Foundation of Classical Electrodynamics
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Lecture – 26
Application of Gauss's Law

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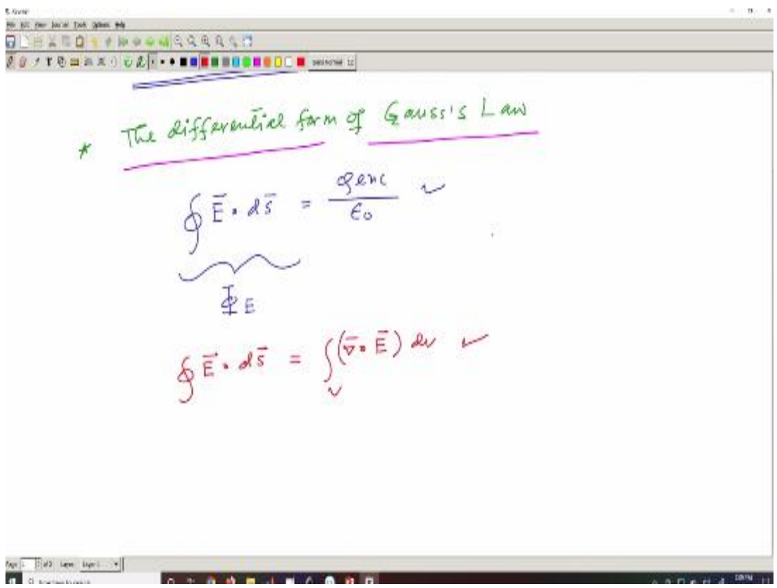

Hello students to the foundation of classical electrodynamics course. So, today we have lecture number 26 where we discuss about some application of the Gauss's law. So, we have class number 26, the first before going to the application few things we need to discuss. First we discuss the differential form of Gauss's law very important. The differential form of Gauss's law. So, we have $\vec{E} \bullet d\vec{s} = \frac{Q_{enc}}{\epsilon_0}$ that is our Gauss's law. And now, this quantity is total flux ϕ_E.

Now, from the integral theorem we already prove this theorem the divergence theorem, which suggests that if I have a close surface integral of a vector field then that is nothing but the divergence of this vector field over the volume through the volume, which is enclosed by this closed surface, so, that is the theorem we have.

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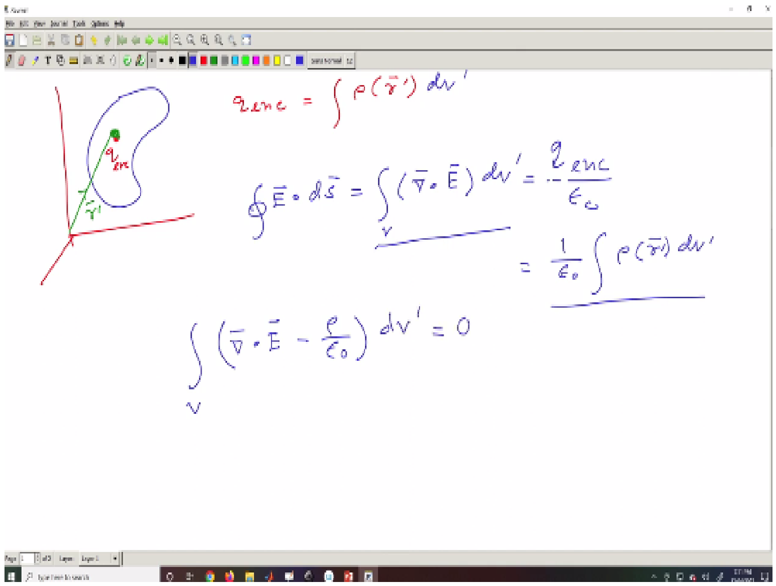

Now, we already know that the Q_{enc} suppose you are having so, this is the picture you are having and inside that you have a charge q. So, this is say q_{enc}. So, q_{enc} I can write in terms of the volume charge density and it simply is ρ, which should be some function of say $\vec{r}\,'$ and if this is my coordinate system and if I say this is the point location of the point charge $\vec{r}\,'$ this is my location of the point charge so, that and then dv' something like this.

