Foundations of Classical Electrodynamics

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Lecture – 03
Use of Levi-Civita Symbol, Coordinate System

So, welcome students to the course of fundamental of classical electrodynamics. So, today we will have lecture 3 and we will continue with the use of this Levi-Civita symbol that we defined in the last class and try to understand few coordinate system.

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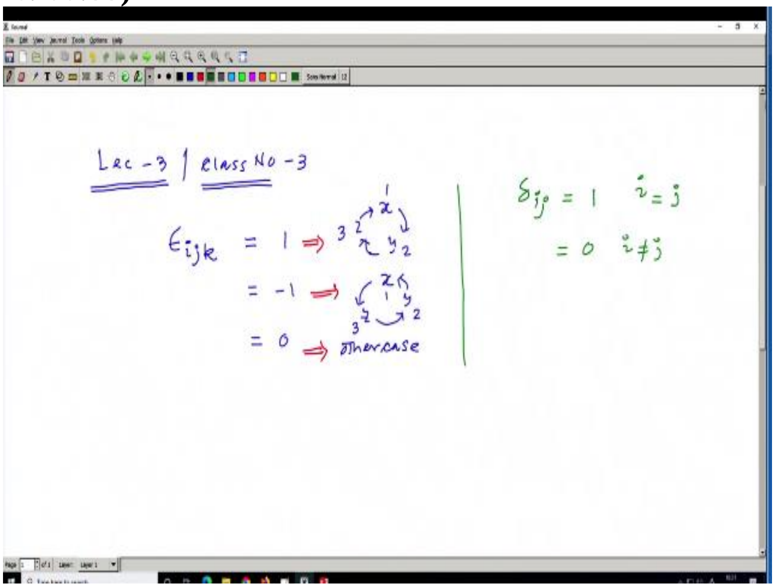

So, today we have lecture 3 or class number 3. So, if you remember last day we define a symbol, which we call Levi-Civita symbol, which is having either value 1, -1 or 0. When we have a cyclic order of x, y, z, then this value is 1. When we have the opposite order, if I say this is 1, this is 2, this is 3, so, it should be right-handed order and similarly, if it is 1, 2 and 3, it is the left-handed order. For this case it was 1, this case it was -1 and all other cases this value is 0.

So, this corresponds to this orientation, this one corresponds to this orientation, and otherwise this value gives us 0. Also, we defined the delta function Kronecker delta rather, which is equal to 1 when i = j and equal to 0 when i ≠ j. So, these were 2 systems 2 delta and 2 symbols that we define in this way and then we did something. So, today we will start we will be going to use these symbols in order to understand few vector properties.

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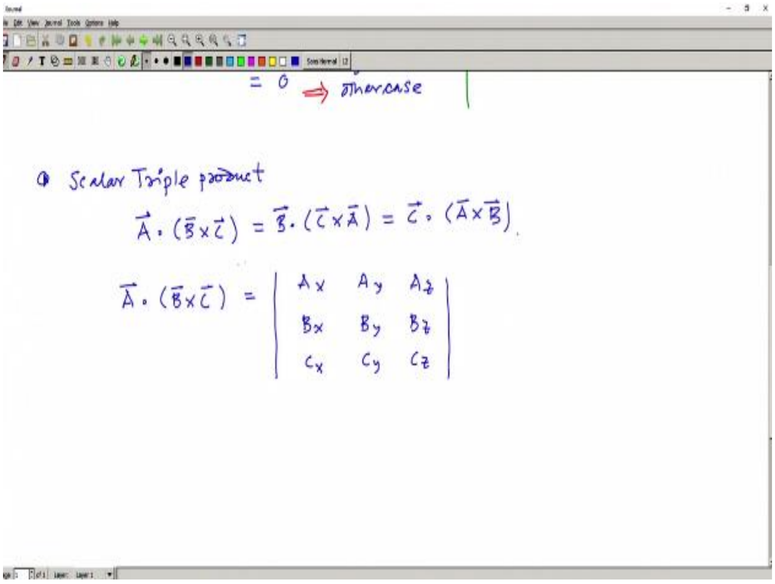

So, let us start with this well-known scalar triple product. In scalar triple product, we know one very well-known identity that if A, B, C three vectors are there, then A dot B cross C, it is equal to B dot C cross A and that is equal to C dot A cross B. So, this identity one can prove easily with using this Levi-Civita symbols. So, what is the meaning of A dot B cross C? This is nothing but A_x A_y A_z, B_x B_y B_z and C_x C_y C_z. So that is by definition.

Now, the next is to understand using this Levi-Civita symbol how to execute how to prove that A dot B cross C = B dot C cross A = C dot A cross B.

