Foundations of Classical Electrodynamics
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Lecture - 30
Electrostatic Energy (Contd.,)

Hello students to the foundation of classical electrodynamics course, founder module 2, today we have lecture number 30.

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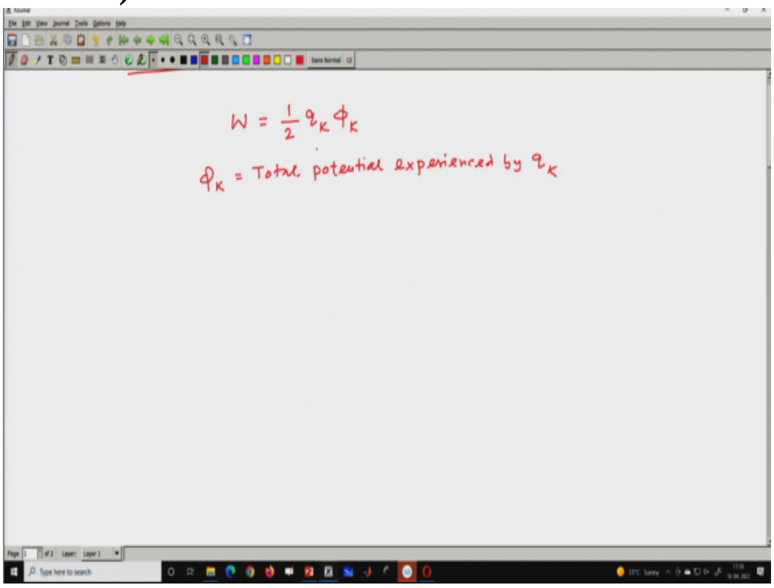

And we will be going to discuss about the electrostatic energy that we started in the last class. So, today we have class number 30. So, we have already discussed that the energy associated with the electrostatic charge when they are bringing from infinite someplace the work done is replaced by this form where ϕ_k is the total potential experienced by the charge q_k. So, this potential total potential is created by all the other charge except q_k. So, that potential going to experience by q_k and that is the form.

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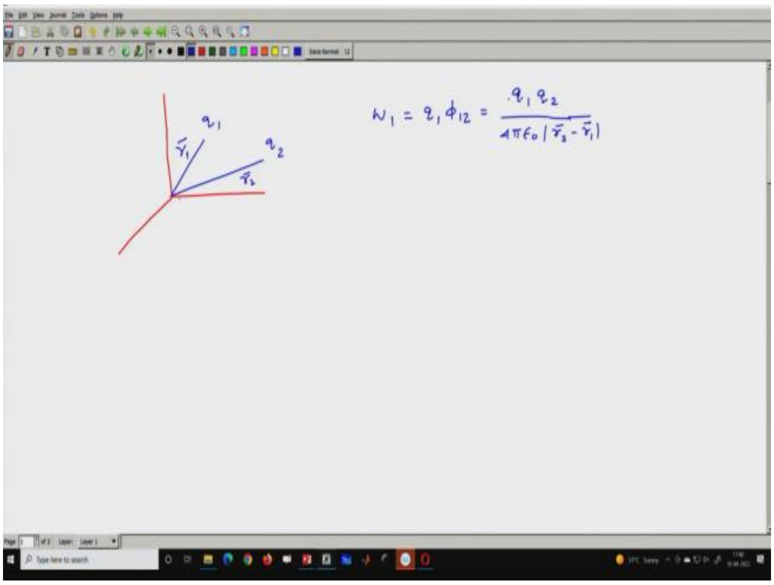

For 2 charge particles for the system of 2 charge particles staying two point $\vec{r}_1$ and $\vec{r}_2$ the charge sitting here say q_1 and q_2 for this system we calculated the potential energy w_1 as to be q_1 ϕ_{12} and that is equal to $\frac{q_1 q_2}{4\pi\epsilon_0\,|\vec{r}_2-\vec{r}_1|}$ and the half term is not there because we are not taking care of another because if you make q_2 ϕ_{21} then it should be you know q_2 q_1 and $\vec{r}_1$ $\vec{r}_2$ that is the same quantity.

I am considering twice so, that is why the half term is not there. So, that is the reason why once we calculate energy of the set of charge particles.