So, now, if I write down this q_{enc}, so, it should be like this. So, my $\vec{E} \cdot d\vec{s}$ that is $\vec{\nabla} \cdot \vec{E}$ over the volume say v and that quantity is $\frac{q_{enc}}{\epsilon_0}$. So, let me write it $\frac{Q_{enc}}{\epsilon_0}$ and Q_{enc} now, this is q so I should use this q here as well not Q, but $\frac{q_{enc}}{\epsilon_0}$ so that I write $\frac{1}{\epsilon_0}$ integration of ρ as a function of r dv for the time being, let us say dv you can also do dv' if here you say this volume is in prime notation, because this is the volume.

So, this is dv'. Now, from this equation and this equation both are volume integrals, so I can write this equation as the volume integral of the $(\vec{\nabla} \cdot \vec{E} - \frac{\rho}{\epsilon_0})$ and then dv' = 0 and this equation is true for all the volume.

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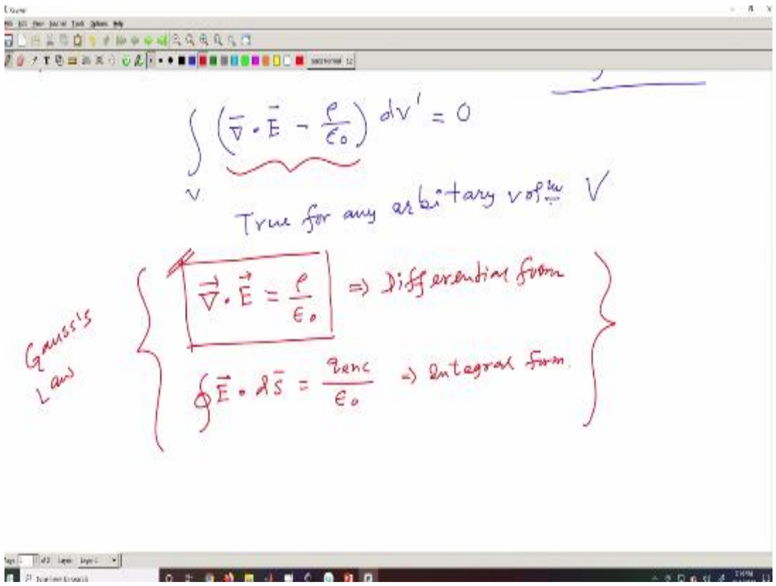

True for any arbitrary volume v if that is the case, then from this equation I can write that this quantity is 0 or I can write $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$, this is basically the equation and it is called the Gauss's law in differential form because this operator is basically a differential operator. So, previously we write the Gauss's law like $\vec{E} \cdot d\vec{s} = \frac{q_{enc}}{\epsilon_0}$. So, this is called the integral form because we are using integration here.

And the same expression you write in a different way and that is called the differential form both are Gauss's law just interpretation in a different way this is the integral form and then we have the differential form this is called differential form so, 2 different kinds of form and this form is very important because this is one of the Maxwell's equation we will discuss later. So, this is eventually the first Maxwell's equation $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ eventually this is a Gauss's law and the differential form.

But also the integral form is there and we calculate from the integral form we calculate that how the differential form will look like. After that we will just test what we have I mean whether it is consistent with the Gauss's law or not because these we check with Gauss's law, so, we are going to check this also. So, how do I check that?

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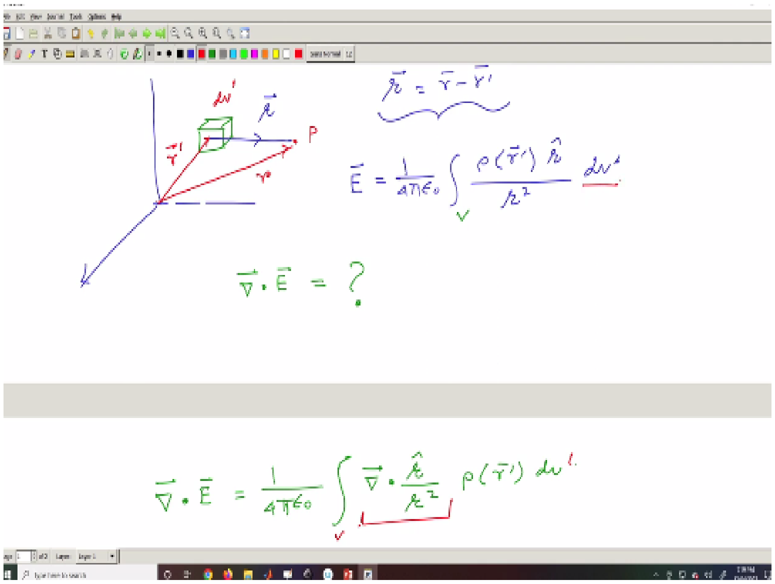