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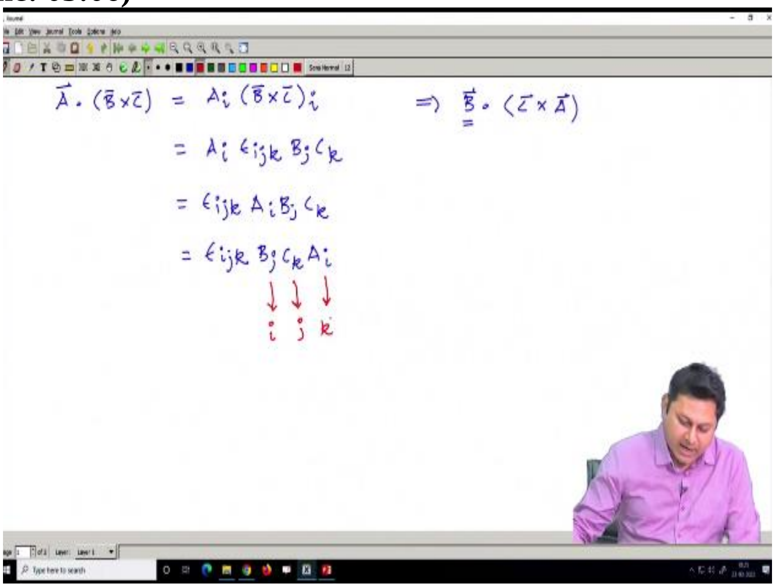

A dot B cross C this is a dot product you can see this is a vector and we know that this dot product can be represented like A_i and whatever the vector we have B cross C and then i. This is the way we define. Now if I expand this so, this is A_i and last day we mentioned that if two vector A cross B is there, then in terms of Levi-Civita symbol that component will be ε_{ijk} B_j C_k. This is the way we can have we can write this.

Now, I manipulate this I write ε_{ijk} because I am now dealing with the components, I put this A inside A_i and then B_j and C_k. The next thing is I just rearrange few things. Let us put this B_j here because I need what I at the end of the day I need to prove that this quantity is equal to B dot C cross A. So, using the Levi-Civita symbols so, this B is sitting here, so, I write B here, I just replace this and rearrange and then C cross A so, I just put C_k here and A here. I just changed the order.

The next thing is very important and that is I now make this because according to our notation, if I have i and i here then it is i then ε_{ijk} j k. So, let us make these things i these things as j and these things said k. By doing that, I just change you know the name of this symbol.

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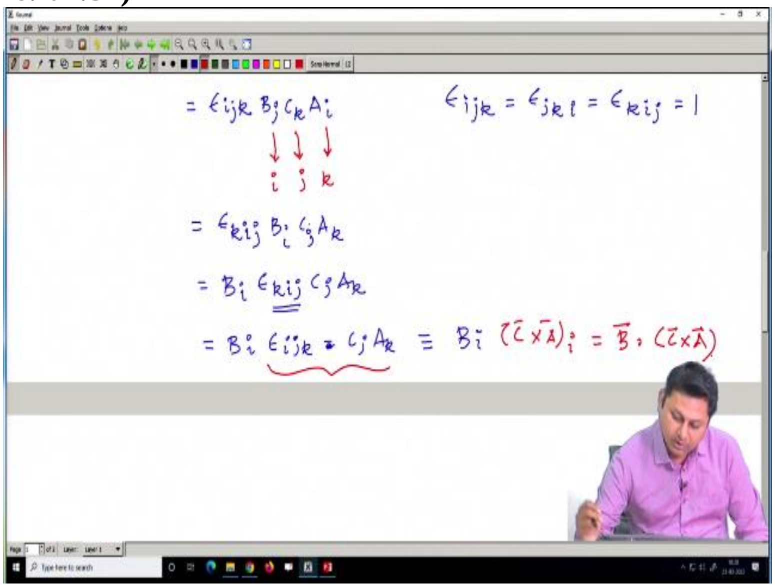

By doing that, I need to change these also in ε and now i become k and then j become i and k become j. And this now I write B_i because I now change this B_i C_j and A_k. So, these things I now rearrange. I write B_i and then ε whatever I am having k_{ij} then C_j A_k. Now, I can rearrange this because the thing that you need to understand that ε_{ijk} is equal to ε if I considered the order jki = ε_{kij} and these values are 1.

So, that means, I can write this as B_i ε_{kij} this one I just replaced ε_{ijk} because the both the things are one and then C_j then here it is a multiplication, C_j A_k. So, this quantity eventually gives us B_i and this section is giving us this part is given us C cross A with the ith component and which is the thing that I wanted to prove here, because A dot B cross C I want you to put B dot C cross A, so, ith component here. So, this is nothing but B dot C cross A. So, I prove this identity just by using this Levi-Civita symbol that is why it is very powerful and very interesting.

