Foundation of Classical Electrodynamics
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Lecture – 34
Multipole Expansion

Hello students to the foundation of classical electrodynamics course, under module 2 today we have lecture number 34 where we discuss the multipole expansion.

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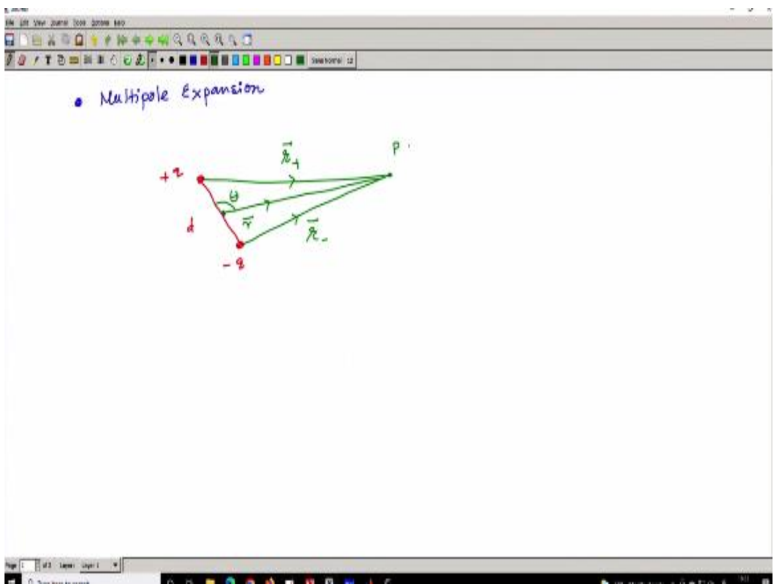

Today we have, so, our topic today is multipole expansion so, we have already discussed the dipole. So, let us quickly recap. So, if we have two equal and opposite charge +q and -q separated by a distance say d then at some point we can find the potential or the electric field due to this dipole. If this is $\vec{r}_+$ this is $\vec{r}_-$ and from here we have $\vec{r}$ say this angle is θ so, this is the P where we try to find out the potential and if I calculate, so, this calculation is already done.

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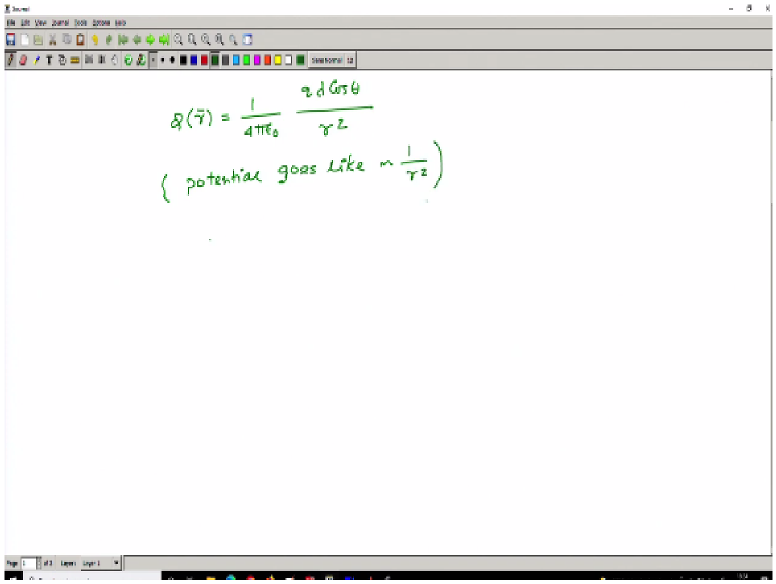

The potential due to the dipole at point $\vec{r}$ comes out to be $\frac{1}{4\pi\epsilon_0}$ and then $\frac{qd\cos\theta}{r^2}$ that is a very important thing. Now, here the potential goes to so, what we find here potential goes like $\frac{1}{r^2}$ form this is the way the potential changes.

