Foundation of Classical Electrodynamics
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Lecture - 28
Electrostatic Potential (Contd.,)

Hello students to the foundation of classical electrodynamics course, under module 2 today we are going to discuss more about the electrostatic potential. So, today we have this is class number 28.

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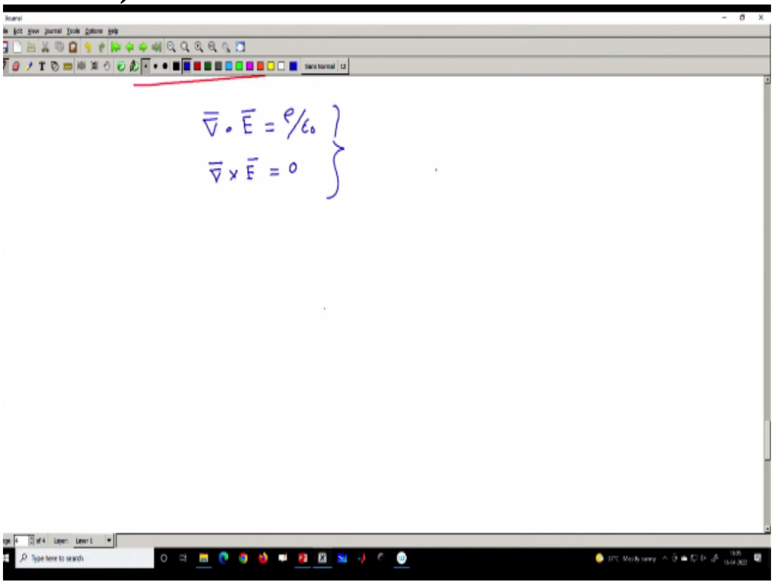

And let us first write what we get in the last class. So, we know that divergence of the electric field is $\frac{\rho}{\epsilon_0}$ and $\vec{\nabla} \times \vec{E}$ is 0 these 2 information lead to the formation of the potential energy exploiting the Helmholtz's theorem and last day we did it.

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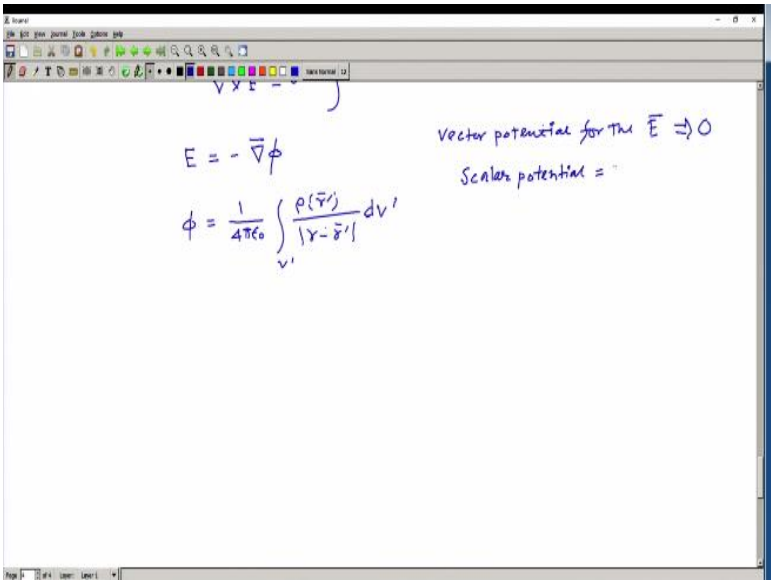

We find that electric field can be represented in terms of a scalar field ϕ, which is the electrostatic potential and ϕ can be written in terms of this. Because this is precisely the Helmholtz's theorem and that is $\frac{1}{4\pi\,\epsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$ dv' prime over v'. So, I can find and also the vector potential for the electric field $\vec{E}$ is simply 0 do not have any vector potential but the scalar potential is very much there and the scalar potential is defined by ϕ, which is this quantity from Coulomb's law we try to understand this.

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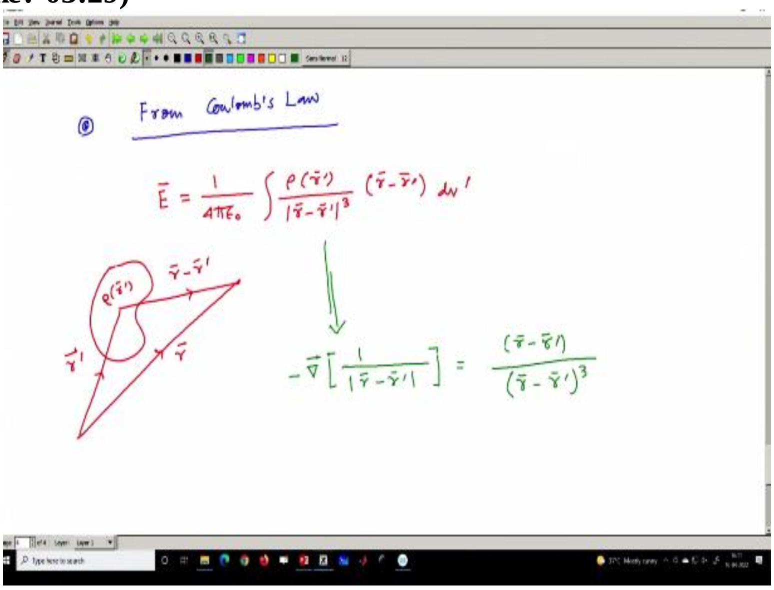

From Coulomb's law, I can write it so, from Coulomb's law now, today we are going to understand the expression of the scalar potential because the Coulomb's law gives us expression of the electric field. So, if I take the gradient of that, if I write this electric field in terms of a scalar potential in this way, then I can extract the information of the scalar potential. So, let us try to do that.