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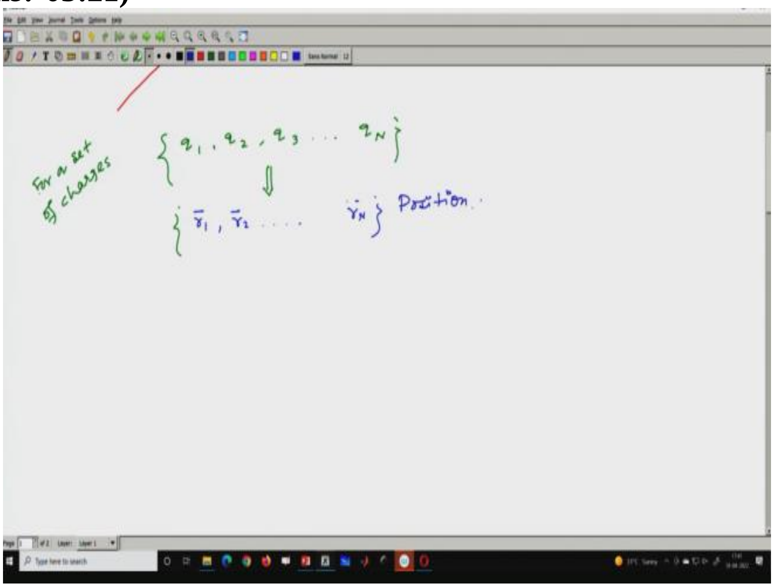

That is q_1 this is for a set of charges q_1 q_2 q_3 and q_N there are N charges are there. So, and with a position having the position they have the corresponding position say $\vec{r}_1$ $\vec{r}_2$ $\vec{r}_N$ these are their position.

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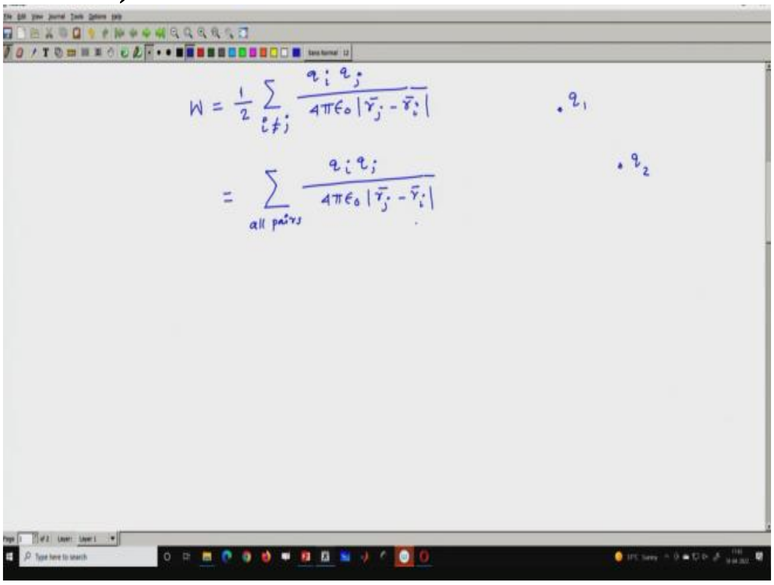

So, now, for that if we calculate the expression should be something like this it should be half this half it ensure that I am not taking account the term twice but i should not be equal to j for i j by $4\pi\epsilon_0$ and $\vec{r}_j$ - $\vec{r}_i$. Another way also it is defined and that is if I do not want to put this half term then it should be simply for all pairs, I am adding these things for all pairs and then it should be $\frac{q_i q_j}{4\pi\epsilon_0}$.

And that quantity and when I write all pairs, so, that means I am taking q_1 q_2 as a pair. So, I am calculating this quantity once not q_2 q_1, but if you calculate q_2 q_1 for example, i can take 1, j can take 2, also i can take 2, j can take 1, so, that means I am calculating these things twice. So, in order to avoid this, that is why I need to make a half here put a half here, but, if I can avoid that, this half by just writing I will do the same calculation for all pairs.

So, that means q_1 q_2 now only behave like one pair, so, only one calculation will be there that is $\frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_2 - \vec{r}_1|}$ it should be the same thing if we just interchange 1 and 2, but in that case, I am adding twice that is why the half term need to be there. So, we will do 1 problem then things will be much clearer.

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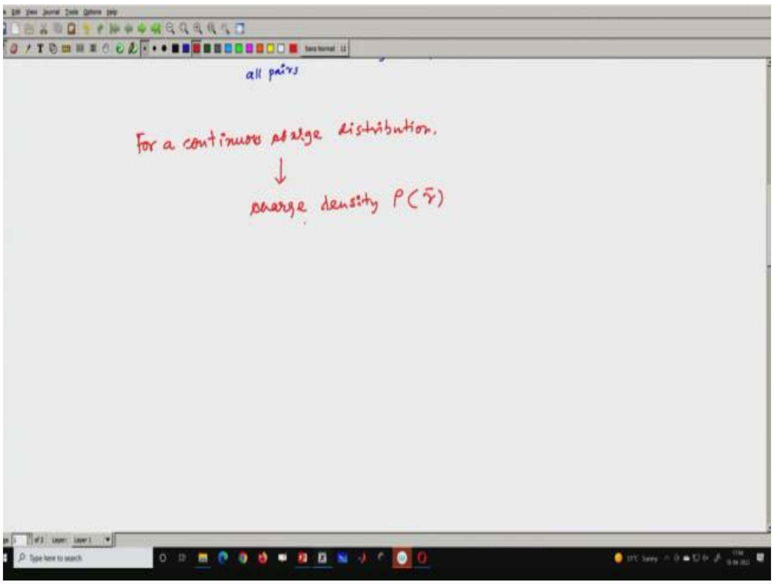

So, now, we will be going to do one thing and that is for a continuous continuum with a charge density. So, for a continuous charge distribution and for that case we have a charge density and that is ρ, which is a function of r.