So, simple way that this is the coordinate system suppose and I have a volume element here small volume element and I want to find out the field at some point here say some point here P. So, this is from here to here, this point is r and from here to here, the volume element is our notation it is prime this volume is dv' and from here to here this quantity is $\vec{\mathscr{r}}$. So, $\vec{\mathscr{r}}$ is simply $\vec{r}$ - $\vec{r}$', now, the Coulomb's law if you want to use the Coulomb's law here what is the field?

Then $\vec{E}$ the total field at point P is simply $\frac{1}{4\pi\epsilon_0}$ and then we should have the integration and this integration should contain the charge density here in this volume and the charge density should be a function of $\vec{r}$' because it is in prime frame and then $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ and then dv' this is simply the Coulomb's law, but in integral way, when we have a continuous charge distribution, we mentioned that in the earlier class. Now, if I directly put the divergence of these things I want to find out this is given over volume.

And I try to find out this quantity $\vec{\nabla} \cdot \vec{E}$. So, what I get will be what? According to the previous line, it should be simply $\frac{\rho}{\epsilon_0}$ should we get $\frac{\rho}{\epsilon_0}$ here that is something we need to check. So, the $\vec{\nabla} \cdot \vec{E}$ if I calculate with the given $\vec{E}$, which is nothing but the Coulomb's law that should be $\frac{1}{4\pi\epsilon_0}$ this is constant integration is there, so, I can interchange this operator integration can be done.

So, here I operate both the side with this divergence operator, but integration is there I can put this operator inside because these are the linear operator. So, I can have simply the divergence of over this $\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}$ and then ρ this is a function of r' and dv prime.

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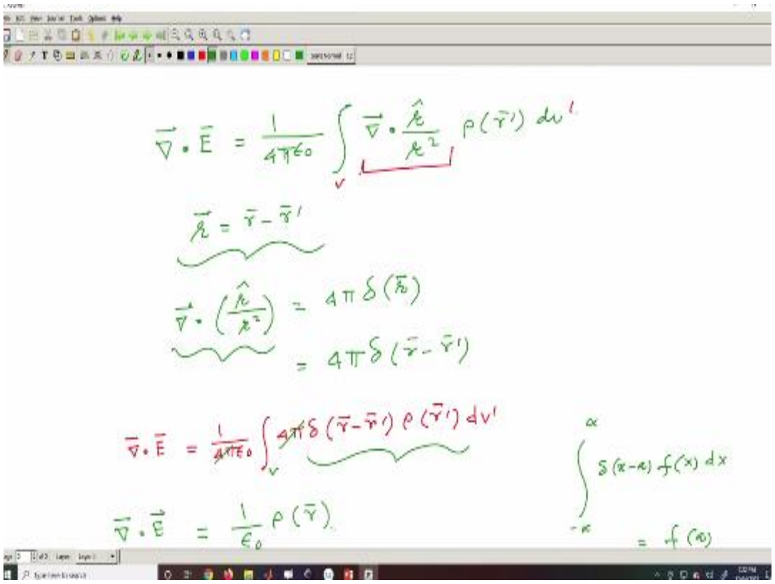

Now, $\vec{\mathfrak{r}}$ you should remember that it is $\vec{r}$ - $\vec{r}$' this is $\vec{\mathfrak{r}}$. So, if you I mean we should remember that when we did the divergence of the function vector field $\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}$ then it was $4\pi\delta$ and that was $\vec{\mathfrak{r}}$ so, here this quantity is simply $4\pi\delta(\vec{r} - \vec{r}')$ that is a value. So, that value I am going to use here because I am having this same term that is sitting exactly in this location this point. So, this is the volume integral by the way and this dv'.