So, my electric field I can write here in this way, which is $\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$ dv' this is the electric field due to the charge distribution ρ at some point. So, if we have a charge distribution here. So, this is the charge distribution and form a given coordinate system, this is $\vec{r}'$ this is $\vec{r}$ and this is $(\vec{r}-\vec{r}')$. So, the field I am calculating at this point and this is the charge distribution we are having so, this is the structure.

For that electric field should be this one I need to integrate. So, ρ is here, which is a function of $\vec{r}'$ and I can calculate the volume integral with this and then the total electric field I can calculate. I can rewrite this expression in a slightly different way by exploiting the fact that minus of divergence of this quantity, which is $\frac{1}{|\vec{r}-\vec{r}'|}$ is simply $\frac{(\vec{r}-\vec{r}')}{(\vec{r}-\vec{r}')^3}$.

So, that we know and if you calculate, then you will find this I suggest you to please check by yourself that whether you are getting this or not very simple calculation. Now, I will go to use I just replace $\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$ with this value here in this equation.

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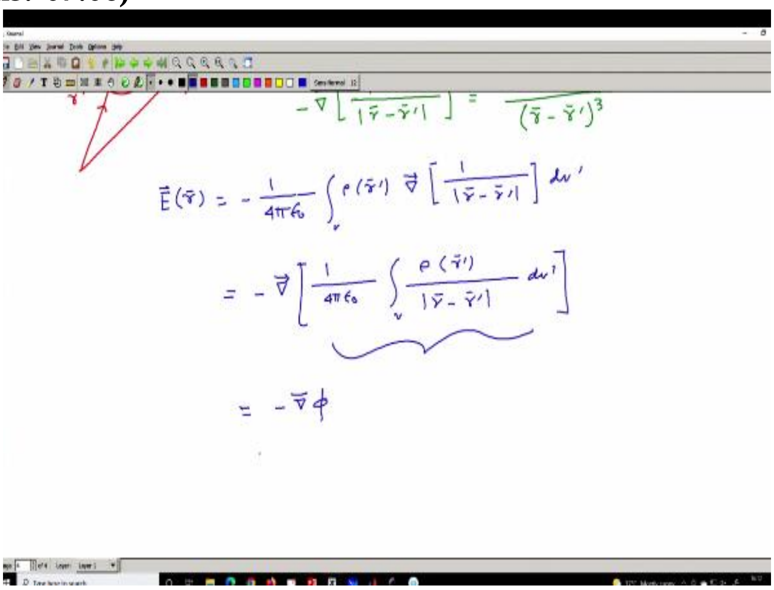

And if I do my $\vec{E}(\vec{r})$ should be $\vec{E}$ function of $\vec{r}$ is simply $-\frac{1}{4\pi\epsilon_0}$ and then integration $\rho(\vec{r}')$ and this quantity I just replace the $\vec{\nabla}\left[\frac{1}{|\vec{r}-\vec{r}'|}\right]$ and dv'. So, this quantity I can make this divergence outside because it is this divergence is operating over $\vec{r}$ so, I can take it outside with a negative sign then it should be divergence of the entire term sorry I can take this gradient I am saying divergence but this gradient outside. So, I will finally have this $\frac{1}{4\pi\epsilon_0}$ this volume integral $\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$

and dv'. So, this quantity is nothing but ϕ so, I can write it as -ϕ.

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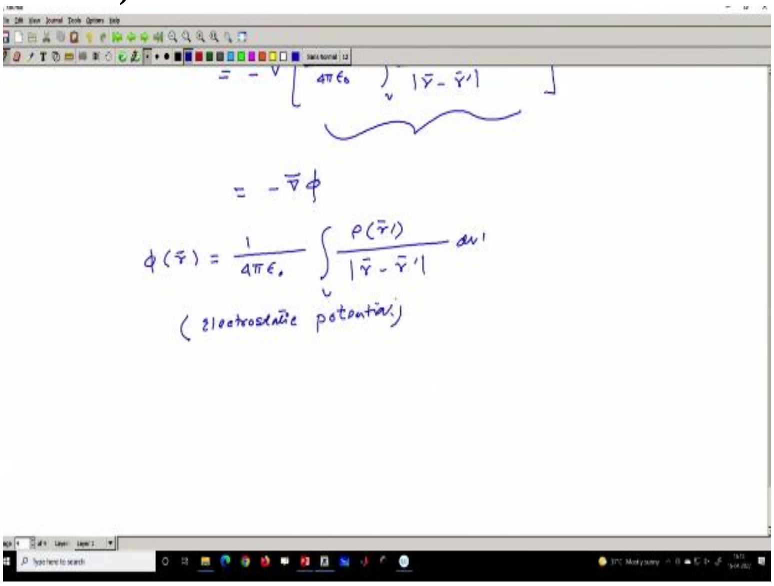

And ϕ becomes this, which we already find exploiting the Helmholtz's theorem. So, this is the electrostatic scalar potential or simply electrostatic potential.

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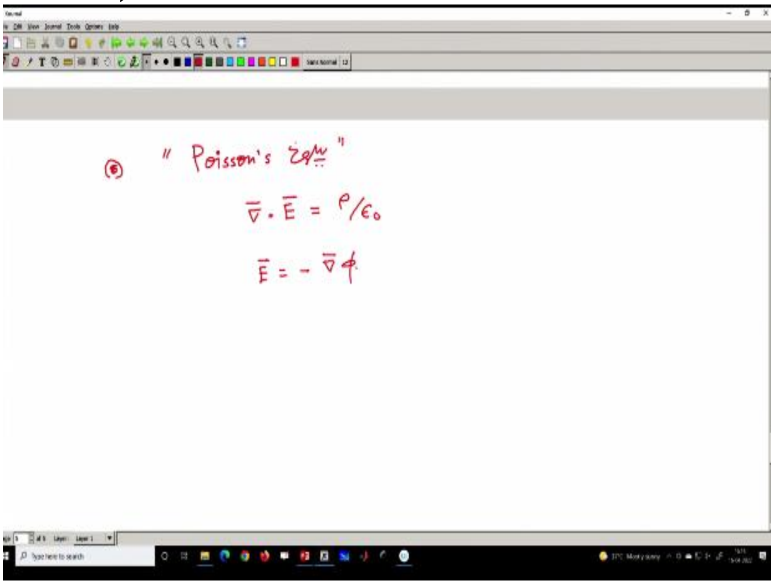

So, the next thing is from the expression is Poisson's equation very important and that we can simply understand that $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ and also $\vec{E}$ can be written as minus of the gradient of a scalar field ϕ, which is called the electrostatic potential.