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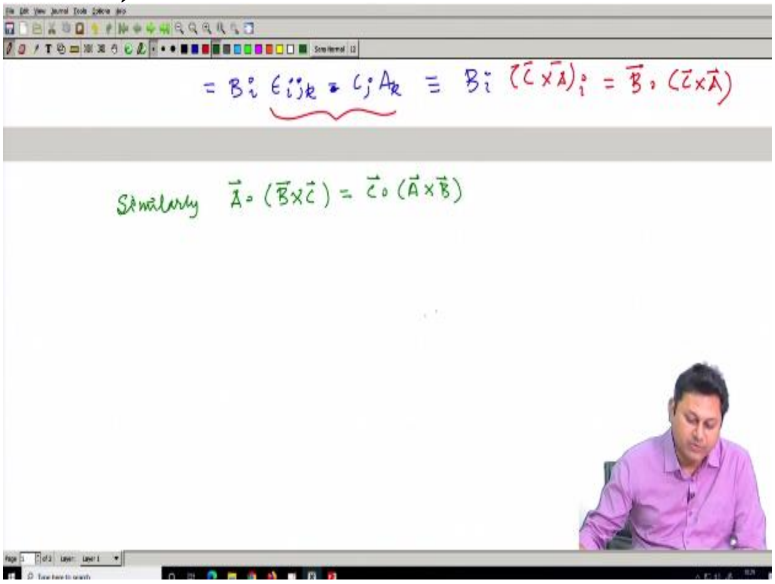

Similarly, you can also prove that A dot B cross C = C dot A cross B. So, I request a student to have a look and please try to you know prove this. I already proved that A dot B cross C is B dot C cross A. So, you can also prove that A dot B cross C = C dot A cross B in the same way using the same process same methodology.

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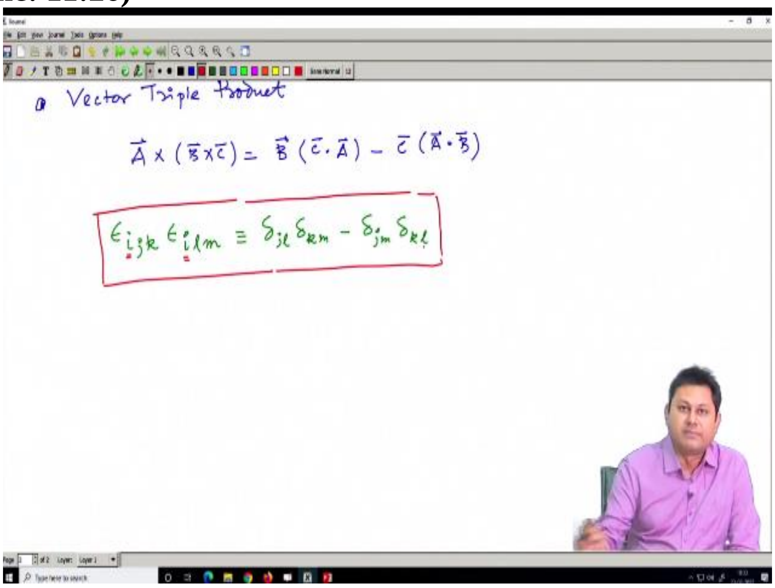

Now, we move forward and now, we understand what happened for a little bit complicated case and that is the vector triple product. For vector triple product we have a very interesting identity A cross B cross C is equal to B and then C A - C A B. So, you can see that in the left-hand side is A cross B cross C. So, eventually this is a vector quantity right-hand side if you look C dot A is a scalar but it is multiplied with a vector B, and A dot B is a scalar, which is multiplied to the vector C.

So, the right-hand side is also a vector quantity, which should be. Now, these things we are going to prove, but before that, I would like to introduce a very important identity you must remember this identity very useful identity and that is ε_{ijk} multiplied by ε_{ilm} that is equivalent to so, δ of jl then δ of km – δ of jm and then δ of kl. So, you need to be careful enough here, but there is a simple way to remember the way I remember it.

So, this is i here, i here so, in the right-hand side it should not be any i, so, when i sitting in the front right-hand side you should not have any kind of i, because I am dealing with the ith component here. Then jl jk sitting here and lm sitting here. So, the first term is the closer one jl jl and then next km km. This pair will be put like this, this order will be like that j first then l first jl, k second m second then km. Now, I reverse the order and in order to reverse the order there will be negative sign you just remember in this way.

