# Foundation of Classical Electrodynamics


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## Lecture - 19
## Helmholtz's Theorem (Recap), Tutorial

So, hello students to the foundation of classical electrodynamics course. So, today we have lecture 19 and we will recap the Helmholtz's theorem that we discussed last day and after that we try to do some problem.

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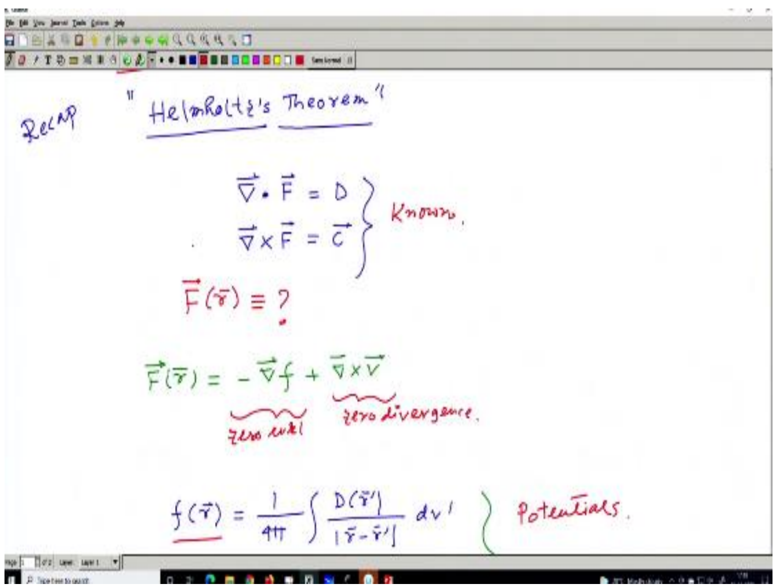


Class number 19 and we will start with making a recap of the Helmholtz's theorem. So, I will just discuss about the statement I am not going to prove the rigorous proof is there, but I am not going to prove it. But, if you just understand the statement at this stage it is fine. So, the Helmholtz's theorem is saying that suppose I know the divergence of a given vector field $\vec{F}$ and curl of a given vector field $\vec{F}$ these 2 are known.

Now, the question is how uniquely I can predict my $\vec{F}$? Where the divergence and curl is known, if I want to predict my $\vec{F}$ so, how I predict that? And what should be the form of $\vec{F}$ in terms of D and C? In order to do that, I mentioned that any given vector field $\vec{F}(\vec{r})$ can be decomposed in two parts in general one is the gradient of a scalar field plus the curl of a vector field v. So, this is the decomposition I can decompose any vector field $\vec{F}$ in these 2 forms. So, kindly note that this is this content 0 curl and this component is having 0 divergence.

So, I am decomposing a given vector field I can decompose into 2 parts one is having a 0 curl and another component is having 0 divergence. So, this is the way in general you can divide any given vector field so, this is I am going to use this. Now, the Helmholtz's theorem saying that I can find my f the scalar field, which is associated with the given vector field in this form.

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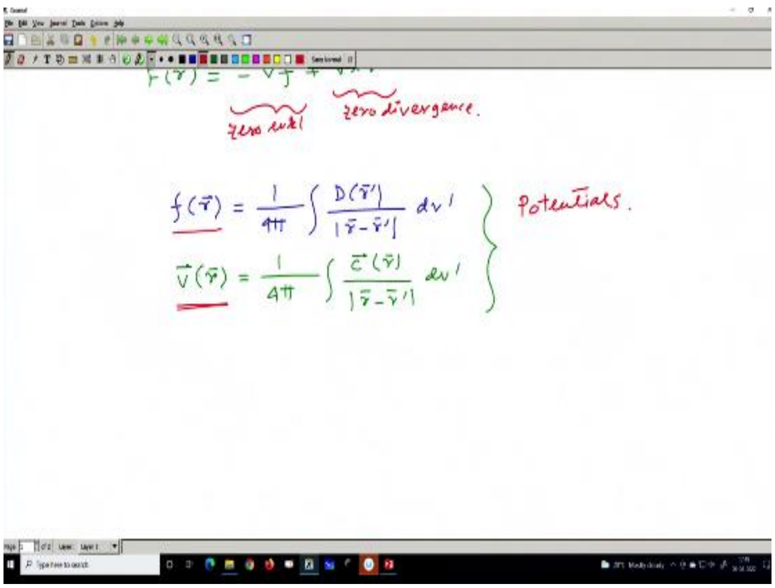


And I can write my vector field v in this form. For any given unknown vector field whose curl and divergence is known, I can construct the f and v in this way. So, these are in general called the potential. Why it is called potential? When we do the example then you will be going to understand. So, these are potentials. So, this is a scalar kind of potential, this is a vector kind of potential because it is a vector quantity. So, why it is called potential? And what is the implication of these treatments? So, these are the overall thing, let me repeat once again.