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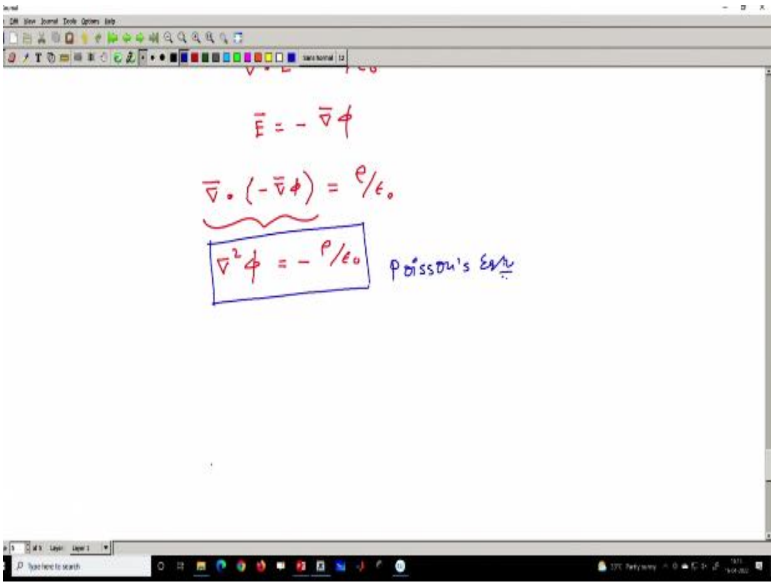

So, if I replace this in the first equation, then I simply have divergence of minus of gradient of the scalar field ϕ is $\frac{\rho}{\epsilon_0}$ this quantity this is not new, we have already discussed this quantity earlier. This is simply forms the Laplacian, which is ∇^2. So, this basically leads to a ∇^2 operator and I can write it as $-\frac{\rho}{\epsilon_0}$. So, this very equation is called the Poisson's equation.

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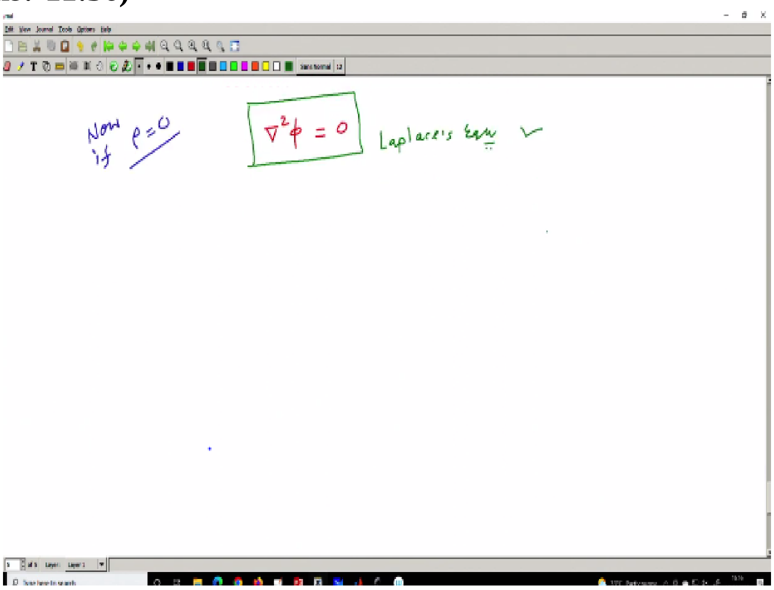

Now, if $\rho = 0$ we are putting the condition that there is a region where the charge density vanishes or there is no charge density then this equation is simplifies to this and this is also a very important equation so, in the region of 0 charge density that is this region and this equation is called Laplace equation. So, we now have an idea about the Poisson's equation and Laplace equation the solution of this equation we will do later. So, quickly let me write down the forms of potential in different charge distribution.

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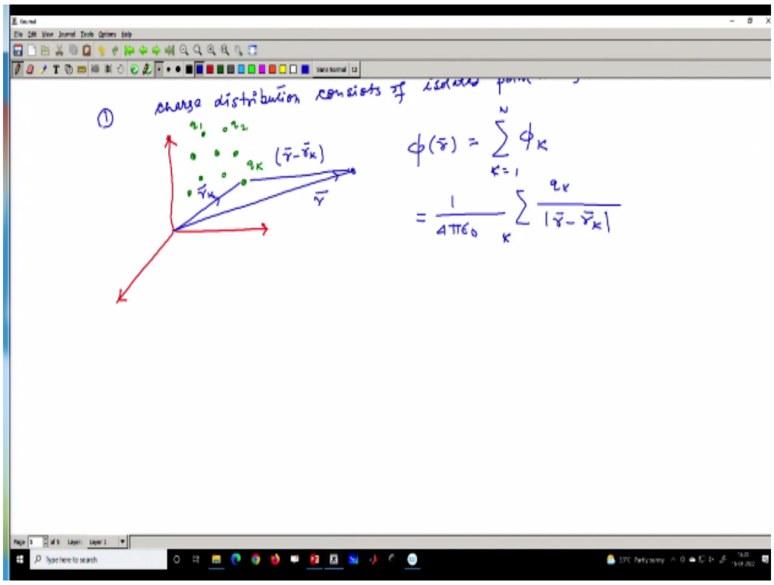

So, next is the different form of potential or simply forms of potential and what is that so, first in case 1 we have discrete charge distribution consists of say isolated point isolated charge, so isolated point charge are forming a charge distribution and for that if I want to find out the expression of the potential what it would be? That is, so this is suppose the coordinate system and in this coordinate system we are having the charge distribution like this, some random charge distributions are there like this and say this is q_1, q_2 and this is q_k.