So, now, you can see that j is here and then the m I now use so m it is not l previously it was l now m then k is here, and the closer one l is so, now, I change the order and when I change the order, there is a negative sign. You just remember in this way, and it should be usefully this identity is very useful in understanding different complicated vector identities. So, I request a student to have a look and seat.

So, ε_{ijk} ε_{ilm}, what is this δ_{jl}, j is sitting here, l is sitting here the first so, I will put it then k sitting here, m sitting here in the second I put it. Now, I change the order so, when I change the order there should be a negative sign. Now j first m is second I put j first and second, and then k second l first I put k and l, that is all. I will be going to use this to prove this vector identity. So, now, what is this quantity, let us quickly understand this δ.

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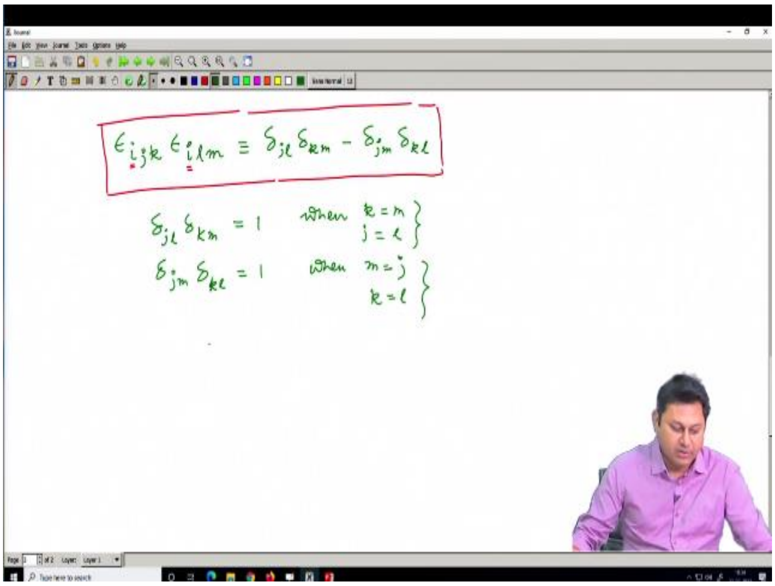

So, δ you know jl δ_{km} this should be 1 when k = m and j = l. This is only condition when this is 1, otherwise it will be 0. Similarly, δ_{jm} δ_{kl} this is 1 when m = j and k = l. This is the condition for making it one, this is the condition for making it one. So, we have to be careful with this, which is related to the delta function. Now, I am going to use this, how I use let us check.

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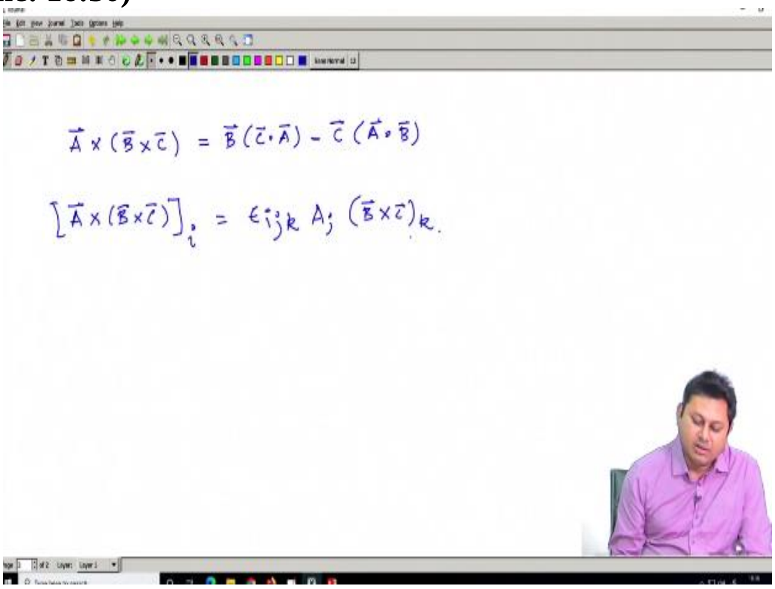

So, I am having A cross B cross C and I need to show on the right-hand side that B C dot A - C A dot B. So, like the way we did in the previous case so, A cross B cross C if I want to find out the ith component of this vector quantity, it will be simply ϵ_{ijk} A_j and whatever I have here this vector quantity B cross C the kth component of that, because A dot A cross B is ϵ_{ijk} A_j B_k. So, in case of B I have another vector B cross C. So, I will just use this expression the way I write A cross B. Now, again one A cross B is B cross C the k th component sitting here, so, that I need to resolve.