So, like here you can see this is dv'. So, I eventually get this $\vec{\nabla} \bullet \vec{E}$ I have $\frac{1}{4\pi\epsilon_0}$ integration of that quantity $\vec{\nabla} \bullet \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}$ is simply replaced by that quantity, which is 4π and then delta function and then $\vec{r}$ - $\vec{r}$' and then ρ and then $\vec{r}$' and dv'. So, this 4π is simply cancel out because this 4π and this 4π is simply cancel out. So, eventually I have $\frac{1}{\epsilon_0}$ and if you look carefully this is over entire volume.

And this is delta function multiplied by a function and we know that so, if I write here that if I have a δ(x – a) and then f(x) dx this is in one dimension from x goes to minus infinity to plus infinity entire region then that value is simply f(a) that we know that is the property of the delta function here we are just using so, in place of f I have ρ, which is at $\vec{r}\,'$ and $\delta(\vec{r} - \vec{r}\,')$, so, I can simply have these things $\rho(\vec{r})$.

So, this quantity $\vec{\nabla} \cdot \vec{E}$ is consistent with whatever the expression we had earlier here in differential form so, that is consistent that is consistent here. So, the next thing is some application we like to introduce here and that application here is this.

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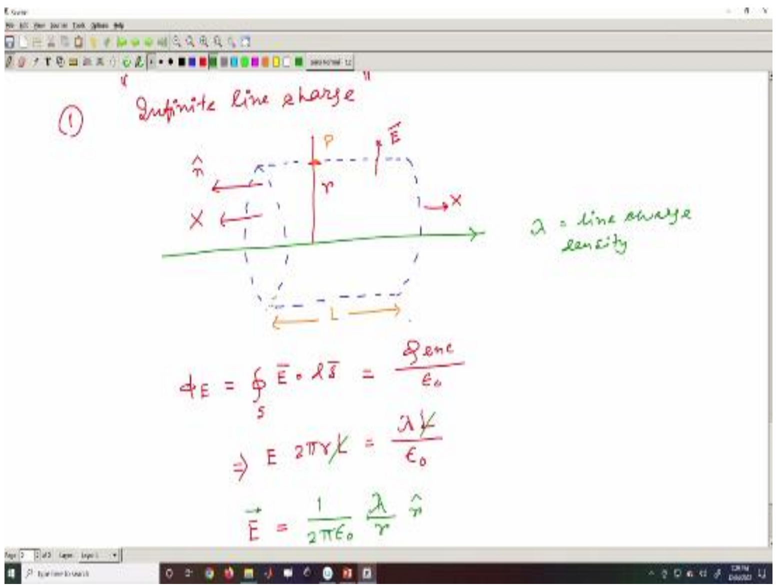

So, application next is application of this Gauss's law. Few standard application we are going to use here this is and our standard application is for example, for infinite this calculation we already done rigorously and the case. So, first application is infinite line charge, so infinite line charge calculation we already done in earlier classes. So, that the same calculation we do in much more easier way by exploiting the Gauss's law. So, suppose this is a line and infinitely extend line with the charge density is say λ.

λ is the line charge density. So, in order to calculate the field at some point r so, my goal is to find the field over some point so, I this is some length here some length at r, which is perpendicular to this infinitely extent line what should be the field at that point and in order to find the field we simply this is a technique that we simply first define a Gaussian surface this is called the Gaussian surface and from the symmetry of the problem, we can see that we can just simply consider a Gaussian surface like this, this is nothing but a cylinder.

A cylindrical surface we consider here whose radius is r and this length so, this is the surface I consider and the length from here to here is say L, this is the length so, that I can calculate the surface area. So, electric field at some point P here say this is the point P where I am trying to find out the electric field due to this infinite long charge. So, I can write I can first draw the Gaussian surface here around this line and then calculate the flux due to this.