So, I want to find out the potential at the point $\vec{r}$ due to all these charge distribution. So, for q I can have this is $\vec{r}_k$ and this quantity is $\vec{r}$ - $\vec{r}_k$. So, I can simply use the superposition principle because ϕ at this point p at point $\vec{r}$ is simply the superposition of all the ϕ that we have for different charges, the k tends to 1 to if there are N charges, it should be like this. So, if I write it, it is simply $\frac{1}{4\pi\epsilon_0}$ and then $\sum_k \frac{q_k}{|\vec{r}-\vec{r}_k|}$.

And that is all k goes to so, this is for each charge particle we have the potential here and if you simply add these we are going to get the total potential at that point, this is for discrete charge distribution. What happened when we have line charge, so, this is case 2.

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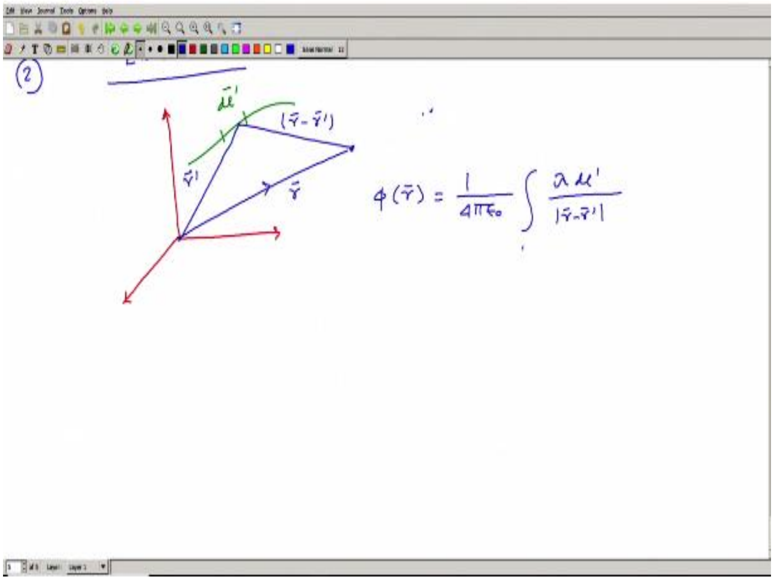

So, case 2 I have for case 1 we have charge distribution with isolated charge point and now, this is simply line charge. So, I have a line charge distribution so line charge so I have again a coordinate system here and a line charge is distributed, this line is given like this, this is say dl' and the coordinates on distributed like this is $\vec{r}$, this is $\vec{r}$' and this is $\vec{r}$ - $\vec{r}$'. So, the point here so, my simply my ϕ ($\vec{r}$) will be so let me write it here. The ϕ at the point $\vec{r}$ will be $\frac{1}{4\pi\epsilon_0}$ integration of line charge distribution λ or the density λ then dl' and $\vec{r}$ - $\vec{r}$' that is all with a line charge integration.

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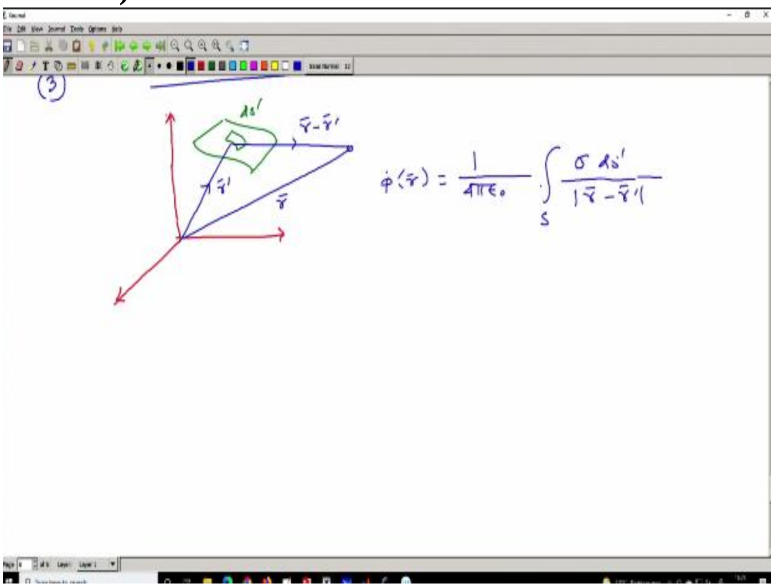

Next, in a similar way if I go on in case 3 for surface charge I have a coordinate system and then I have a small surface element here which we call ds' and like before we have these coordinate points this is $\vec{r}$' and this is $\vec{r}$ - $\vec{r}$', the potential here should be simply ϕ at $\vec{r}$ for this kind of distribution is $\frac{1}{4\pi\epsilon_0}$ integration of σ, which is a surface charge density ds' over $\vec{r}$ - $\vec{r}$', over surface charge integration and finally the volume charge that we already discussed, but still let me write it.