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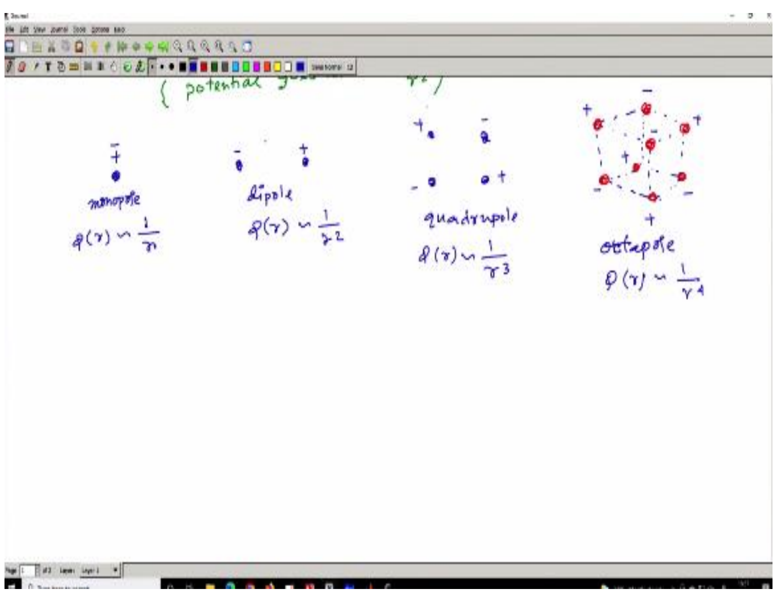

Now, if we look carefully for other cases like a monopole a single charge either plus or minus this is called a monopole the potential ϕ(r) varies like $\frac{1}{r}$ so, now, we have dipole where instead of having one charge you have 2 equal and opposite charges say minus and plus and this is called dipole the potential now varies as $\frac{1}{r^2}$. Now, we can extend these 2 other kinds of charge system like if I have 4 charge points having say plus and then minus than minus plus these kinds of charge distribution we call it quadrupole.

And here the potential will vary as $\frac{1}{r^3}$ also we can have like a structure with the charged particles sitting here 1 2 3 4 5 6 7 another charge is sitting here 8. So, this kind of system is say this is minus, this is plus, this is minus, this is plus, this is minus, this is plus, this is minus, this is plus, this is the charge distribution we can have and this is called octapole where the potential changes like r^4. So, different kinds of charge system you can consider and this is the way the potential is changing.

Now, the next thing is we will systematically find out the potential of an arbitrary charge distribution and later we find that it can be the contribution of monopole, dipole and quadrupole, octapole and so on. This is the way we can so, I have an arbitrary charge distribution here. So, let me do that.

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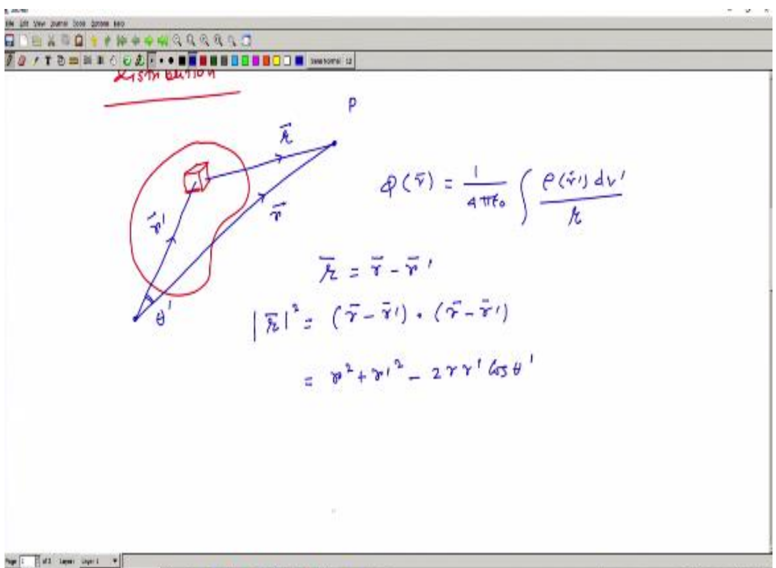

So, now, we will do the systematic expansion of an arbitrary localized charge distribution so, that thing so, I should have a region where we have charge distributed and take a small piece of that and our goal is to find out what is the potential for this small piece at some point P here and suppose I have origin here at some point so the location from the origin to this small portion is say $\vec{r}$ ' and this location of this is $\vec{\mathscr{r}}$ and from the origin to the point P, say, this is my point P this is $\vec{r}$.