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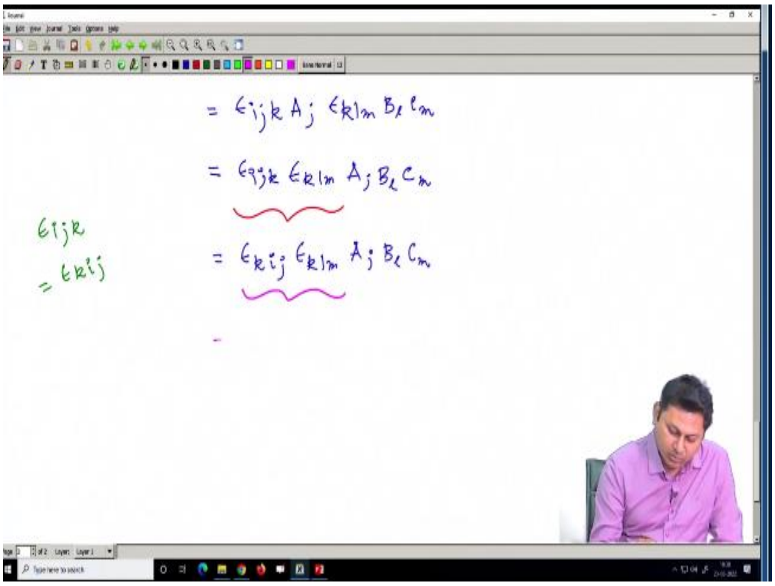

So, now, I have ε_{ijk} and then A_j and this quantity I need to write that ε_k and then I need to use another indices lm because it is another and then B_l C_m. Now, I put all ε together ε_{ijk} ε_{klm} and then I write A_j B_l C_m. Now, you can see that I am having a quantity this ε_{ijk} ε_{klm}, which I already defined here ε_{ijk} multiplied by ε_{ilm} but here you can see that it is ijk and klm. So, here it is i is sitting here, but here you can see that k, which is a common term sitting here and here.

So, I need to you know, I need to make it symmetric. So, I write this term I write ε, you know kij and ε_{klm} and the rest of the term is as usual, it is A_j B_l and C_m. So, you can see that here, I am making a note that ε_{ijk} is same as ε_{kij} by cyclic rule. Now this quantity once I get this quantity now I am in a position to expand this the way I expand it, because I know that this is the rule of this.

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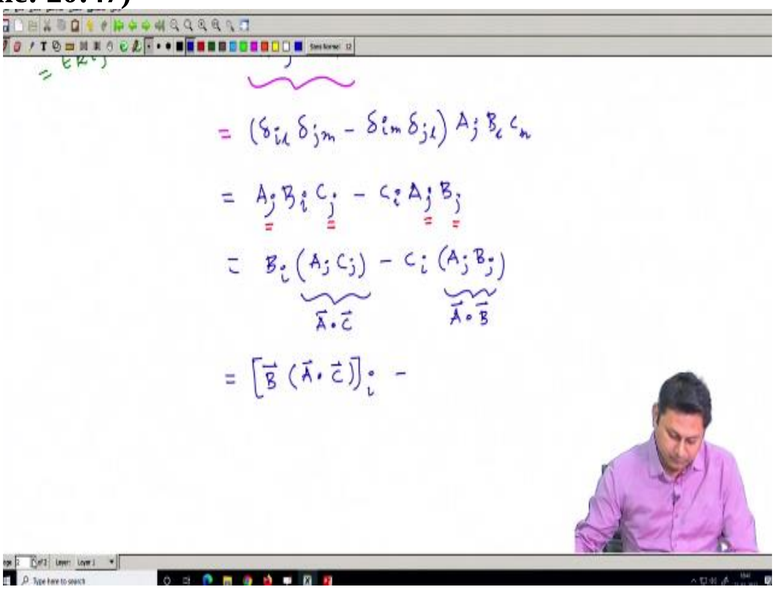

So, I am going to expand this and it should be δ_i because it is a k th is here so il then δ_{jm} - δ_{im} δ_{jl} and the rest of the term. Now, this quantity i everything is changing. So, I need to understand that when it is so, when i = l then and when j = m or m = j, so, I the first term I can write is $A_{j\,m}$ where it is there, I need to put it as j. So, B_l is there so, it should be B_i because when l = i that is a meaningful and I have C_m is there so, it should be equal to j then only it is a meaningful.

So, I have A_j B_i C_j. In a similar way, the next term, when i = m then it is a meaningful thing. So, I can write it like C_i and then A_j is there. So, j let us put j here l I now need to put like B_j. Now, here you can see that I am having j here j here and also, I am having j here j here. So, I can write it like B_i this quantity I write B_i and in the bracket A_j C_j - C_i I can write in a bracket A_j B_j. Now, this quantity A_j C_j is nothing but A dot C.