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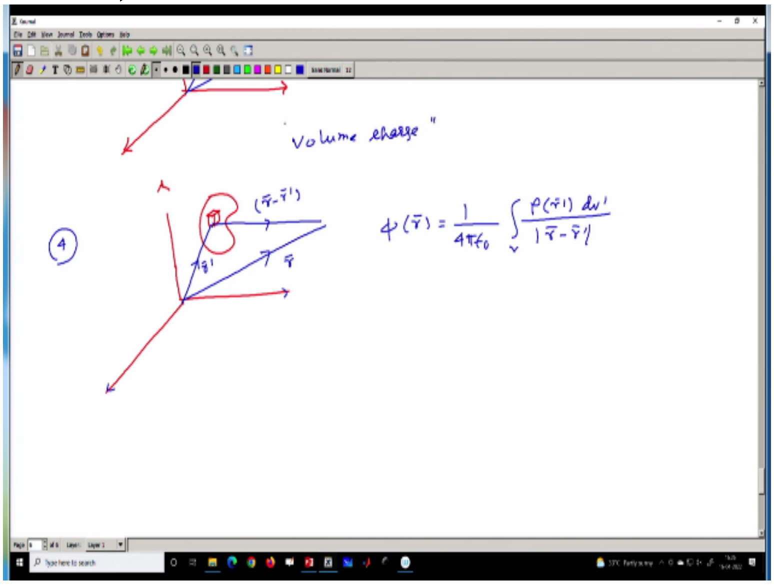

So, now, this is 4 so we have a volume charge element here like the way we find the electric field we are now. So, here we have a volume element and coordinates are defined the way we have defined so far for all other cases that this is $\vec{r}$, this is $\vec{r}\,'$ and this is $\vec{r}$ - $\vec{r}\,'$ so the potential that one can expect at point $\vec{r}$ is simply $\frac{1}{4\pi\epsilon_0}$ then volume integral $\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$ dv' that is all. So, this is for volume charge. So, after all these distributions so let us now quickly do few problems one is potential due to uniformly charge wire of length 2l say.

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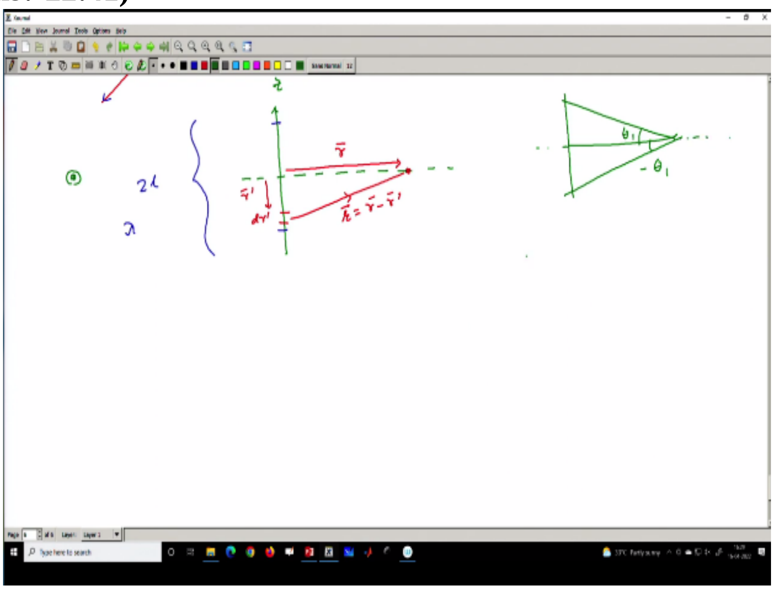

So, we will try to do 1 quick problem and try to find out the wire so, this is the wire we are having but this is a finite wire say this is the z direction and we have the wire of 2l this length is 2l and uniformly charge means the line charge density is λ here. Now, the point is what should be the thing that we want to find is what should be the potential at some point say here at $\vec{r}$. So, from here to here this is say, $\vec{r}$ and what should be the potential at this point mind it the potential is the scalar quantity.

So, if I want to find out the potential so small element I will calculate for all the small element I will calculate what the potential here is and then sum it over. So, let us take a small segment here and maybe this is say dr' and from here to here, this is $\vec{r}\,'$ and this length from here to here this is $\vec{\mathcal{R}}$, which is $\vec{r}$ - $\vec{r}\,'$. From the geometry we can have one thing let us first do so that when we integrate. So, this is the 2 points of the wire if this angle is θ_1, so this angle has to be $-\theta_1$ because I am measuring from this axis angle axis.

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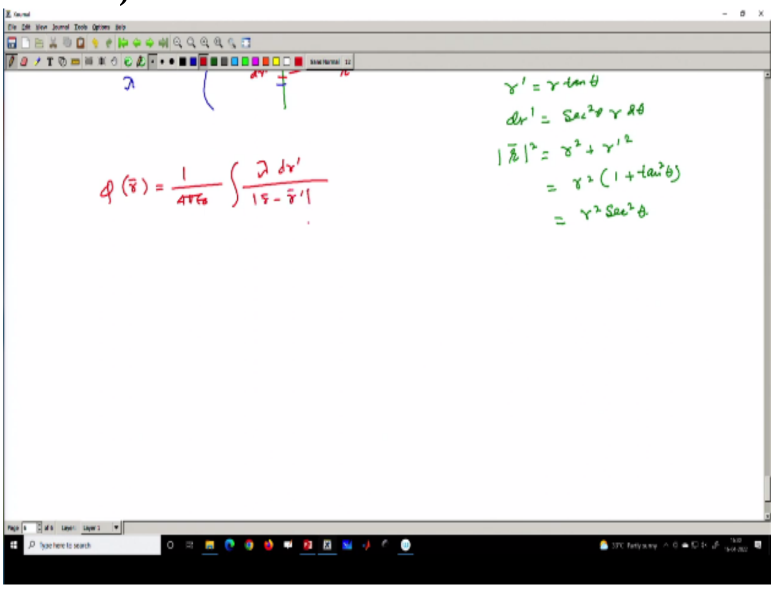

And $\vec{r}\,'$ = $\vec{r}$ tan θ, because this is $\vec{r}\,'$, this is $\frac{\vec{r}}{\vec{r}'}$ is tan θ. So, $\vec{r}\,'$ = $\vec{r}$ tan θ. So, dr' is $\sec^2\theta$ and then r dθ and $|\vec{\mathcal{R}}|^2$ is simply $r^2 + r'^2$, which gives us $r^2(1 + \tan^2\theta)$, this is nothing but $r^2\sec^2\theta$, you can also find it to using the simple geometry what is there, it is just I calculate because we are going to use this later on. So, now what is my ϕ at point $\vec{r}$ so, this is $\frac{1}{4\pi\epsilon_0}$ and then $\int \frac{\lambda}{|\vec{r}-\vec{r}'|}$ dr' .