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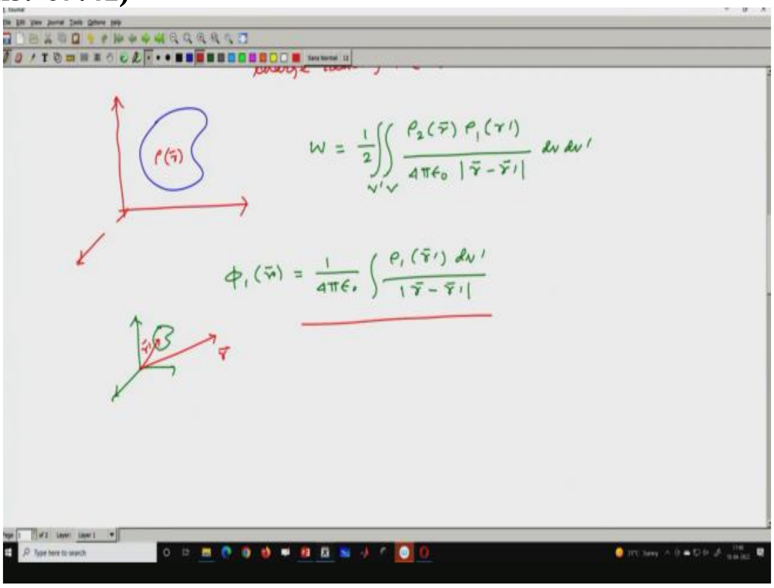

So, what happened if I have a continuous charge, so far we are dealing with discrete now, for a continuous charge distribution where we have the charge density ρ as a function of r. So, this is the distribution we are having for some coordinate system and for this charge distribution I want to find out the energy. The energy simply will be $\frac{1}{2}\int\frac{\rho_2(\vec{r})\rho_1(\vec{r}')}{4\pi\epsilon_0\,|\vec{r}-\vec{r}'|}$ dv dv'.

This so, if I compare with this equation, which is shown here I just replace everything in terms of the charge density ρ. So, then I need to put some volume integral and this volume integral will give us the corresponding charges. So, there should be 2 volume integral here. So, now so, I can in principle put another line here to make it 2 volume integral one is over v and analysis over v'.

So, now, I can find the potential for say ϕ_1 at $\vec{r}$ and that is simply $\frac{1}{4\pi\epsilon_0}$ integration of ρ_1, which is placed at $\vec{r}$' point dv' by $\vec{r}$-$\vec{r}$'. So, now that we know because the potential for a charge distribution ρ_1 at some point $\vec{r}$ is simply this. So, this we know, so, I have a charge distribution here, which is at $\vec{r}$' point and if they want to find out.

What is the potential at $\vec{r}$, at $\vec{r}$ the potentials should simply comes up to be in this form. So, now, if that is the case, so, I can simply you know inside the W if you look carefully I already have this term in the W I have ρ_1 ($\vec{r}$') then $\vec{r}$ - $\vec{r}$' and then we have dv'. So, all these term is in principle there, so, I can write this W like this.

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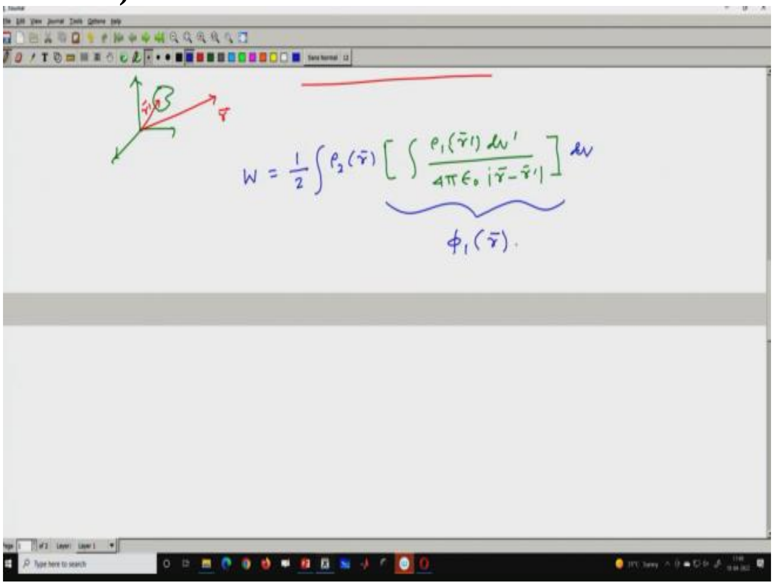

So, this is half I can extract out ρ_2, which is say at $\vec{r}$ and then group all this term that I just wrote here. So, that is $\vec{r}_1$ ρ($\vec{r}$') then dv' by $4\pi\epsilon_0$ then $\vec{r}$ - $\vec{r}$' bracket close and then for this I already have dv. So, this term I just group and these term is nothing but ϕ_1 at the point $\vec{r}$.