Because if you expand A dot C by using the Einstein notation it should be simply A_j C_j. In a similar way, A_j B_j is nothing like A dot B. So, I can write it like this is a vector B and then A the i th component of that quantity and minus now I am having another term so, I can write it as C vector B dot A or A dot B. This is the i th component. So, this is the thing actually I wanted to prove. You can see that A dot B A cross B cross C is B C A – C A B. So, here is a line I do not know from where this line came so let me erase this.

So, B C A – C A B so, we prove this A dot B and B dot A are same. So, here you can see that B C A - C A B, so, this is the way by using the Levi-Civita symbol, you can prove the different vector identities and you find that it is very useful and I request you to please note this identity, which is in this red box is a very useful identity and in future will want to use this. After that let me now quickly jump to the next very important topic and that is the coordinate transformation.

We have a preliminary idea about the vector operation vector and, how using the δ function and Levi-Civita symbol we can play with these vector identities later we are going to prove more and more vector identities again by using this Levi-Civita symbol, but I request you to practice there were a couple of problems given to you. So, I believe you are practicing them, but now, let us jump to the new topic, which is the coordinate system.

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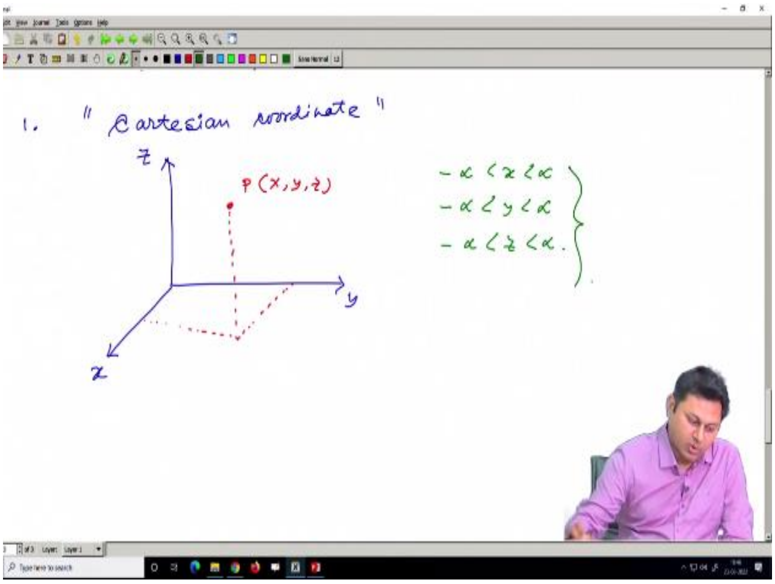

So, coordinate system and transformation. So, let us start with the most useful or most well-known coordinate system, which is the Cartesian coordinate system. For Cartesian coordinate we have, these are the x, y, z. They are perpendicular to each other and any point here P, which is a function of x, y and z, and we can have this point by simply calculating the x point and then y point and the z point this is the way we calculate.

So, what is the restrictions of this coordinate system? So, the restriction is x is in between this, y is in between -∞ to +∞, and z lies in this range. These are the range of the coordinate points x, y, z, A vector.

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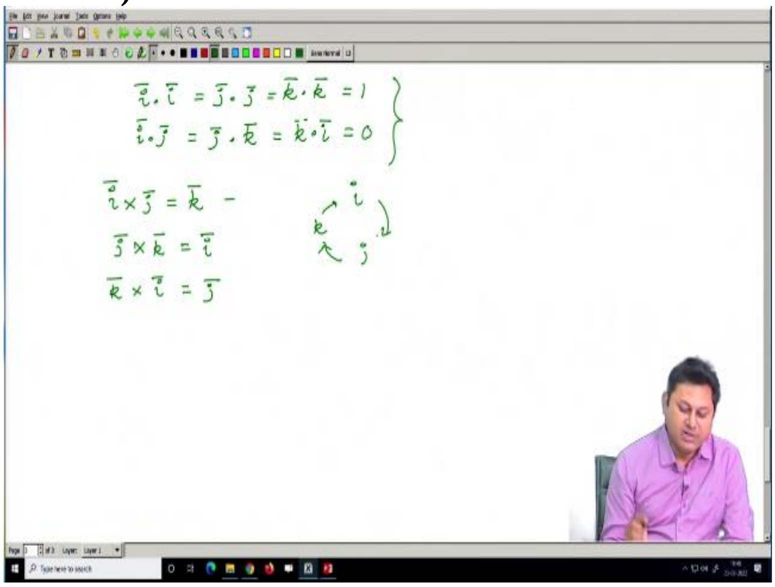

Now, if I want to define a vector in this coordinate system, say vector A that is $A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k}$ and i j k also follows certain rule and that is i dot i = j dot j = k dot k = 1 and i dot j = j dot k = k dot i = 0 orthonormal. They are forming orthonormal basis not only that i cross j = k and then j cross k = i and then k cross i it should be j. They are forming the cyclic thing. i cross j it should be k, k cross i it should be j, and then i cross j it should be k, j cross k it should be i and k cross i it should be j.