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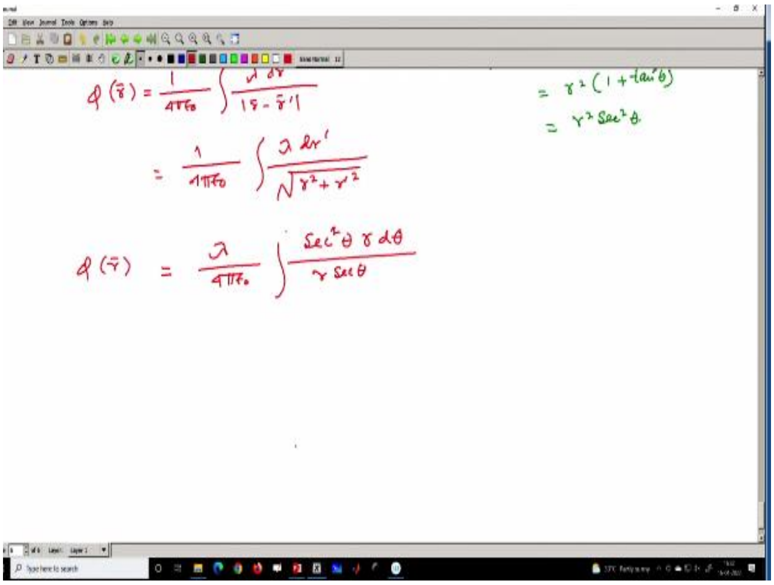

So, that quantity is $\frac{1}{4\pi\epsilon_0}$ integral I need to put the boundary here, the value here. So, dr is goes from -l to +l that we will put later. So, λ dr*'* and this quantity is $\sqrt{r^2 + r'^2}$. So, if I go on with the calculation whatever we are doing so, at point $\vec{r}$ is $\frac{\lambda}{4\pi\epsilon_0}$ because λ is a constant I can take it out and dr*'* we calculate here.

So, that value we will be going to put it is $\sec^2\theta$ r and then dθ divided by r^2 this quantity I already calculate. So, this value is r of sec θ so lsec θ simply cancel out. So, this and this will cancel out.

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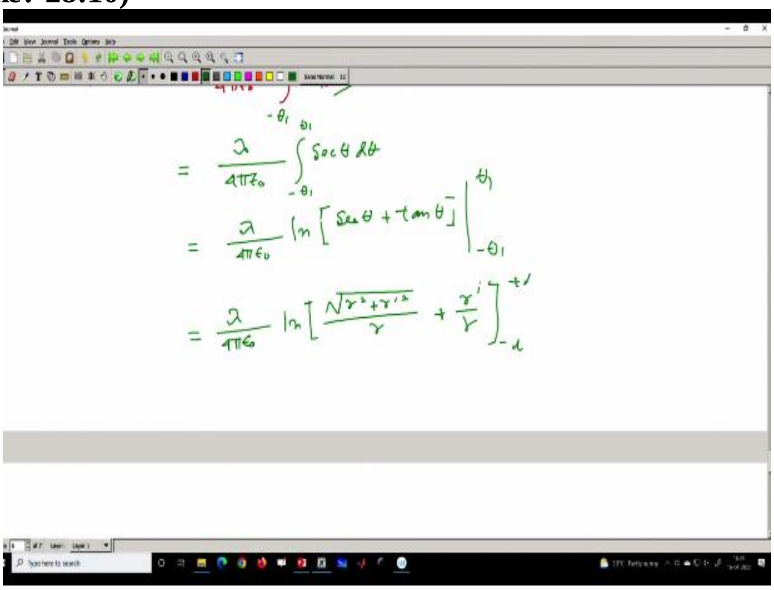

So, eventually we get $\frac{\lambda}{4\pi\epsilon_0}$ then integration and also the r r will cancel out. So, simply I get simple integration and that is sec θ dθ and we know from the standard integrals this value is now into put θ because θ here is $-\theta_1$ to $+\theta_1$ so, that after putting this I can put it so, my limit is $-\theta_1$ to θ_1, this is ln and then it should be sec θ + tan θ that will be executed at $-\theta_1$ and θ_1.

So, this if I now go back to the other real coordinates, which is r and all this so, it is simply ln then sec θ I calculated it is $r^2 + r'^2$ divided by r and tan θ also I calculated it is $1 + \frac{r'}{r}$ and that thing is from –l to +l because now I removed my θ. So, this value if I put so I will get my result it is a straightforward but a bit lengthy calculation but anyway.