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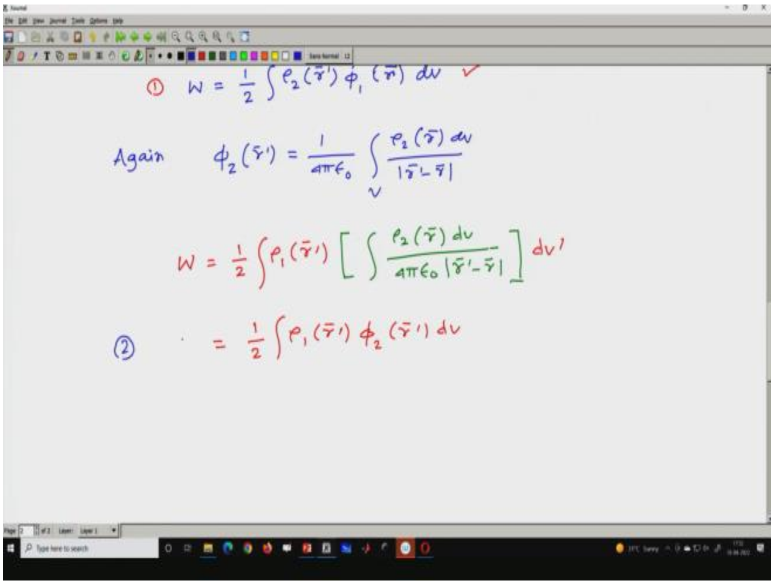

So, my W I can write it as half integration of ρ_2 at $\vec{r}\,'$ point and then the potential ϕ_1 at $\vec{r}$ and this is over dv. So, that is typically the form of the W. Now, if we look carefully again what we have I can write ϕ_2, which is say function of $\vec{r}\,'$ and that potential can also be written in the similar fashion integration of ρ_2 ($\vec{r}\,'$) dv by $\vec{r}\,'$ - $\vec{r}\,'$ and that should be the potential at $\vec{r}\,'$ in the similar way due to the distribution of the charge at some location $\vec{r}$.

So, that means, I can over the volume integral v. So, I can again write my W by taking half integral of ρ_1, which is at $\vec{r}\,'$ and then I just group this term, which I just wrote here $\frac{\rho_2(\vec{r})dv}{4\pi\,\epsilon_0\,|\vec{r}\,'-\vec{r}|}$ and then dv'. So, this quantity simply comes out to be half of you know ρ_1 ($\vec{r}\,'$) and then ϕ_2 ($\vec{r}\,'$) dv. So, you can see I already have an expression of W here in terms of ρ_2 and ϕ_1.

And again I am having this is say equation 1 this is say equation 2 so, from 1 and 2, I can write in expression like this.

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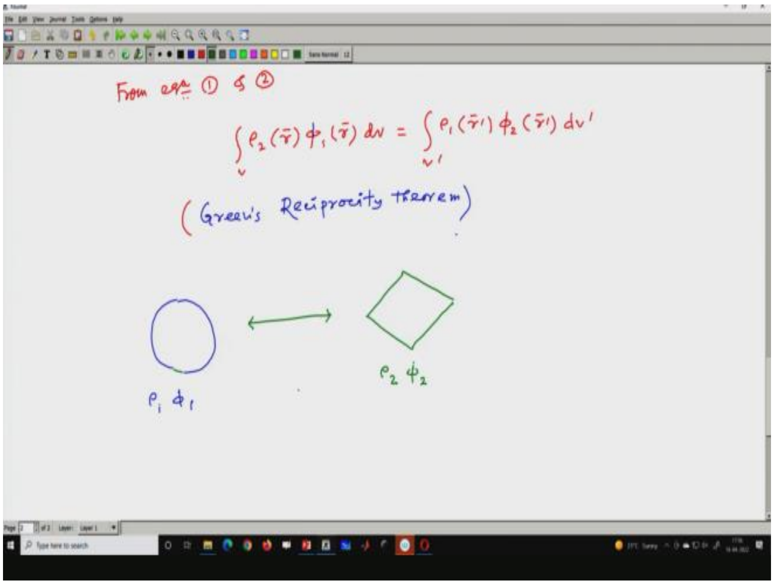

So, from 1 and 2 what we get? We simply get integration of ρ_2 $(\vec{r})$ and ϕ_1 $(\vec{r})$ dv is simply equal to ρ_1 $(\vec{r}\,')$ ϕ_2 $(\vec{r}\,')$ dv' so, this is an identity and this identity is called the Green's reciprocity, this is a theorem and this theorem is called the Green's reciprocity theorem so, what it says let me write it down and then briefly explain. So, from the expression we can see that I can have 2 charge distribution and for 2 charge distribution.