So, we know that I mean this is a well-known problem suppose this angle is given and this is say θ' and this problem is a well-known problem, because we already know what is the potential for this charge distribution at point P and that is nothing but $\frac{1}{4\pi\epsilon_0}$ integration of the charge density of here which is ρ r' dv' divided by the distance from here to here which is r. Now, this $\vec{\text{Л}}$ is $\vec{r}$ - $\vec{r}$'. What is the magnitude of this quantity or if I want to find out the dot product.

So, it should be ($\vec{r}$ - $\vec{r}$') • ($\vec{r}$ - $\vec{r}$'), which simply gives us $r^2 + r'^2 - 2\, r\, r'$ angle between these 2, which is cos θ'.

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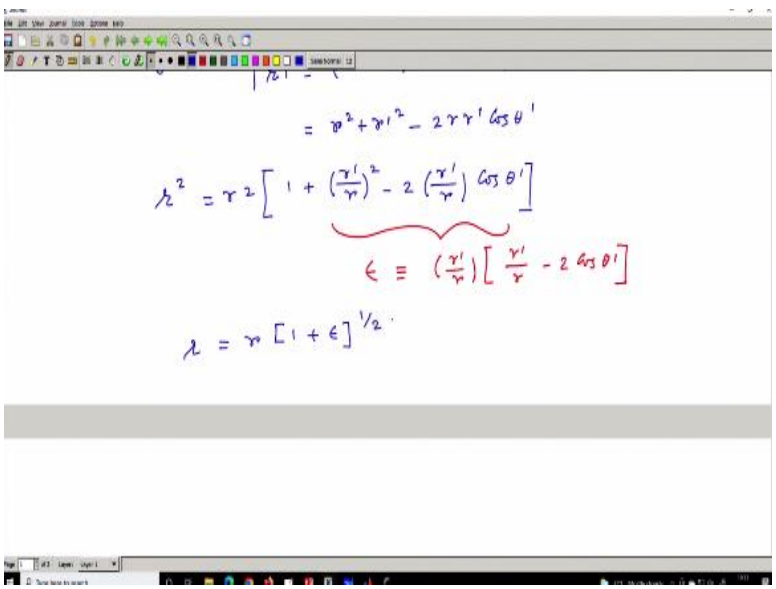

So, r^2 I can write as r^2 taking common $1 + (\frac{r'}{r})^2 - 2\,(\frac{r'}{r})\cos\theta'$ just take r^2 common and we are going to get this one. Now, let us consider this part to be ϵ, which is essentially $(\frac{r'}{r})\,[(\frac{r'}{r}) - 2\cos\theta']$ I just write it as ϵ to make like simple, then my r will be simply $r\,[1 + \epsilon]^{1/2}$.

So, now, I just need to expand this Л this technique we already use to calculate you know this when we are calculating the potential of a dipole.

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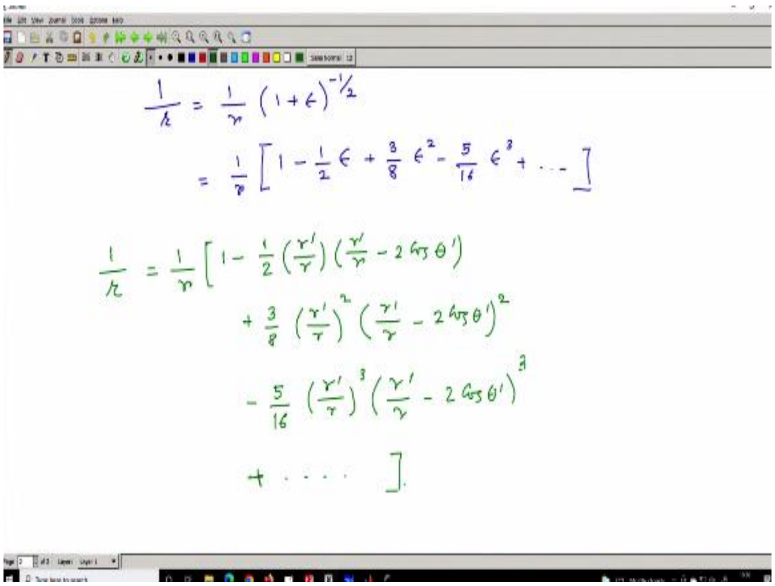

So, the same technique will be going to use that so, $\frac{1}{\mathcal{R}}$ is simply $\frac{1}{r}$ $(1 + \epsilon)^{-1/2}$ and I can make a binomial expansion like $\frac{1}{r}$ $[1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3$ + so on] this is the expansion we can have. Now, I can put the value of the ϵ here and if I do that, then I should get some terms, which is very interesting, why it is interesting we will be going to discuss, so, let me write it.