So, the flux if I calculate total flux that should be integration of $\vec{E} \cdot d\vec{s}$ and that is $\frac{Q_{enc}}{\epsilon_0}$ according to the Gauss's law. Now, here I know what is for this surface if I use this whatever the surface I choose this is my surface. So, if I use this surface, then it should be E and the surface area if I calculate it should be 2πrL kindly note that 2πrL is only the surface in this I should exclude the surface, which is here there are 2 surface, this side and this side so, this surface I will exclude and this surface I will exclude.

Because the electric field and the surface they are perpendicular in these directions. By symmetry we can see that here if it is a line charge the electric field is going outward. So, the direction of the electric field should be in this direction. And the surface here in this direction the direction of the surface is this. So, if I calculate $\vec{E} \cdot d\vec{s}$ for these 2 surface I will going to get 0 value. So, that is why I only consider the surface here and then it should be 2πrL. What is Q_{enc}? The Q_{enc} is the amount of charge that is enclosed by this surface.

And that is simply the line charge density λ multiplied by the line amount of segment $\frac{L}{\epsilon_0}$ that is all. So, from this equation, I do not need to put this vector sign anymore because I already make this dot product here. So, from this we can calculate the magnitude of the electric field and it should be L, L will going to be cancel out. So, you can see that there is the geometry is not there and I get simply $\frac{1}{2\pi\epsilon_0}$and 1 divided by sorry $\frac{\lambda}{r}$.

If I want to find out what is the vector? Then vector will be at the unit direction of r as well. So, this calculation we already did for infinite long wire and meticulously we calculate this, but using the Gaussian surface you can see that how easily you can execute the amount of electric field for this. Next we will do for infinite surface again for infinite surface we calculate.

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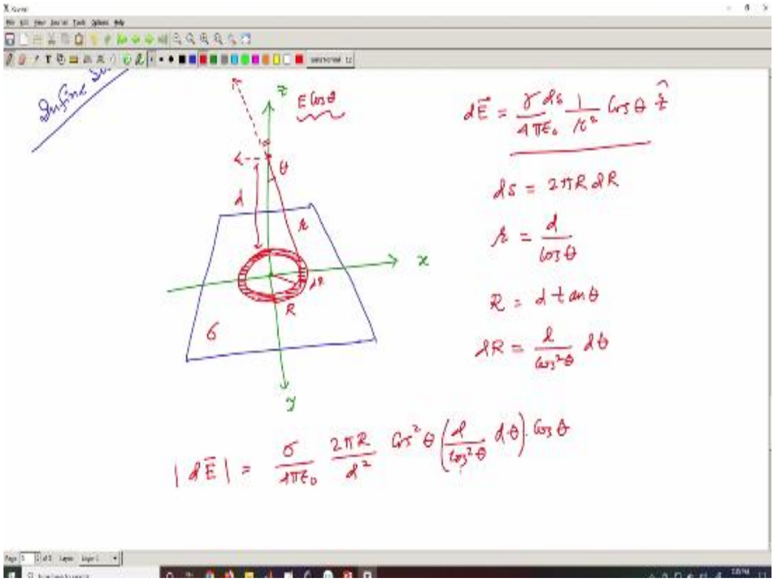

So, this is for infinite surface, so, you may remember the calculation, but still let me do it once again quickly, meticulously, we can calculate the surface. So, suppose I can have a surface like this I will do here in a different way the way we did last time, you can several ways possible. So, suppose this is a surface here I have a coordinate here this is one coordinate, this is another coordinate and another coordinate is perpendicular to that.

So, this is the z coordinate I am having. Suppose, and this is say x y, so, the surface in x y surface, this surface is parallel to xy plane. Then I can have a section here, a small surface section like this, where from here to here it is say R and I want to find a field somewhere here and I can have a section here I can join this. And now you can see by symmetry if this angle. So, suppose this field this is the direction of the field so, I can divide this reduction of the field into 2 parts.

One is along this direction and another is along z direction. So, obviously, this horizontal component will cancel out because of the symmetry. So, if this is θ so, along this direction we should have E. So, this angle, will θ as well. So, E cos θ will be the component that will contribute the sin component will cancel out. So, if I calculate the dE here the small amount of the electric field due to the strip whatever the strip I have. So, dE so, it is simply $\frac{\sigma}{4\pi\epsilon_0}$ and the σ is surface charge density.