F suppose is unknown, but I know what is the divergence and what is the $\vec{\nabla} \times \vec{F}$, which is unknown, but these things known D and C, the thing is that would it be possible to construct $\vec{F}$ with the knowledge of D and C and in order to do we first divide this vector field any vector field into these 2 parts, which is the gradient of a scalar field and the curl of a vector field. So, combination of these 2 can give me F and F should have 2 components as I mentioned, one is 0 curl and another is 0 divergence.

So, this is the way in general we can divide a vector field, but f and v now, these 2 quantities if we know then I can predict my F and these 2 quantities one can find in terms of D and C in this way. So, D, C is known, so, right-hand side is known, so, that means, I know f and v and once we know f and v I can find out what is my F. Now, this is in general. So, now, we are going to use this for the electric field E and magnetic field B to find something.

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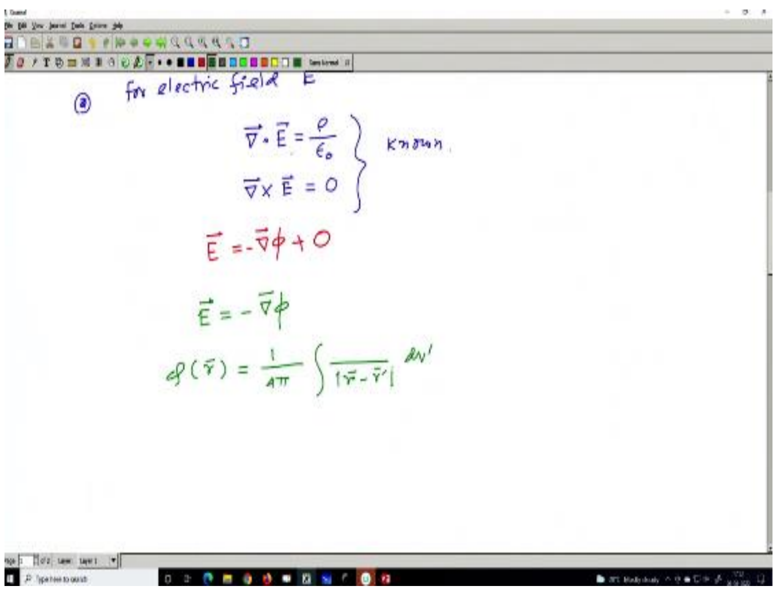


So, we will do that later also, but right now, I just like to show that how for a given electric field and magnetic field it works. So, for electric field $\vec{E}$ I have 2 Maxwell's equation that is known. So, even though I did not covered this part, but let us consider that you know it that $\vec{\nabla} \bullet \vec{E}$ is $\frac{\rho}{\epsilon_0}$ where $\rho$ is a charge density and the $\vec{\nabla} \times \vec{E}$ that is also suppose we know and that is 0. So, this is known. Now, it is exactly the same problem that $\vec{\nabla} \bullet \vec{F}$ and $\vec{\nabla} \times \vec{F}$ is known here the $\vec{\nabla} \bullet \vec{E}$ and $\vec{\nabla} \times \vec{E}$ is known one is 0 mind it.

So, now, if I want to construct my $\vec{E}$ because this is 0, so, if I want to construct my $\vec{E}$ like the way we construct so, I should have like $\vec{E}$ is say minus of gradient of some scalar function plus curl of some vector function, but here we can see that this quantity is 0. So, that means whatever the vector function I take here which is depends on the value of the $\vec{C}$ but here for electric field $\vec{C}$ is zero. So, I should not have anything here so, vector function no longer be present in defining the electric field. So, this is 0.

So, electric field simply defined in this way and that we know the electric field I can write in terms of potential. And this is exactly if I use the Helmholtz's theorem I am going to get this one. Now, what is φ? That is other interesting because I know the recipe to find φ. So, φ, which is a function of r, according to Helmholtz's theorem it should be $\frac{1}{4\pi}$ and then integration divided by (r – r') dv' and here we put in this region we put whatever the value I have here during the divergence, because if you know the divergence D. So, D is present here.