So, $\frac{1}{r}$ was my first term and then I write 1 minus this is $\frac{1}{2}\epsilon$, $\frac{1}{2}\epsilon$ I just write like $\frac{1}{2}\epsilon$ I already know this is $(\frac{r'}{r})$ and then multiplied by $[(\frac{r'}{r}) - 2\cos\theta']$ that is my first term. What is my second term I just write it here $+\frac{3}{8}$ and my second term is a square of that, so, $(\frac{r'}{r})^2$ of it and $((\frac{r'}{r})$ - 2 of $\cos\theta')^2$. What is my next term?

It is $-\frac{5}{16}$ and cube of that so, I have $(\frac{r'}{r})^3$ and then $((\frac{r'}{r})$ - 2 of $\cos\theta')^3$ and so on.

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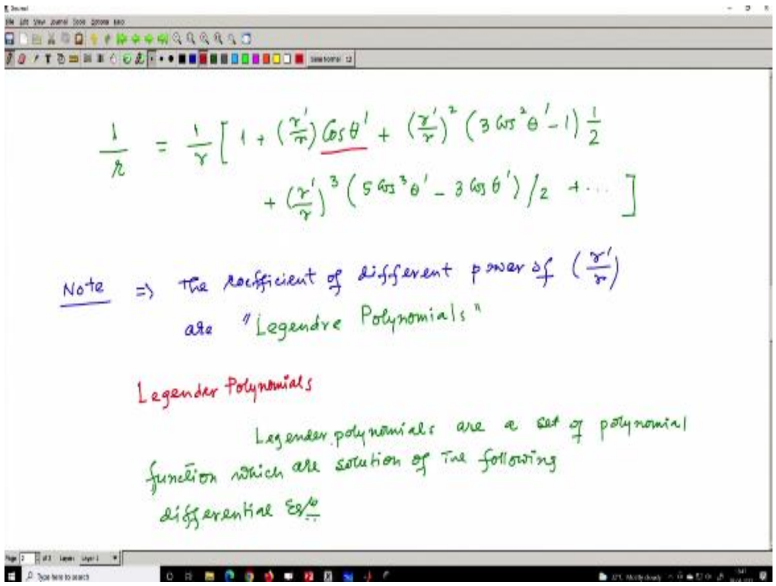

So, now, in the next step what I do is very important that I gather the term based with the power of cos θ. So, if I do that, then $\frac{1}{\text{Л}}$ in the left-hand side it should be $\frac{1}{r}$ and then I have a bracket the first term is 1 then I have $\frac{r'}{r}$ and the cos term is here. So, first order is cos θ, so, I should have simply cos θ*'* then I gather all the term associated with $\cos^2$ θ and if I do then I should have a term like + r^2 $(\frac{r'}{r})^2$, because the $\cos^2$ θ is sitting here.

So, you have I mean we have like this, no I just gathered all the order of this r this let us separate out the order of $\frac{r}{r'}$ the power of $\frac{r}{r'}$ let us do that, so, here we have cube, so, when I make a cos θ, then its power is 1 then I have a square here and then I have a square here also so I need to adjust this. So, if I take r – r^2, then the rest of the term should be say there is a 3 there, so, I should have 3 $\cos^2$ θ*'* - 1 with a $\frac{1}{2}$ term.

And if I gather the cube term cube of $(\frac{r'}{r})$ this quantity so, + $\frac{r'}{r}$ next term is associated with q and if I take a common then it should be $\frac{5\,cos^3\theta' - 3\cos\theta'}{2}$and so on. So, what I do let me explain once again so, I have this expansion and after having the expansion now, I am gathering all the term, which is the power of $\frac{r'}{r}$.