So, if I want to find out what is the amount of charge here? And that is simply σ into ds, ds is the area of this segment and then 1 divided by if I say this is my Л from here to here then $\frac{1}{Л^2}$ and cos component is there and also it should be along z direction that is the electric field we can have for only this segment whatever a segment we are having this segment let me shade it quickly for this segment. Now, I quickly I need to integrate.

So, I quickly change so, my ds is how much? ds is 2πR and if this width is dR then it should be 2πR dR. What is r here? r is by geometry from here to here the length is d then it should be $\frac{d}{\cos\theta}$ and R should be again in terms of θ and the constant d is d tan θ. So, dR should be d $\sec^2$ θ so, that I write $\cos^2$ θ and dθ. So, everything I now make in terms of θ now I can write the magnitude of dE is simply $\frac{\sigma}{4\pi\epsilon_0}$ and then $\frac{2\pi R}{d^2}$.

And then we have $\cos^2$ θ d divided by I just change everything whatever we have. So, mind it $\frac{1}{r^2}$ I change $\frac{d^2}{cos^2\theta}$. So, $\cos^2$ θ is going up and d^2 is going down because it is $\frac{1}{r^2}$ ds I just simply write 2πR dR, 2πR I write dR is $\frac{d}{cos^2\theta}$ and dθ that part and on top of that, we have cos θ that is already multiplied. So, this portion is simply my dR and rest of the part I just to replace.

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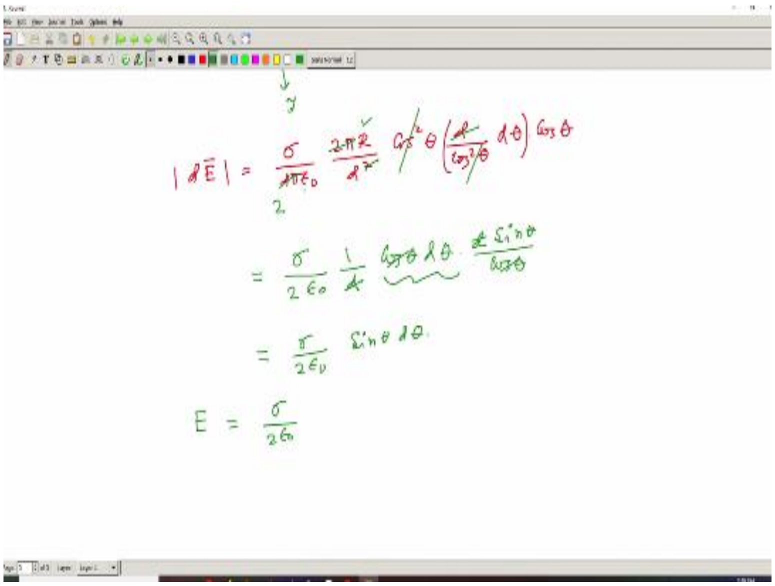

Now, few things will cancel out because this $\cos^2\theta$ this $\cos^2\theta$ we will going to cancel out and we simply have and also 2π and this 4π will give you 2. So, I simply have $\frac{\sigma}{2\epsilon_0}$ one d is also going to cancel out so, it should be just a minute there should be dR so dR is $\frac{d}{cos^2\theta}$ and R is d^2. So, we have this d here and one d should be there so, 1 over d is there and then I have cos θ and dθ so, this is the thing we are getting.

And now, if I want to find out what is the value of the entire thing? So, I can make it also slightly $\frac{\sigma}{2\epsilon_0}$. So, this d should be cancelled out with something like me check that everything is written properly or not. So, our length is d so, R is replaced by so, $\frac{\sigma}{4\pi\epsilon_0}$ then ds replace 2π R dR, dR I replace like $\frac{d}{cos^2\theta}$ dθ that is dR and then $\frac{1}{r^2}$ I replace by d^2 and then $\cos^2\theta$.

And what else we are having here so, σ and then dθ $2\epsilon_0$ is there and we have d. So, one R is still there r is $\frac{d}{cos\theta}$, so, let me check σ ds $4\pi\epsilon_0$ this ds is 2πRdR, R and dR is in place of R I missed one because there is ds in R is there, so, I missed one d here and also the tan θ. So, that means, I should have d that I missed here. So, these 2π R if I replace this R in terms of d and tan θ it should be d of $\frac{\sin\theta}{\cos\theta}$.