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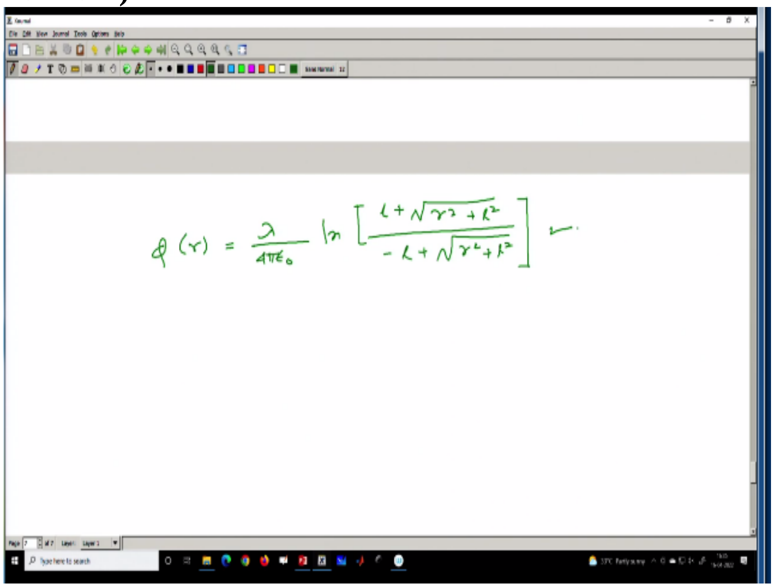

So, r you will get the value at $\frac{\lambda}{4\pi\,\epsilon_0}$ ln and then it will be $\frac{l + \sqrt{r^2+l^2}}{-l + \sqrt{r^2+l^2}}$. So, this is the value of the potential for a charged wire having the length +l to –l. So, now we are going to exploit this result to find out the total electric field at some point r that we also calculated using the Gauss's law earlier so, that we cross verify.

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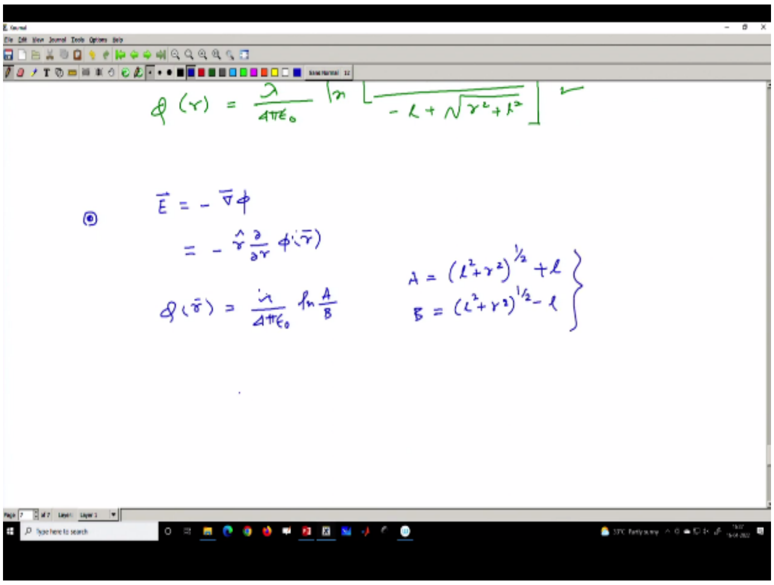

So, my $\vec{E}$ now, so, my ϕ is given. So, now, I calculate exploiting this ϕ I will be going to calculate the $\vec{E}$, so electric field is minus of this quantity. So, the electric field I can write as a minus of the potential ϕ and ϕ is now given but this ϕ is given in terms of r so, this is a function of r. So, I can write this operator as in spherical coordinate and that should be $\hat{r}\,\frac{\partial}{\partial r}$ and then ϕ as a function of $\vec{r}$. So now, ϕ ($\vec{r}$) already I calculate so, I can write in a simpler way.

So, ϕ ($\vec{r}$) is like $\frac{\lambda}{4\pi\epsilon_0}$ and then ln $\frac{A}{B}$. What is A here? A is simply $(l^2 + r^2)^{1/2} + l$ and B is simply $(l^2 + r^2)^{1/2} - l$, this is my A and B. So, now if I want to find out this $\vec{E}$, I need to make a partial derivative with respect to r to this ϕ.

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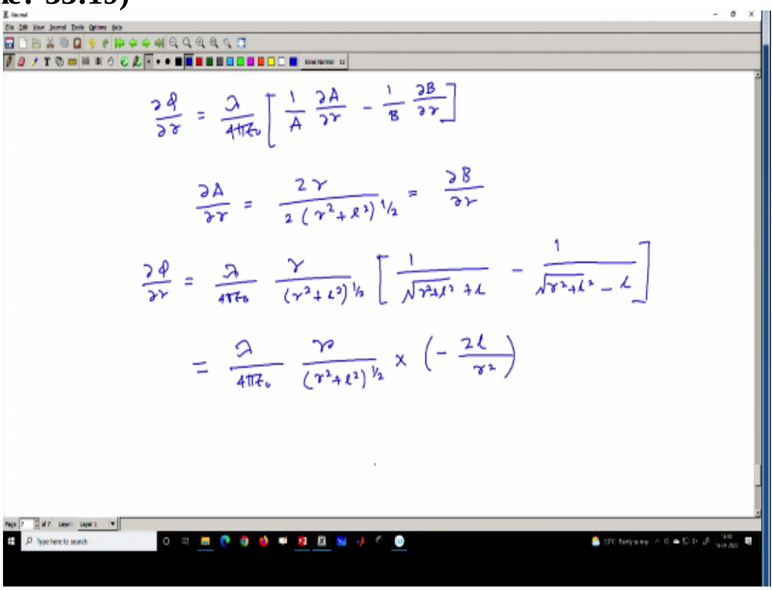

So, $\frac{\partial \phi}{\partial r}$ comes up to be $\frac{\lambda}{4\pi\epsilon_0}$ will be constant and stay like this and I have $[\frac{1}{A}\frac{\partial A}{\partial r} - \frac{1}{B}\frac{\partial B}{\partial r}]$. Now $\frac{\partial A}{\partial r}$

from the expression it is easy to calculate and that is nothing but I have 2r here divided by 2 into $(r^2 + l^2)^{1/2}$. So, it is simply equal to because for B because this is a negative sign and constant, so this quantity is simply equal to $\frac{\partial B}{\partial r}$.