So, if $\frac{r'}{r}$ is x, so, I first take the term x then the term associated to the x^2 then the term associated with the x^3 and so on. So, now, after having all these terms, let us now do something else because this term we need to identify why this term is there this is important term. So, this is a note. The note is the coefficient of different power whatever we are getting here the coefficient of different power of $\frac{r'}{r}$ are very special term, which is called the Legendre Polynomials.

So, what is Legendre polynomial let us quickly understand, so Legendre polynomial is nothing but the set of parameters, so, let me write it here. So, Legendre polynomials are in principle are a set of polynomial functions, which are solution of the following differential equation so, you have a specific differential equation and solutions are basically.

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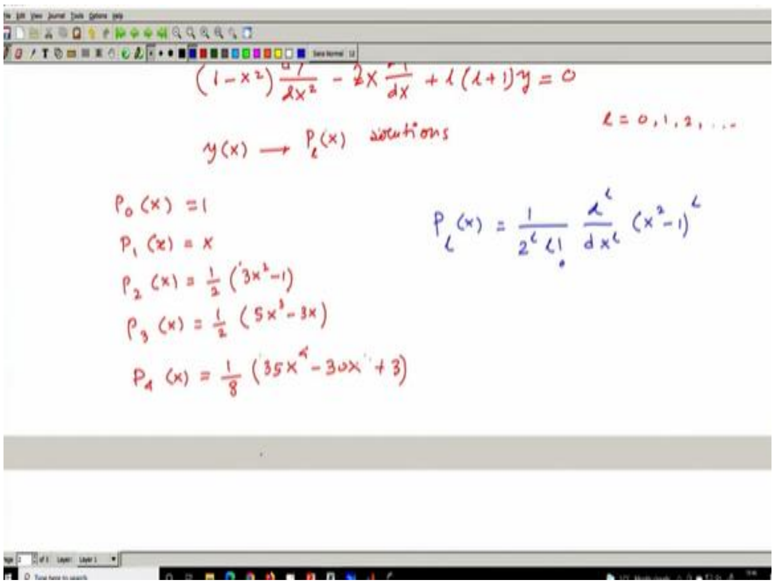

So, let me write down the differential equation very famous one $(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l + 1)y = 0$. This is a very famous differential equation in many branches of the physics, you will be going to encounter this equation and the solution here y as a function of x and we write the solution like $P_l(x)$ these are the set of solutions and T of l means different values of l here is 0, 1, 2, 3 etcetera. l here is 0, 1, 2 integers. So, I have $P_0(x)$ as 1, I have $P_1(x)$ as x, I have $P_2(x)$ as $\frac{1}{2}(3x^2 - 1)$, I have $P_3(x)$ as $\frac{1}{2}(5x^3 - 3x)$.

And then I have P_4 (x) would be something like $\frac{1}{8}$ $(35x^4 – 30x + 3)$ and so on. In general if I write what is P_l? So, it will be like this. So, P_l, which is a function of x can simply be written as $\frac{1}{2^l l!}$ and then $\frac{d^l}{dx^l}$ and the function $(x^2 – 1)^l$. So, if you now put the value of l and try to find out what is P_l exploiting this, you will get all these results.

Now, what is why I am doing that, because now, if you carefully look with these expressions that we have here, the coefficients especially, then let me just highlight this coefficient here. So, this is the first coefficient I am having, then I am having the second coefficient, which is this one from here to here. And this is the third coefficient, which is exactly the same that value we are having here, this is the coefficient, this is the coefficient this is the coefficient.

So, these are matching these, these so, all coefficients that we are getting are essentially the Legendre polynomials of different orders. So, I can write my $\frac{1}{\mathscr{r}}$ after having the knowledge.