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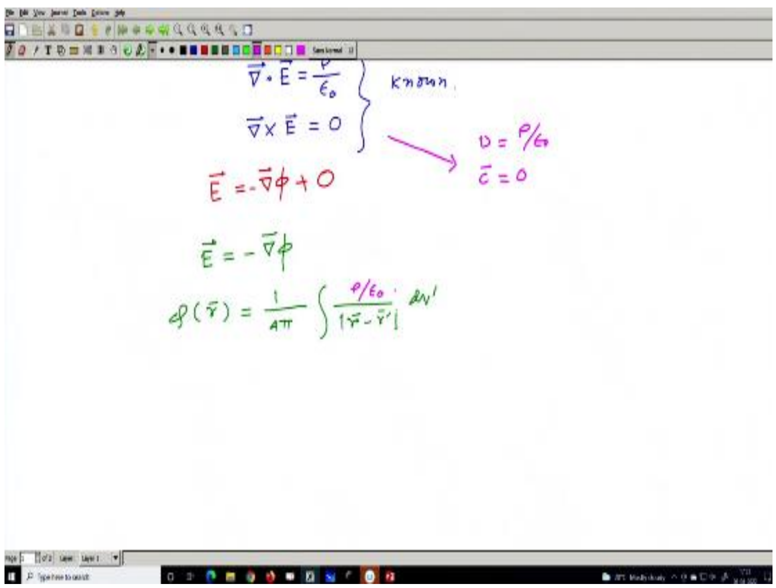


So, according to this equation, my D is $\frac{\rho}{\epsilon_0}$ and sorry, D should not be a vector because it is a scalar quantity and my C is simply 0 from this I can get, so, I will put this value here $\frac{\rho}{\epsilon_0}$ and that is basically the solution of φ.

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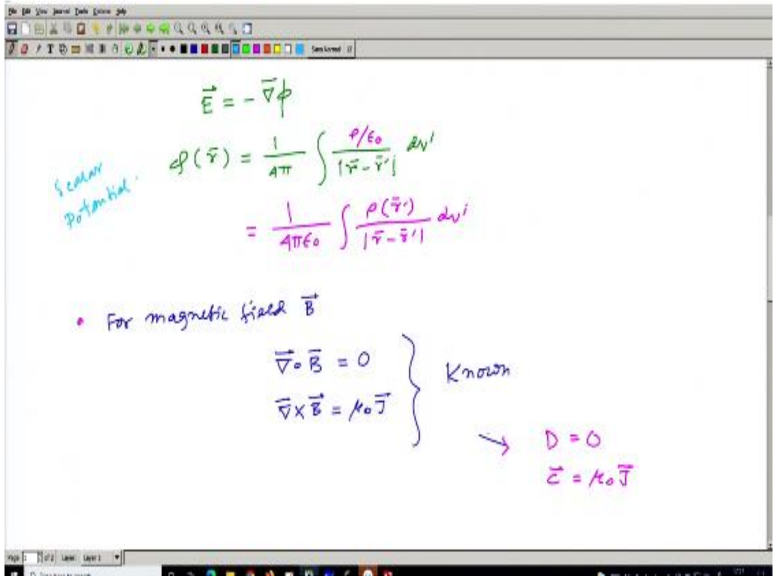


So, when you solve φ and I will get this I will get $\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$ dv’ in a similar way for magnetic field I can extend this. For magnetic field B what we have the 2 Maxwell's equation for magnetic field does not have any divergence and if I do the curl of magnetic field it should be $\mu_0$J. So, this is again known. So, if this is known, then I know so, this tells me that my D is 0 here and $\vec{C}$ is $\mu_0$J.

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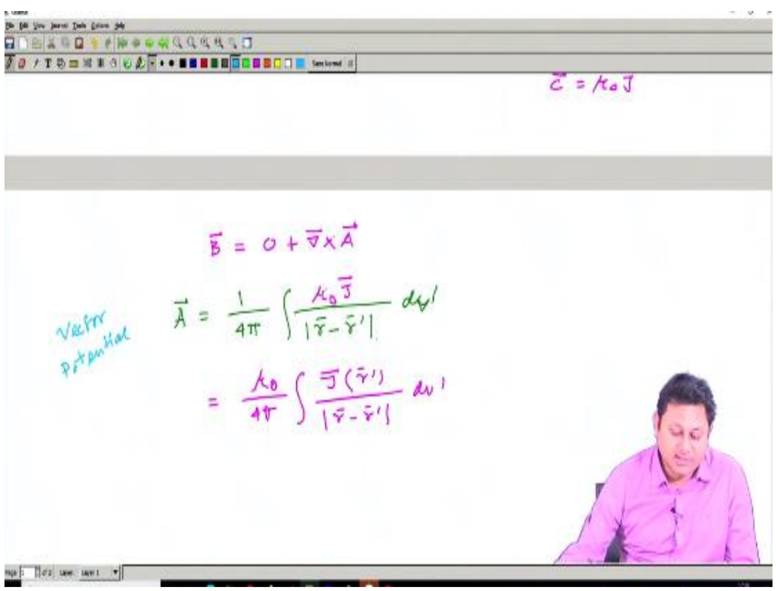