So, suppose I have a charge distribution here and say another charge distribution here, I am just drawing 2 different symbols to make sure that 2 different kinds of charts distribution you can realize. So, for this case I have this is you know the distribution is ρ. Say, ρ_1 and potential is ϕ_1 and for this case the distribution is ρ_2 and potential is ϕ_2. So, these are 2 you know 2 different systems, but here from the Green's reciprocity theorem.

You can see that at some point $\vec{r}$ whatever the density you are for example, here whatever the charge density you are having that multiplication the potential due to the charge distribution of the ρ_1 at that particular point if I multiply it and integrate it over the volume, the value is exactly same, if I do the opposite that means, if I calculate at some point $\vec{r}\,'$, if I calculate the potential due to the charge distribution of the ρ_2 and then multiply the potential and multiply the charge density at that point and integrate.

So, these are 2 isolated systems 2 different system, but they are related to each other with this theorem, which suggests that I mean whatever the value you find here with just multiplication of ρ_2 and ϕ_1 with the integration of making a volume integral, you will get the same thing if you do the opposite way. So, let me write it down then things may be clear.

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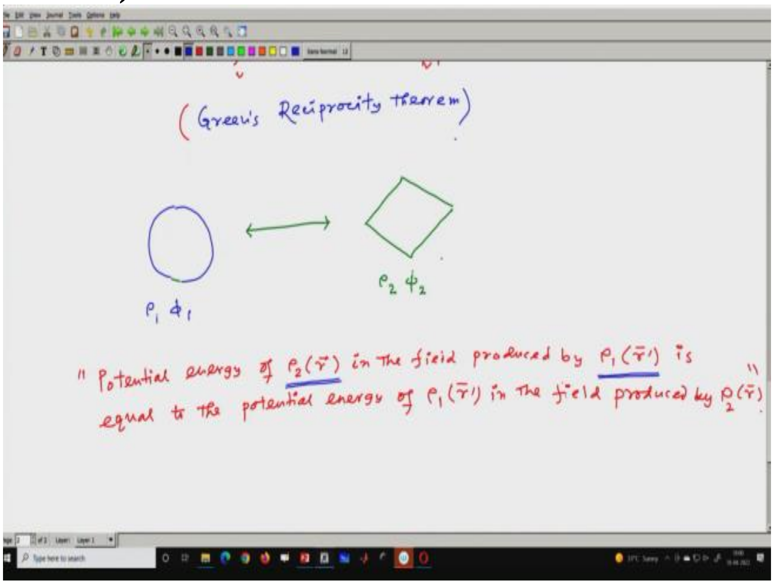

So, what I say is this so, potential energy of ρ_2 say at point $\vec{r}$ in the field produced by ρ_1, which is at some point $\vec{r}\,'$ is equal to the potential energy of ρ_1 ($\vec{r}\,'$) in the field produced by ρ_2, which is at $\vec{r}$. So, the statement is straightforward that whatever the potential energy we have so because of the charge distribution of ρ_1 and ρ_2, the potential energy of created by the charge distribution ρ_1, energy of the potential energy of ρ_2 in the field produced by the ρ_1.

So, ρ_1 because of the presence of ρ_1, if I have a put in a ρ because of the presence of the charge distribution ρ_1 I have a field and on this field if I put ρ_2 then this ρ_2 we are going to have some potential energy. So, that value is equal to the same thing in opposite way that means, the same value is when ρ_2 produced the potential energy and ρ_1 is experiencing that energy ρ_2 is producing the field and ρ_1 is experiencing that potential energy.

So, that is the meaning of this Green's reciprocity theorem in other branches also we have these kinds of reciprocity theorem this is one of the example we have in electrostatic where this theorem is valid. We can make use of this theorem in certain problems 1 just typical problem I like to which is difficult to calculate in other way, but very simple when we use this you know, this Green's reciprocity theorem.

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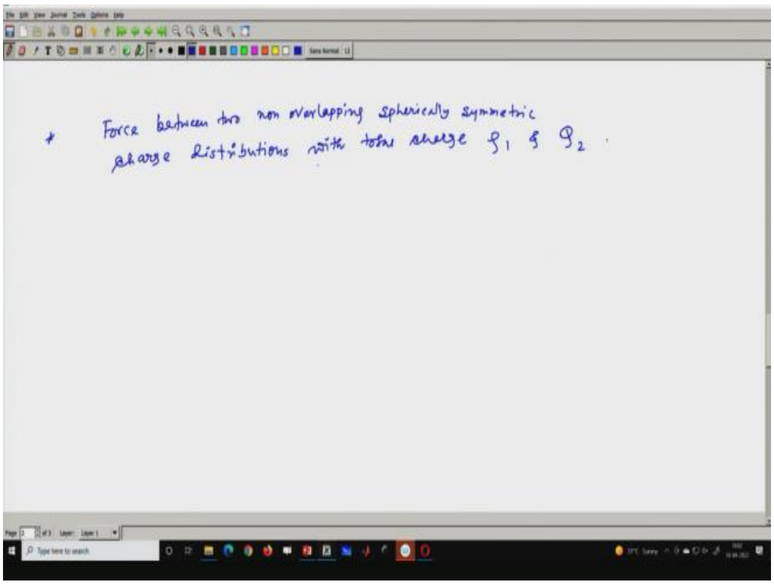