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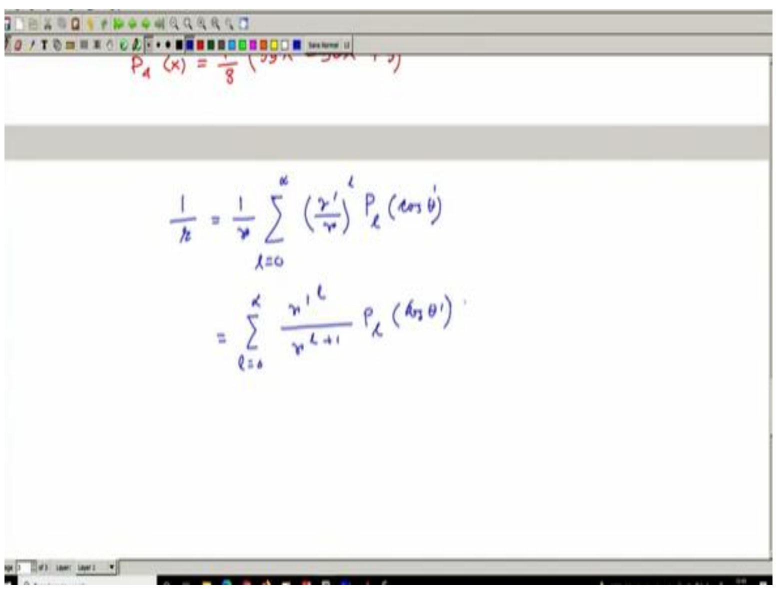

I can now write the term $\frac{1}{\mathscr{r}}$ as $\frac{1}{r}$ then summation l can goes to 0 to infinity and the coefficients of $\frac{r'}{r}$ important. And then these are the Legendre polynomials with the argument cos θ. So, if I replace x to cos θ here, if I replace x to cos θ, then I will simply go back and check and we will find that I will get this one cos θ means θ' because everything in prime, so, I should put a prime here.

So, that I simplify, so I can write it as l = 0 to infinity. So, here I should have l because this ordered is also changing $\frac{r'^{l}}{r^{l+1}}$ and the Legendre polynomials, this is the way you can expand your $\frac{1}{r}$ term. This is quite exciting that you just expand $\frac{1}{r}$ term in this way and you will be going to get.

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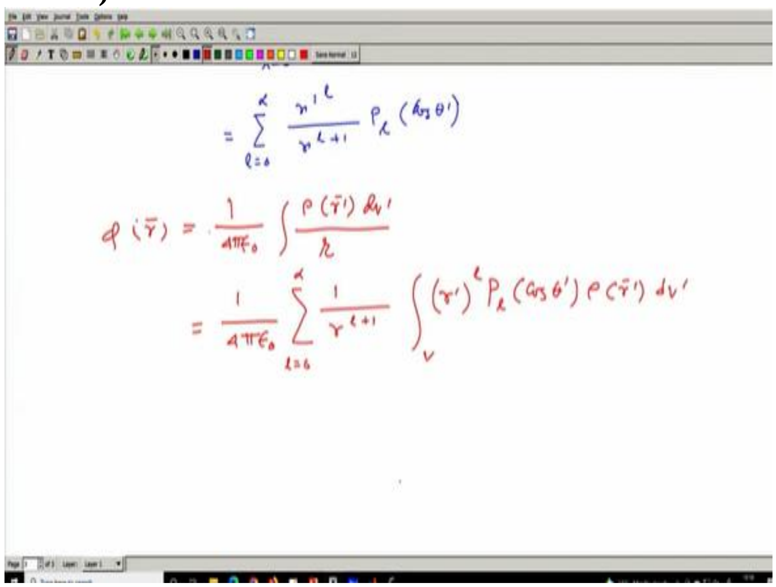

So, what should be the form of the potential then? The form of the potential ϕ ($\vec{r}$) let me write once again it is $\frac{1}{4\pi\epsilon_0}$ and then integration of $\frac{\rho(\vec{r}')dv'}{\text{Л}}$ now, this $\frac{1}{\text{Л}}$ whatever the $\frac{1}{\text{Л}}$ I have, I just write it here in this way $\frac{1}{4\pi\epsilon_0}$ and then I can have the summation sign here, where l goes to 0 to infinity and $\frac{1}{\text{Л}}$ I write like $\frac{1}{r^{l+1}}$ this term, which can I take outside the integral because this integral is over prime.

And then integration $(r')^{l}$ and then P_l (cos θ') then ρ is there already, which is a function of r' and dv' over the volume integral dv.