After that we have a coordinate system, which is the cylindrical coordinate system. Very important in electrostatics.

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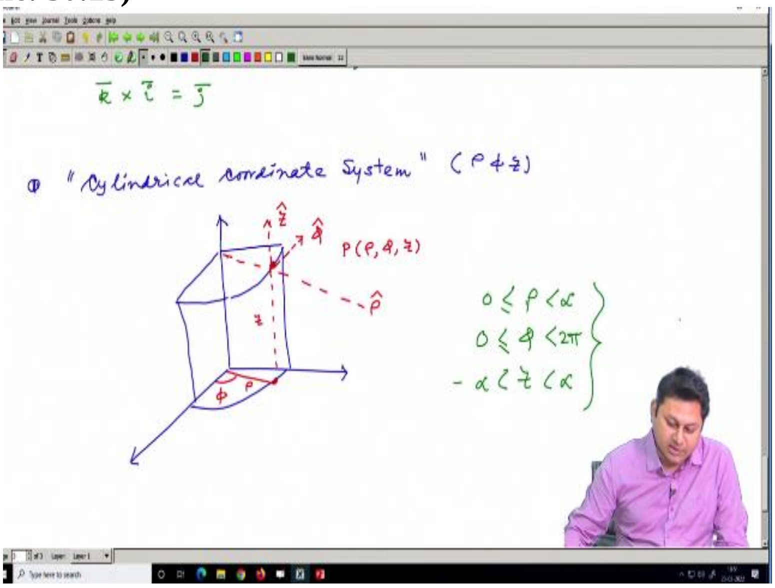

And this is the next coordinate system is normally we call it ρ, φ, z coordinate system in ρ, φ, z coordinate system if I want to you know define a point I have x here not x now, we are not dealing with so, I should not write it x, y, z rather I just make it like this and then. So, a point here should be like over a cylinder so, I will have cylindrical shape here first. Now, I define the point here, this is the point say and this point.

If I make it perpendicular here so, this is my ρ making an angle φ here and from here it is parallel. So, this is my ρ it is along this direction I have ρ unit vector and this is the value of the ρ, this is the φ, and this length is z from here to here this length is z. So, if I now define the unit vectors, so, z will be as usual in this direction this is a unit vector z, this is ρ, and φ will be along this tangential direction.

So, this will be my φ unit vector. The point here P should be defined with these 3 coordinates that is ρ from here to here and then φ the angle ρ is making and the length from here to here z. So, any point in space can be defined by these 3 parameter ρ, φ, z. Now, quickly try to understand that what is the range of this the way we did previously. So, $0 \leq \rho < \infty$. $0 \leq \varphi$

$< 2\pi$ and z it should be in between the ∞ and $-\infty$ like before. So, these are the ranges that we have for the coordinate system this Cartesian coordinate system.

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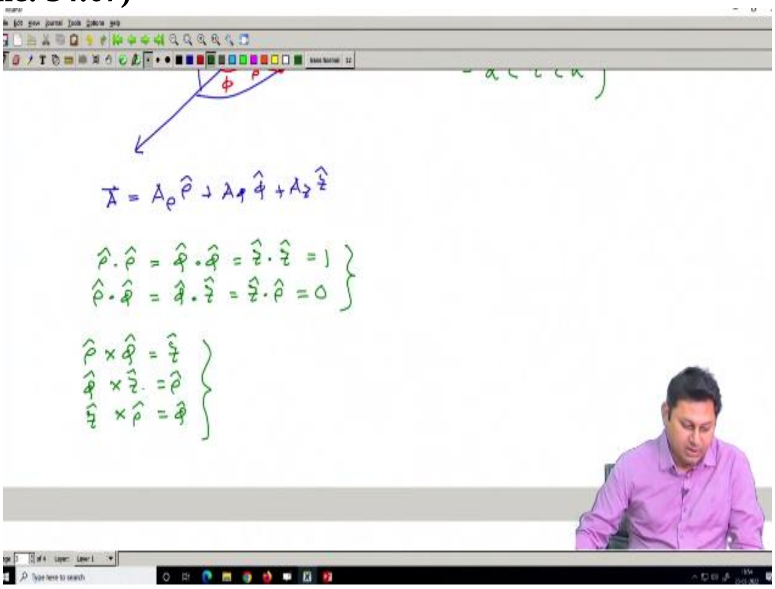

The vector quickly I define the vector how you write this vector A? So, A vector can be written as the component $A_\rho \hat{\rho} + A_\varphi \hat{\varphi} + A_z \hat{z}$. Now, the same vector I defined previously in a different way A_x i A_y j A_z k and the same again the same vector say A I can define in this way. Note that the basis are now changed not only that based on the change basis, we also have to change this component A_ρ A_φ A_z but at the end of the day we are defining the same vector.