So, these R I missed. Now, I think we are in a place so, d d will now cancel out and these cos θ and cos θ cancel out. So, we simply have the sin θ and dθ. Now, if I want to find out the total electric field I need to integrate and this integration of this θ for infinite field is simply 0 to $\frac{\pi}{2}$.

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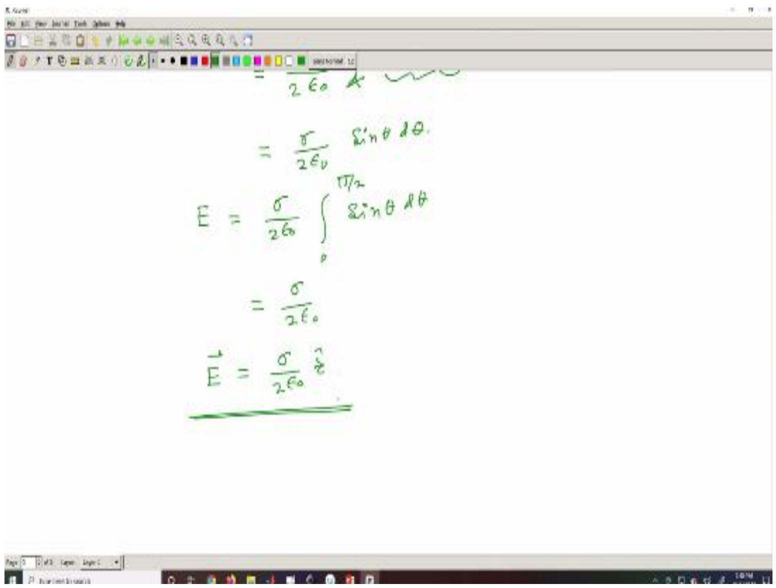

So, if I integrate it from 0 to $\frac{\pi}{2}$ and then I have sin θ dθ, then this value is $\frac{\sigma}{2\epsilon_0}$. And that is all and if I want to find out because this is one this integration if I find out the field this is $\frac{\sigma}{2\epsilon_0}$ and $\hat{z}$. So, these results already figured out I mean this is the second time I am doing the same problem. So, it is already there. So, quickly I apply the Gauss's law to find out.

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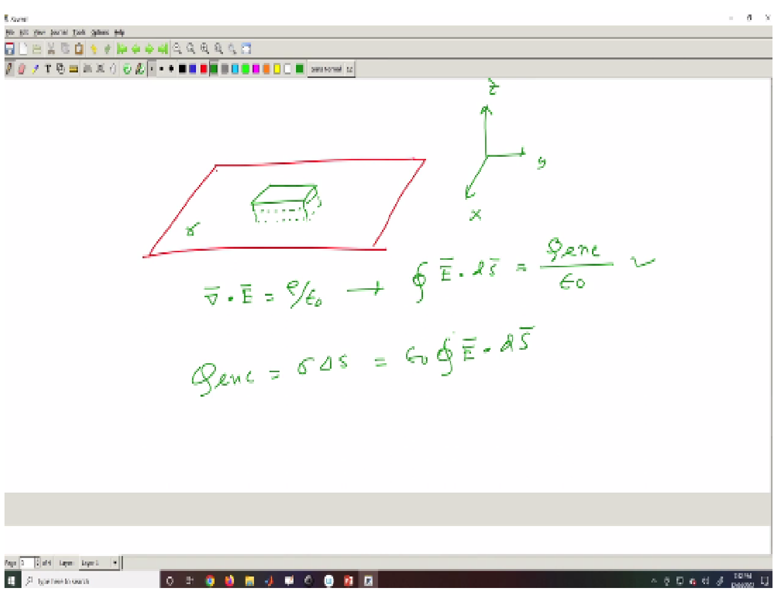

That is quickly I need to find out this. So, suppose this is my infinite surface and in order to execute the Gauss's law, the first thing that I need to do is to put some surface here. So, I put a Gaussian pillbox here a surface like this the lower side and doted one is going to the lower side a surface like this. So, it is along this direction z this is y, so, this is x. So, in xy plane the surfaces is. Now, we know according to the divergence theorem $\vec{\nabla} \cdot \vec{E}$ is $\frac{\rho}{\epsilon_0}$.