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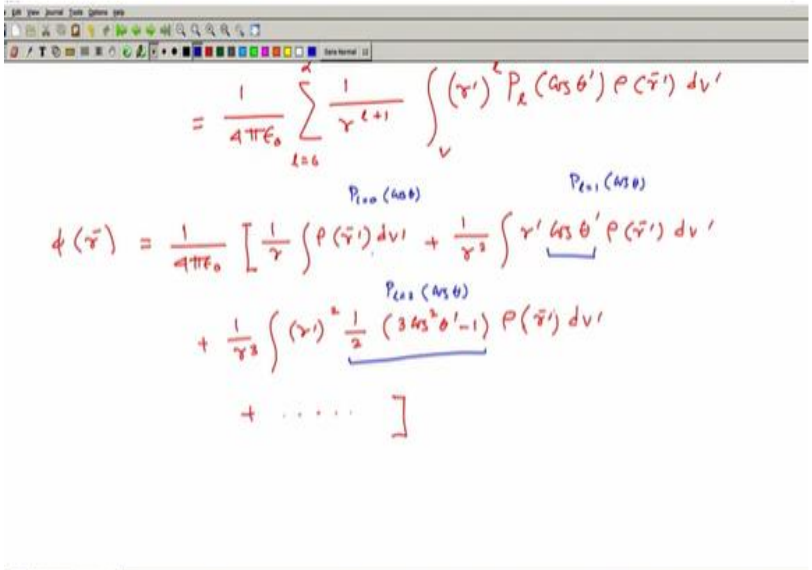

So, I can have a series here the potential can be defined can be expanded in terms of some series. And if I expand this, I will get the term one by one. So, let us do that $\frac{1}{4\pi\epsilon_0}$ and the first term is $\frac{1}{r}$ by taking $l = 0$, I have $\rho(\vec{r}')dv'$ this is my first term, what is my second term? $\frac{1}{r^2}$ $l = 1$ $\frac{1}{r^2}$ integration of r' and cos θ' $\rho(\vec{r}')dv'$ this is my second.

Here I use the P_l so, this value is associated to $P_l = 0$ with cos θ argument and this value is associated to $P_l = 1$ cos θ, which is simply cos θ. So, that value is here and so on we have then we have plus $\frac{1}{r^3}$ then $(r')^2$ then $\frac{1}{2}$ $(3 \cos^2 \theta' - 1)$ and then $\rho(\vec{r}')dv'$ and so on. Well, this term again is $P_l = 2$ cos θ argument. So, that value is this one from here to here, from here to here this is this one and this is one.

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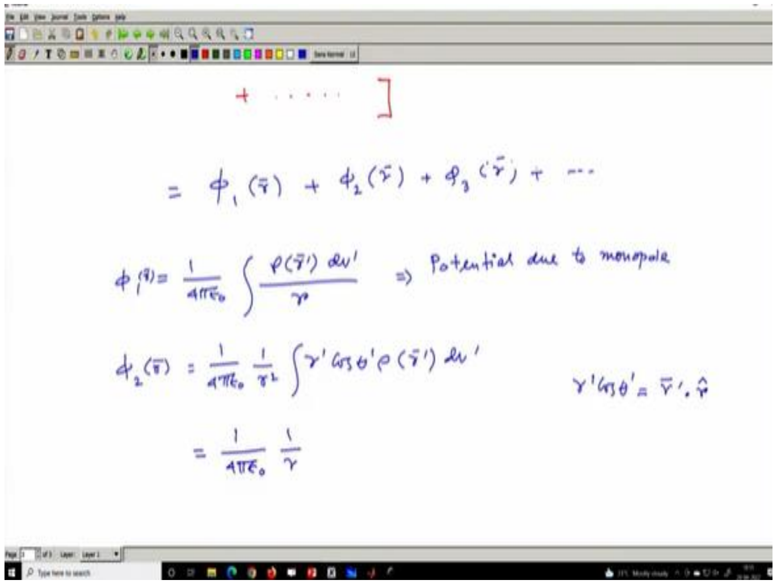

So, now, I can write these things individually like ϕ_1 ($\vec{r}$) + ϕ_2 ($\vec{r}$). So, entire potential is divided into different ϕ's, what these different ϕ's ϕ_1, ϕ_2, ϕ_3 corresponds to the values like this. So, what is my ϕ_1 here if I look carefully, it is $\frac{1}{4\pi\epsilon_0}$ integration of $\frac{\rho(\vec{r}')dv'}{r}$ this is a known term, so this term is nothing but potential due to the monopole contribution of the potential.