So, suppose I want to find out the force between 2 non-overlapping spherically symmetric charge distribution with total charge Q_1 and Q_2 so, electrostatic potential if you calculate then from that you can also calculate the force so, our strategy is to find out the electrostatic potential that is created due to the system.

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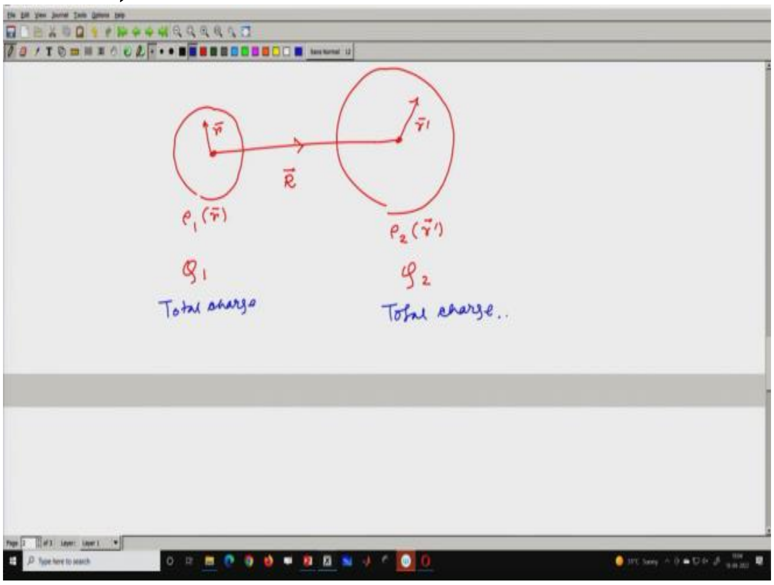

So, the system that means if I have a spherically symmetric charge distribution and they are not overlapping so, I have 2 spherically symmetric charge distributions obviously their radius is different. So, suppose the charge distribution here is ρ_1 and that should be a function of $\vec{r}$. So, that means, from here to here whenever you have some r value you will have a charge distribution that is ρ_1 that is defined. In the similar way, I have a charge distribution here which I say ρ_2 and say from here to here this is same coordinate is $\vec{r}\,'$.

So, if I go from here to here at $\vec{r}'$ point I should have a distribution and this is you can see this is a function of $\vec{r}$ this is not constant this is a function of $\vec{r}$ and suppose the distance between these 2 is $\vec{r}$. So, obviously, r is greater than the summation of the radius of these 2 that is why they are not overlapping and the total charge Q is here Q_1 and the total charge is this is mentioned this is my total charge. So, now, the electrostatic potential let us now check 1 by 1 what I can get from this.

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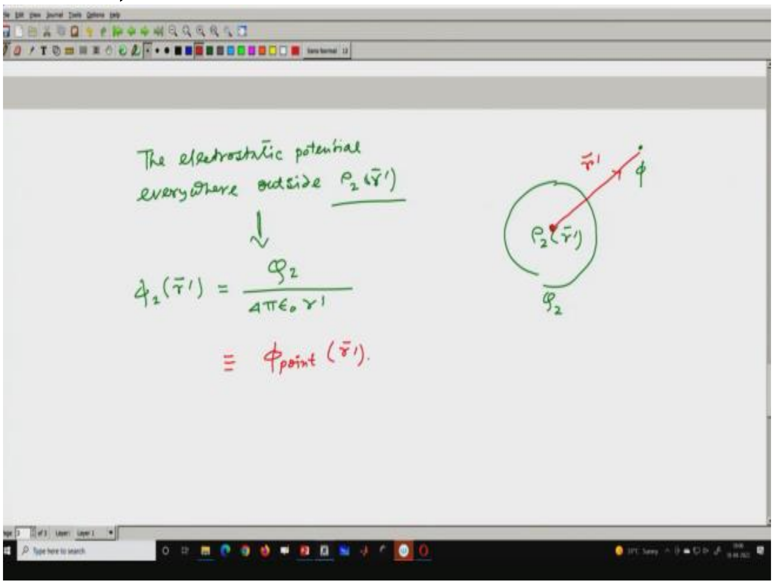

So, the electrostatic potential everywhere outside the distribution ρ_2 ($\vec{r}'$) so, I want to find out the distribution at so, that thing is simple so, I have a charge distribution here this is ρ_2 ($\vec{r}'$), but if I want to find out a point here what is the potential then this potential ϕ_2 that is simply the total charge divided by here the total charge is Q_2 because Q_2 is given divided by $4\pi\epsilon_0$ and then the point here to here, which is $\vec{r}'$.