Now, what is the relationship between these unit vectors that we are having here. So, let us try to quickly write it the properties. So, here so ρ dot ρ = φ unit vector dot φ unit vector = z unit vector dot z unit vector = 1. Similarly, ρ dot φ = φ dot z that is equal to z dot ρ is 0. That means they are forming an orthonormal basis not only that, this is the right-hand system. So, ρ cross φ give you z, φ cross z give you ρ, and z cross ρ give you φ unit vector.

So, this is a right-hand orthonormal they are forming right-hand orthonormal basis. So, today I do not have that much of time. So, I will in the next class I will you know, before concluding the class, let me at least write the transformation that the relation between x, y, z and because that is related to this.

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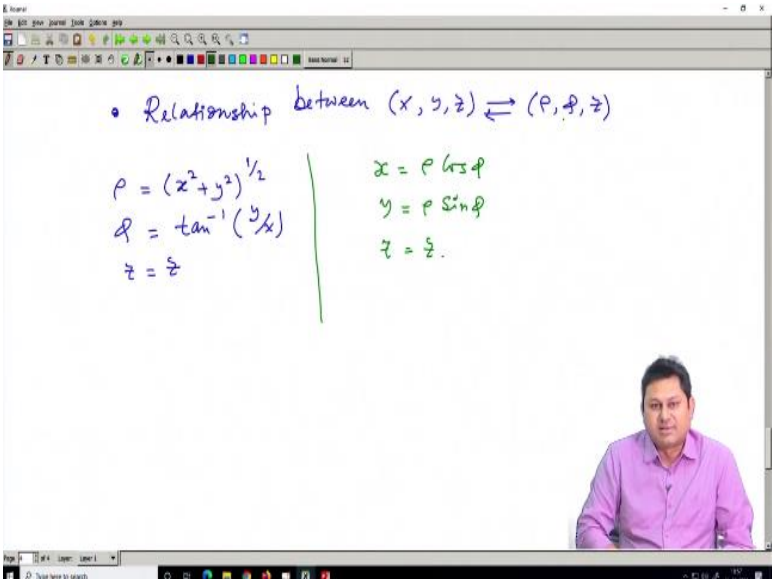

So, the relationship between the coordinate system x, y, z and ρ, φ, z. What is the relation? The relation if you know most of the students, I believe they are aware of that x, y, z can be represented to ρ, φ, z with a relationship and the relationship I am writing here. So, $\rho = (x^2 + y^2)^{1/2}$, φ is $\tan^{-1}y/x$ and z is z. This is the relationship to the left-hand side we have ρ and right-hand side we have x, y, z.

So, if the x, y, z is given to you of a point you can find out what should be the value of the ρ, φ, z in cylindrical coordinate system. In the similar way, I can have a reverse relationship and that is $x = \rho \cos\varphi$, $y = \rho \sin\varphi$ and z = z. So, that is quite trivial I mean if you just look in this coordinate system, what is the value of the ρ if this is x, y, z the line I am drawing here, if I am putting x, y, z here.

Then you can find out and this point is also defined to x, y, z then you know what is the relationship between the x, y, z point and ρ, φ, z point. If ρ, φ, z is given you can calculate the x, y, z if x, y, z is given you can calculate the ρ, φ, z and these are the relationship they will follow and this is quite trivial if you know the ρ then you will find what is the value of x, what is the value of y, $\rho \cos\varphi$, so, you will find the x value, $\rho \sin\varphi$ you will get the y value, and z is z.

So, very trivial relationship they are following and from that also you can find out that what is the value of ρ, φ, z if x, y, z is given. So, with this note, I am not having much time today. So, I like to conclude here and see you in the next class, where we will extend more and try to find out the relationship between the unit vectors. So, now we are having the relationship between the x, y, z coordinate point to ρ, φ, z coordinate point.

Now, the unit vector is also related. If I know i j k how should, I know the unit vector, ρ φ and z. With this note, let me conclude. So, thank you for your attention see you in the next class.