What about ϕ_2 here which is a function of r again the function of r, so that is $\frac{1}{4\pi\epsilon_0}$ and then $\frac{1}{r^2}$ integration of r' and then cos θ' $\rho(\vec{r}')dv'$ I can manipulate I can write this in a different way $\frac{1}{4\pi\epsilon_0}$then $\frac{1}{r}$ and this cos θ I can write it as r' cos θ' I can write it as $\vec{r}$' • $\hat{r}$ as of the figure if you look careful in this figure whatever the figure I draw so this is θ' so $\vec{r}$' • $\hat{r}$ is simply r' cos θ.

So that I replace here in this equation, so this is r^2 and $\hat{r}$ dot integration of r' $\rho(\vec{r}')dv'$.

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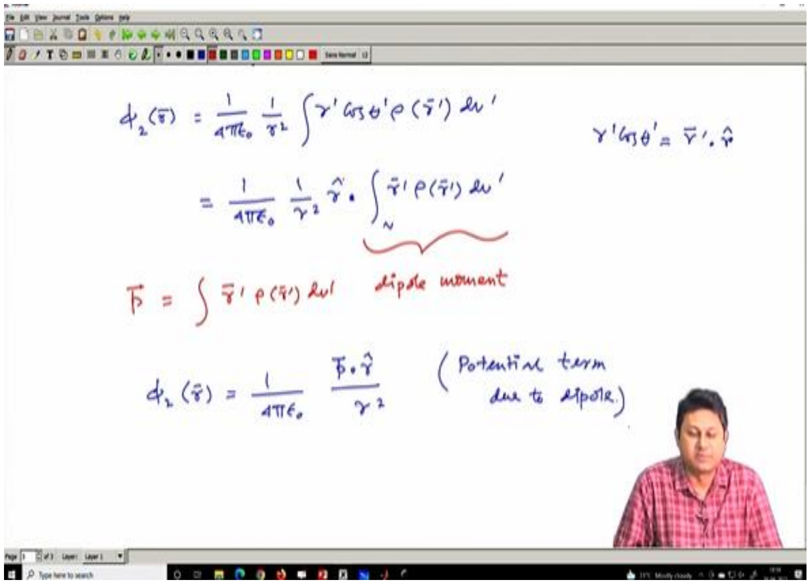

Now this is known term again and when we discuss the dipole moment then we mention that integration the dipole moment is essentially integration r$'\rho(\vec{r}')dv'$, so this is nothing but the dipole moment, so this is the dipole moment. So, once we recognize this is a dipole moment my ϕ_2 can be written very nicely and that is simply $\frac{1}{4\pi\epsilon_0}$ and then this is dipole moment p so I should have $\frac{\vec{p}\bullet\hat{r}}{r^2}$, so this is nothing but the potential due to dipole.

So, now I believe you can understand that when I divide this ϕ systematically I can have the contribution here which is the contribution for monopole, I have a contribution here which is due to dipole, I have another contribution, which is if I do the similar thing I only did for 2 cases ϕ_1 and ϕ_2 but in principle I can also find out what is ϕ_3 and if you do that you will find that this is for quadrupole and so on.

So that means when I for a distribution of charge, when I have a distribution of charge then whatever the potential it is exerting on some point r that is the summation of that it seems it is a summation of all the charge distribution here and this charge distribution is seems systematically we find that it is a contribution of the monopole here, it is a contribution of the dipole here, there is a contribution of the quadrupole here and so on if I add all together then I will have the entire contribution here due to this charge distribution.

But a systematic when we systematically expand this term we find the components of the monopole, we find the component of the dipole, we find the component of the quadrupole and so on. So today I do not have much time so in the next class I will try to understand more about the dipole moment and how to calculate the dipole moment and monopole moment this term the first term is called the monopole moment, the monopole moment is ρ r dv this term is called the monopole moment.

What is the meaning of monopole moment from that you can readily understand that it is nothing but the total charge q, so total charge q is your monopole moment and so on. So, I would like to conclude here today and in the next class I am going to discuss about this how the dipole terms will be there and what should be the consequence of the dipole term, how you calculate the dipole term for redistribution of the charge and dipole moment calculation for different distribution of the charge etcetera. So, thank you for your attention and see you in the next class where we discuss all these issues.