So, I can construct my B, B should be how much? The gradient of any scalar field, but here the gradient portion will not be there because the divergence of B this equation is 0. So, I will have 0 here but the curl is very must there. So, the second part I should write and I should write that it should be curl of some value A. Now, if I want to construct A according to the Helmholtz’s theorem A it should be $\frac{1}{4\pi}$ then integral divided by (r – r') dv' and here in this region I should put whatever the curl value that is given.

The $\vec{\nabla} \times \vec{B}$ is $\mu_0$J, which is C here, I need to put C here C is $\mu_0$J. So, I should put $\mu_0$J here. So, eventually my A becomes $\frac{\mu_0}{4\pi} \int \frac{J(\vec{r}')}{|\vec{r}-\vec{r}'|}$ dv'. So, here the scalar quantity φ is called the scalar potential. This is also a potential so, it is called the scalar potential and this quantity A is called the vector potential. So, that we will discuss in the later class obviously, we are going to discuss in detail but in the context of Helmholtz's theorem, which is a generalized theorem.

You can see that for electric and magnetic field, how one can apply that and after the application you can find out the scalar potential and vector potential just by using the recipe of this Helmholtz's theorem. So, you should remember that, because in the later part, we will discuss again in more rigorous way. So, now, after that I am planning to have some tutorial today, so, few problems I like to solve. So, let us do that.

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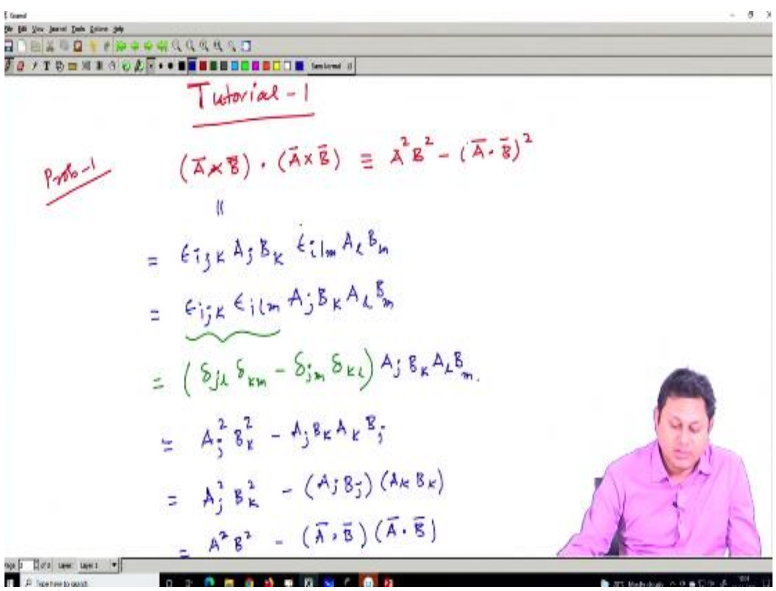


So, it is say tutorial 1 so, problem 1 is this also find out show that $(\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = A^2 B^2 - (\vec{A} \cdot \vec{B})^2$, a vector identity you need to prove by using the Levi-Civita symbol that we learn in earlier classes. So, let us do that how to do it. So, this quantity if I write the $i^{th}$ component so, the

$i^{th}$ component will be $\varepsilon_{ijk}$ then it should be $A_j$ $B_k$ and then it should be multiplied by the $i^{th}$ component of the next term, which is $\varepsilon_{ilm}$ we are not going to use the jk anymore ilm then $A_l$ $B_m$.