So, that means, as if the total charge is confined to the centre and this point charge is confined to the centre here and from here to here. This is r and we know from the Gauss's law that if I have a charge distribution, and if I want to find out the field outside this charge distribution, then the field of the potential can be simply like whatever the charge distribution we have and considered the charge due to that charge distribution at that origin point and then just simply calculate that, so, I am just trying to do that.

So, this quantity is equivalent to ϕ I should write another quantity ϕ_{point}. So, ϕ_{point} means, as if I compress all the charge to this point and then calculate the potential, which is the same thing.

Now, this is considering all the total charge is replaced by a point charge at the centre as I mentioned, so, that is the potential. So, both the cases you will get the same result.

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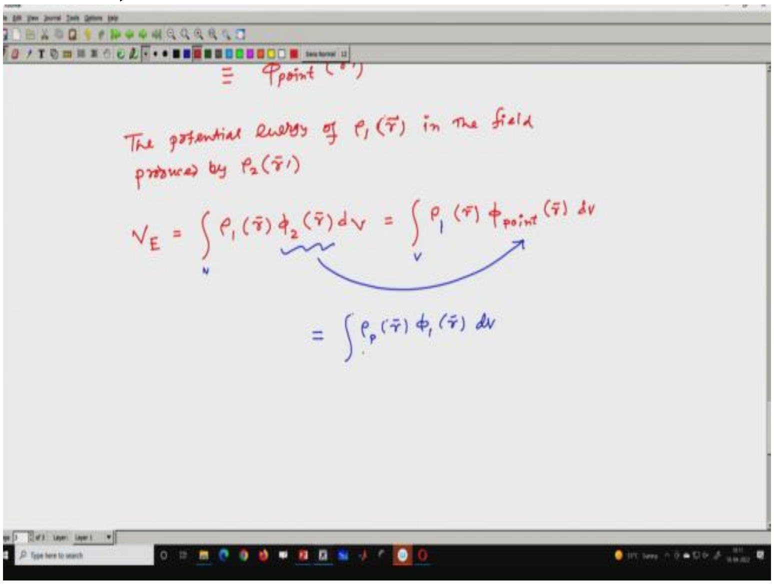

Now, the potential energy of ρ at some point $\vec{r}$ this is the potential energy that ρ_1 should experience in the field produced by the charge distribution ρ_2 the potential energy of ρ_1 in the field that produced by the charge energy produced by the charge distribution ρ_2 can be simply written as, $V_E = \rho_1$ $(\vec{r})$ and then ϕ_2 produced by ρ_2 at point $\vec{r}$ and dv. Now, according to the reciprocity theorem, this can be simply represented by this that ρ produced by ϕ_2.

So, ϕ_2 is created ϕ_2 if I write so, this is ρ_2 and then that is at $\vec{r}$ and the potential is ϕ and r dv so, this is not I am just simply writing the same thing, so ϕ_2 I just replace by these ϕ points. So, this has to be not ρ_2 this has to be simply 1. So, this is the same thing because that quantity is simply replaced by this, which I already mentioned here. Now, I am going to use the reciprocity theorem for this equation, and that is here in a different colour I am writing so this is over volume.

So, if I use the reciprocity theorem for this, so, now, the point charge density will experience the same thing due to ρ_1. So, I am writing this ρ_1 and dv. So, the potential this is for sorry, this is not potential this is point in this point, so, this point potential, I can replace in terms of the ρ, which is the charge distribution of point charge Q_2.

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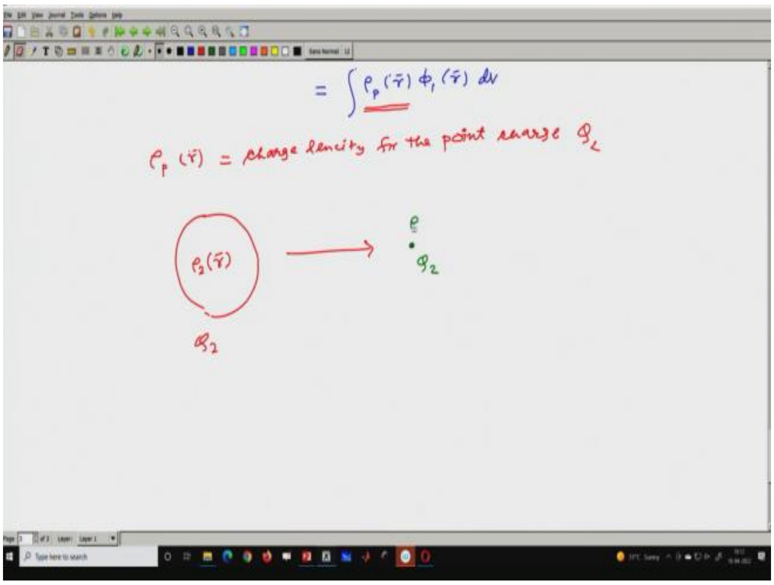

So, ρ_P is $\vec{r}$ is simply the point this is the charge density for the point charge Q_2 try to understand once again that I can make this Q_2 here distributed having the distribution the charge density like this, this is equivalent to I can make this point charge Q_2 sitting here this charge distribution entire charge Q_2 to make a point charge here with the density of this point charge to be ρ_P this is the point charge, charge density for this point charge ρ_P.

