just put this value of $\vec{r}$ and the σ is already given.

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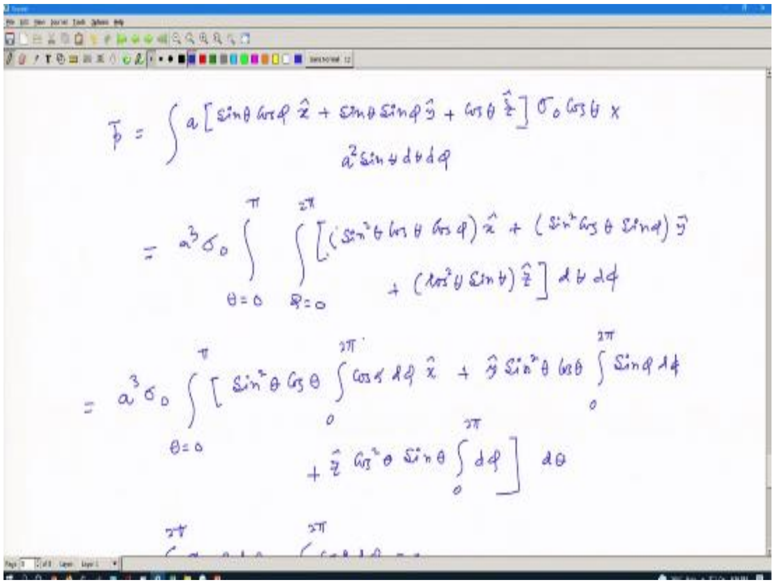

So, simply my $\vec{p}$ will be integration of $a$ the whole quantity that is sin θ cos φ x unit vector + sin θ sin φ $\hat{y}$ + cos θ $\hat{z}$ that is my $\vec{r}$ and ρ that is given $\sigma_0$ cos θ and then ds, ds again and calculate and it is $a^2$ sin θ dθ and then dφ so, this is the entire integration I need to now integrate. So, let us take this $a^2$ outside it becomes $a^3$ because another $a$ sitting here, so, simply it become $a^3$ and $\sigma_0$ and now, the integration has so, different parts, so, let me do one by one.

So, θ = 0 to $\pi$ and then I have φ = 0 to $2\pi$ and then let me do one by one. So, it is $\sin^2$ θ cos θ cos φ and that is unit vector and then plus I have $\sin^2$ θ and then cos θ and sin φ this is $\hat{y}$ and plus $\cos^2$ θ sine θ $\hat{z}$ and over dθ and dφ. So, this is a lengthy integration, but very much durable. So, well I first need to calculate let me do that. So, $a^3$ $\sigma_0$ and if I do the integration for θ first θ = 0 to $\pi$.

And let us take $\sin^2$ θ cos θ out and I do the integration for φ only. So, it should be cos φ and dφ and $\hat{x}$ plus say $\hat{y}$ let me write the unit vector first then it is $\sin^2$ θ cos θ integration 0 to $2\pi$ and then sin dφ and finally, I have $\hat{z}$ $\cos^2$ θ sin θ integration 0 to $2\pi$ and dφ that is the total integration and then over that I have dθ.

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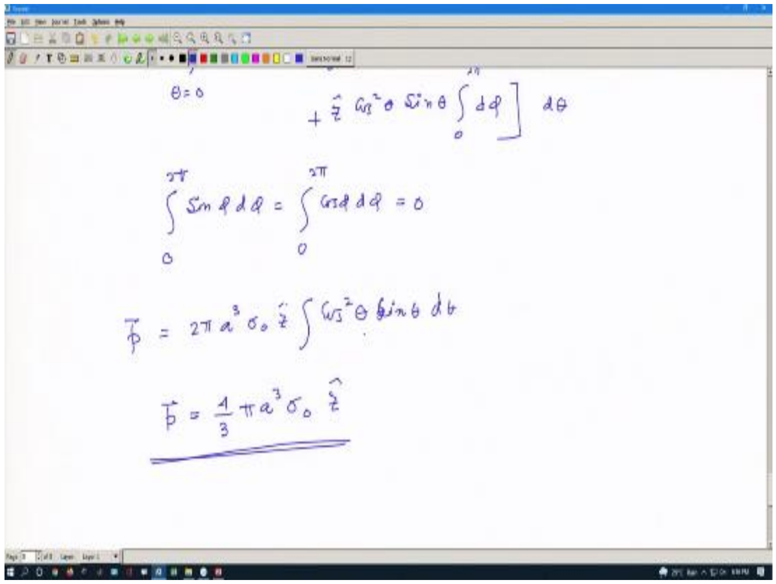

So, you can see that many terms will go to vanish here because integration 0 to $2\pi$ sin φ dφ that is integral from 0 to $2\pi$ cos φ dφ these 2 should vanish then only remaining term is this. So, it is simply $2\pi a^3$ $\sigma_0$ z direction integration of $\cos^2\theta$ and then sin θ is sin θ dθ. Now, if you do the integration you simply have so, I am not going to do the entire calculation please do this integration this is so, you will get a result, which is $\frac{4}{3}$ and then $\pi$ $a^3$ $\sigma_0$ along the direction.

So, you can see that so, almost all the calculation I have done here the problem is very interesting because we have a charge distribution here it is not a single charge or multi charge but this is a continuous distribution of the charge. And for this distribution of the charge you are asked to calculate the monopole moment, which is nothing but the total charge and it is not easy to find the total charge by symmetry however, you can see that the total charge should be 0.

But the important thing is to find out the dipole moment and in order to find the dipole moment you need to use the formula or the definition of the dipole moment in this integral form where the continuous charge distribution are here, little bit lengthy but very much doable calculation and hope you can understand this calculation how to do in today's class hope you understand how to calculate the dipole moment for discrete charge and continuous charge I give 2 very important problem and there are different kinds of problem you can find in different books.

And please practice that to find out the dipole moment and monopole moment for different charge distribution. So, thank you very much for your attention and see you in the next class.