Then I have $\varepsilon_{ijk}$ $\varepsilon_{ilm}$ and then I have $A_j$ $B_k$ $A_l$ $B_m$. So, now this we know that when we have $\varepsilon_{ijk}$ and $\varepsilon_{ilm}$ then you associate it with a delta function and this delta function is this $\delta_i$ if it is i then it should be $\delta_{jl}$ $\delta_{km}$ - $\delta_{jm}$ $\delta_{kl}$ multiplied by the rest term, which is $A_j$ $B_k$ and then $A_l$ $B_m$. So, if I execute this what I get? So, j j is there l so, the first term is $A_j$ $B_k$ so, j and k j and l here so, let me write it here. So, $A_j$ I right and then this l has to be j.

So, I can write $A_j^2$ here if I write $B_k$, but this m has to be k so, I get the first term will be simply $A_j^2$ $B_k^2$ whatever the next term minus so, $A_j$ is there I can put $A_j$ so, then m should be j. So, $B_k$ is also there so I put $B_k$ so $A_l$ should be $A_k$ and $B_m$ here should be $B_j$. So, this quantity is simply $A_j^2$ $B_k^2$, but this one if I write properly here I am making a mistake. So, m should be so, so, here it is I am having m. So, this m should be j here then delta function is one m should be j here.

So, I have $A_j$ $B_j$ and then I have $A_k$ $B_k$. So, this quantity is simply $A^2$ $B^2$ without any doubt but this quantity is $\vec{A} \bullet \vec{B}$ and then again another $\vec{A} \bullet \vec{B}$ because $\vec{A} \bullet \vec{B}$ is written as $A_k$ $B_k$ or $A_j$ $B_j$. So, that is the thing we wanted to prove, because this is the thing we wanted to prove and we prove it this is one kind of problem let us go to the next problem. The next problem is problem 2.

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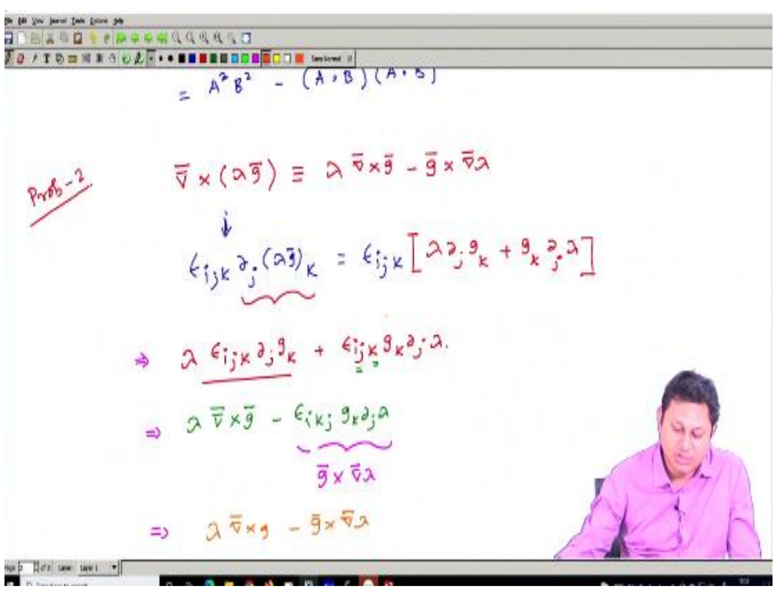

So, problem 2 is saying that if I have $\vec{\nabla} \times \lambda$ is a constant and g so, that this quantity is equivalent to $\lambda\vec{\nabla} \times \vec{g}$ - $\vec{g} \times \vec{\nabla}\lambda$. So, we need to show this identity. So, again let us start with the left-hand side whatever we have so, we have a curl here, so, we know how to represent curl so, curl is $\varepsilon_{ijk}$ and then $\partial_j$ and then $\lambda g_k$ component that we have.

So, this if I elaborate because this is a derivative so, this if I elaborate we have ijk then this portion I am now writing so, lambda then $\partial_j$ $g_k$ + $g_k$ $\partial_j$ $\lambda$ so, $\lambda$ is a scalar. So, from here I can write $\lambda$ this from this, then $\varepsilon_{ijk}$ $\partial_j$ $g_k$ so, this is forming something and plus other term $\varepsilon_{ijk}$ then we have $g_k$ $\partial_j$ $\lambda$. So, these you can see quickly that this is nothing but $\lambda$ and then $\vec{\nabla} \times \vec{g}$ $\lambda$ whatever the next term because here we have j and k here we have k and j.