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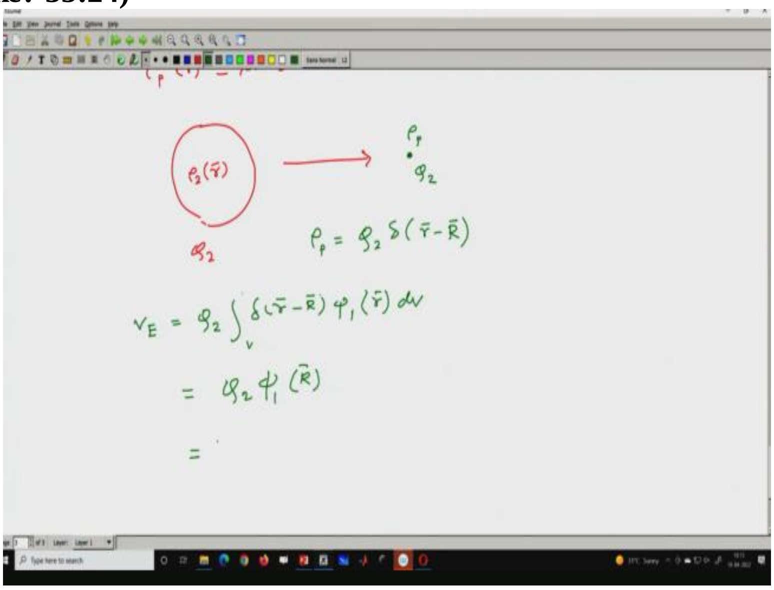

So, now ρ_P what is ρ_P? Point charge density this is we know from the delta function definition this is $Q_2\ \delta\ (\vec{r} - \vec{R})$ because now Q_2 is here, Q_2 is sitting here so, which is in $\vec{r}$ so, when I shrink everything to the point well then my V_E simply Q_2 integration of because this is here I ρ_P just replaced by Q_2 delta function then it should be $\delta\ (\vec{r} - \vec{R})$ and then ϕ_1 then $\vec{r}$ and then dv so, this quantity is Q_2 and this integration is all with the delta function. So, simply I can have this value as whatever the ϕ_1 at $\vec{R}$.

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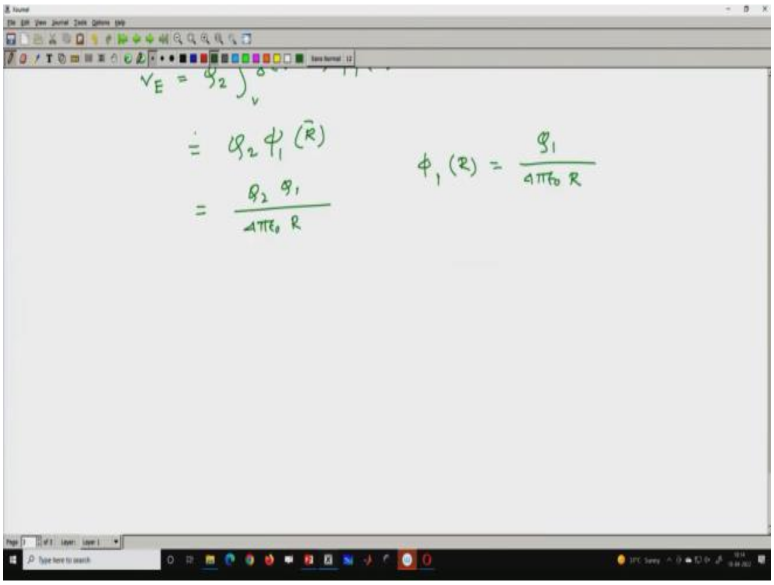

So, this quantity simply comes up to be Q_2 and the potential due to the distribution of the ρ_1 and that is also having a charge Q_1 so, it is $\frac{Q_1}{4\pi\epsilon_0 R}$ note it that quantity is total charge $\frac{Q_1}{4\pi\epsilon_0 R}$ that is the distribution the potential due to the charge distribution here whatever we have. So, that is the today I do not have much time to discuss. So, with that note I like to conclude and in the next class I will start again this.

The discussion on the electrostatic energy and try to calculate field distribution discrete charge and some continuous charge distribution and try to find out mathematically what should be the potential energy for that specific charge distribution or that for specific distribution so few problem I like to solve in the next class. So, thank you very much for your attention, so see you in the next class.