So, eventually, what I find is $\frac{\partial \phi}{\partial r}$ is $\frac{\lambda}{4\pi \epsilon_0}$ that is there and I can take these $\frac{r}{(r^2+l^2)^{1/2}}$ common because both the term I will be going to get that and only I have $\frac{1}{\sqrt{r^2+l^2}+l} - \frac{1}{\sqrt{r^2+l^2}-l}$, because this is $\frac{1}{B}$ - l. So, eventually that leads to the term like $\frac{\lambda}{4\pi \epsilon_0}$.

Then $\frac{r}{(r^2+l^2)^{1/2}}$ multiplied by minus of B if we calculate this and this will cancel out. So, 2l and A + B into A - B, so, that is $A^2 - B^2$. So, I simply have r^2 here.

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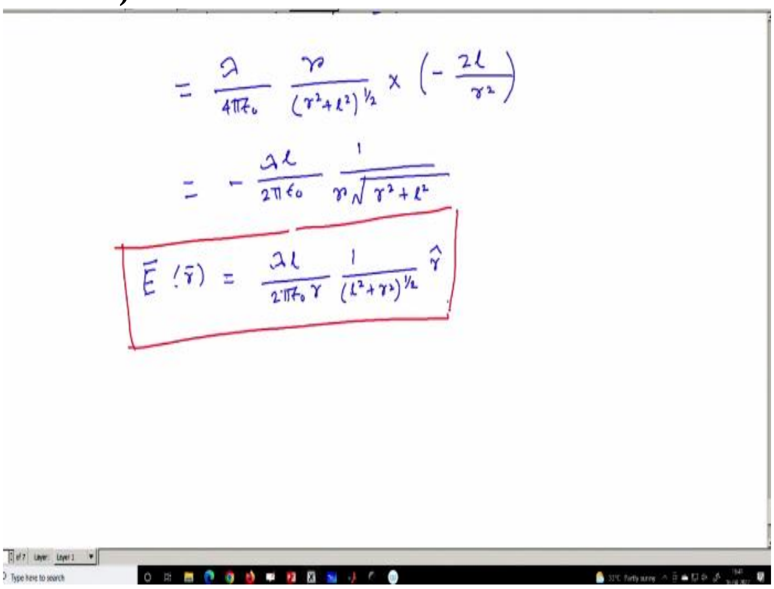

So, eventually this term is $-\frac{\lambda l}{2\pi \epsilon_0}$ this 2 will cancel out with this 4 so, that gives me 2 here and then we have $\frac{1}{r\sqrt{r^2+l^2}}$. So, that should be my $\frac{d\phi}{dr}$ and how the $\frac{d\phi}{dr}$ is related to $\vec{E}$? $\vec{E}$ is simply $-\frac{d\phi}{dr}$ with $\hat{r}$. So, my electric field $\vec{E}$ here is at point $\vec{r}$ due to the wire having length 2l is simply $\frac{\lambda l}{2\pi\epsilon_0 r}$ and then $\frac{1}{\sqrt{r^2+l^2}}$ and $\hat{r}$.

So, that is the value of the electric field for a finite length of r so, finite length of 2l of a charged wire. Now, we calculated earlier if you remember the electric field for infinite wire.

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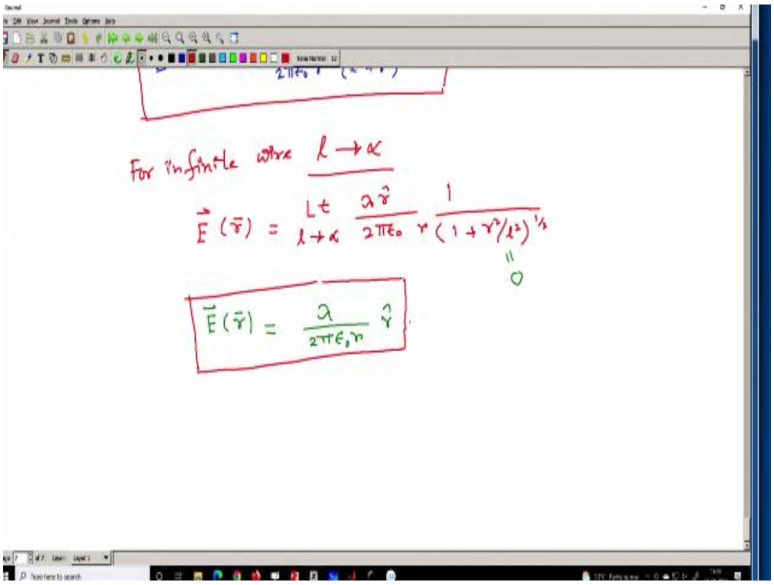

So, for infinite wire I have l tends to infinity that is the condition, for infinite wire I can put l tends to infinity. So, if I put l tends to infinity, I will get back my old result that we have already calculated for infinite wire in earlier classes l tends to infinity $\frac{\lambda\hat{r}}{2\pi\epsilon_0 r}$. So, if I take r^2 common then it should be $(1 + \frac{r^2}{l^2})^{½}$ one l was there in the numerator, so, that I put lower side and to make $\frac{l^2+r^2}{l^2}$.

So, this quantity will be 0 if l tends to infinity. So, eventually I will get the standard result that the electric field for infinitely charged wire, infinitely extended charge wire with uniform line charge density $\frac{\lambda}{2\pi\epsilon_0 r}\,\hat{r}$, so, that is the result we already figured out. So, I just verify this figure once again exploiting the expression of the ϕ that we calculate. So, ϕ first we calculate the expression of the potential ϕ and then from that we calculate the electric field.

So, with this note today I do not have much time to discuss so, I like to conclude here. So, today we are going to learn that how the potential can be calculated for different cases. In today's class I just calculate for 1 problem that for a finite wire but in tomorrow's in the next class maybe I can go on with this calculation and we will be going to calculate what happened when we have a charged sphere and what should be the charged disk and what should be the potential for that and these kinds of calculation we will do in meticulously.

So, with that note, let me conclude today's class. Thank you for your attention and see you in the next class.