And that is integration of the surface close surface integral is equal to $\frac{Q_{enc}}{\epsilon_0}$ this is the Gauss's law 2 form. Now here Q_{enc} if I calculate quickly for this Gaussian pillbox, if σ is a surface charge density then Q_{enc} will be $\sigma\ \Delta s$ and that is ϵ_0 and close integration of $\vec{E} \cdot d\vec{s}$. Because Q_{enc} is ϵ_0 and Q is this quantity. So, now I can write this part here.

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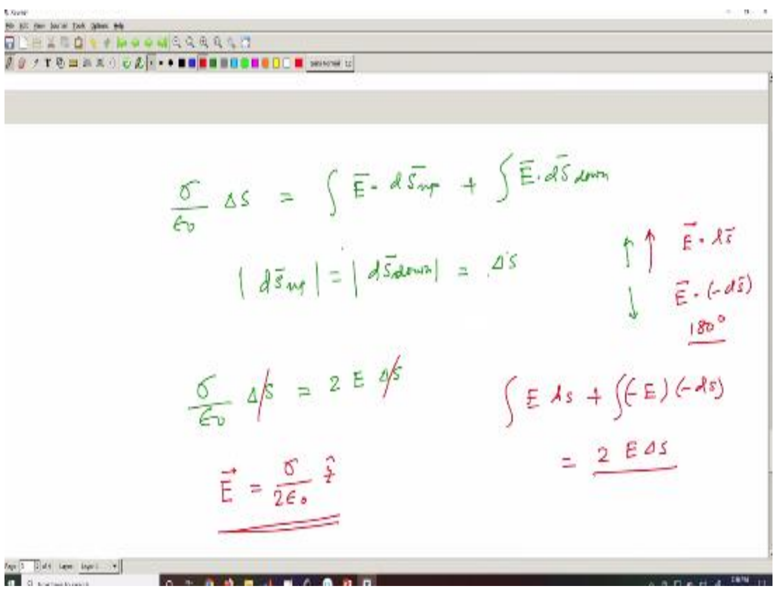

So, $\frac{\sigma}{\epsilon_0}$ and then Δs and this closed surface integral I can divide into 2 parts one is E dot the surface that is the upper surface and another surface is E dot the down surface I can make this surface this I can reduce this to infinity so that this surface will vanish. So, only upper and down surface will contribute here. And the amount of upper surface the value of the upper surface and the value of the down surface, the magnitude is same and that value is nothing but say ds.

So, that value is nothing but Δs because Δs is the surface we took. So, this is simply I put Δs. So, the right-hand side we simply have $\frac{\sigma}{\epsilon_0}$ Δs and the right-hand we simply have 2E and then the surface that we are having. Mind it in upper surface and lower surface we are having the direction opposite, but when you make a dot product with E, then one case electric field is up and another case is electric field down. So, that is why we have this plus sign and this plus sign makes both these negative will make positive so, we will have this.

So, I can cancel these Δs and from here we can have the electric field the magnitude of the electric field is $\frac{\sigma}{2\epsilon_0}$ and if I want to find out what is the z direction, so, that should be the direction of the z if electric field is considered along z direction along this direction. So, in one case, this is $\vec{E} \cdot d\vec{s}$ and in other case electric field along this direction, so, $\vec{E} \cdot (-d\vec{s})$ but the angle between these 2 is 180°.

So, when you make the cos value then one minus sign we will have and that is why you should have these 2 things. So, something like E ds + (- E) and then (- ds) that gives you 2E Δs. So, that is the value we find exploiting the Gauss's law. And you can see that the same calculation you calculate from the first principle and both the cases were getting the same result. So, this is the application one of the applications of the Gauss's law.

So, with that note I like to conclude here and maybe in the next class I would like to show another application like over what about the charge is spherical charge distribution. That problem we did not do meticulously but using you know Gauss's law we will show that how easy this problem I mean, how easily we can resolve this problem. With that note I would like to conclude here and thank you for your attention and see you in the next class.