So, if I now change it by putting a negative sign then it should be $\varepsilon_{ikj}$ and then $g_k$ $\partial_j$ $\lambda$. If I do then I can have orientation here and this orientation is $\vec{\nabla} \times \vec{g}$ sorry not curl because here we have an orientation like g cross this quantity this is the $\vec{\nabla}\lambda$. So, overall what we find? So, overall we find that my final result is $\lambda$ $\vec{\nabla} \times \vec{g}$ - $\vec{g}$ $\times \vec{\nabla}\lambda$ this is precisely we need to prove this is the thing we need to prove and we prove it here. So, this is a second kind of problem. Now, let us go to another problem.

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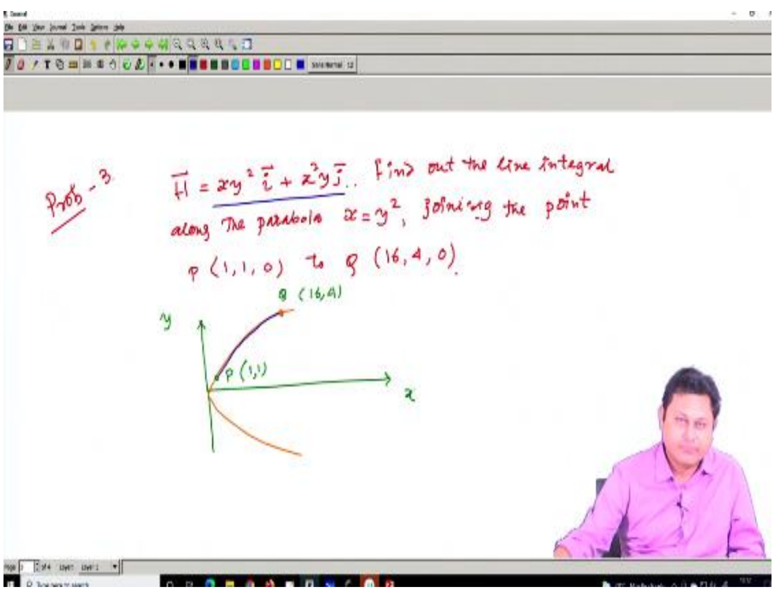


The problem 3 suggests that suppose H is $xy^2$ i + $x^2y$ j this is a vector field so, the question is find out the line integral along the parabola $x = y^2$ joining the point P one point is P, which is 1, 1, 0 to

Q another point, which is 16 and then 4 and then 0. So, it is in 2 dimension because z axis z point is 0. So, how the path will look like let us first determine the path and then we will execute this line integral for a given vector field.

So, this is my y axis and this is my x axis, x y and if I plot it, it is a parabola so, it will be something like this. So, 2 points are mentioned one is here say this point is Q and another is here this point is say P the coordinate of this point is 16, 4 and coordinate is 1, 1 my drawing is not in the scale but you can understand that. So, I need to integrate it over this point P to Q. So, this is my line element from P to Q and I need to integrate the given function this one this is the given function given vector field. So, I know how to how to integrate this.

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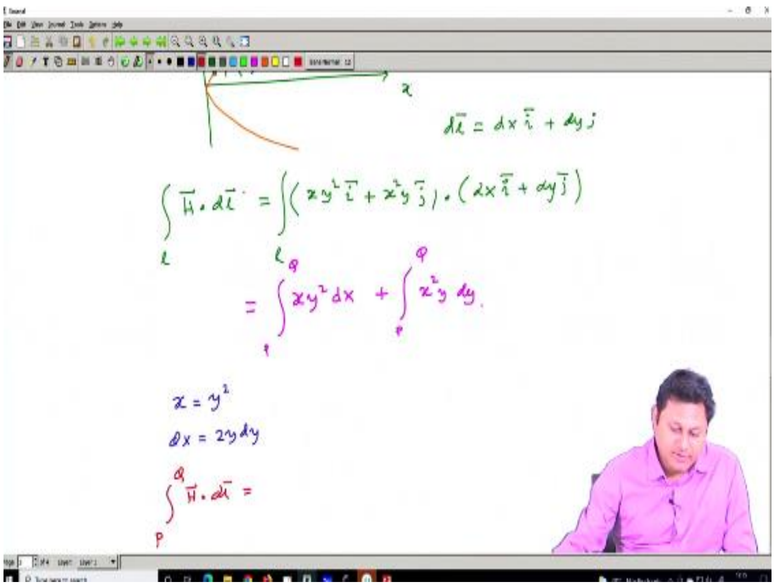

So, the line integral if I calculate, it is $\vec{H} \bullet \vec{dl}$, which is so what is dl this is a Cartesian coordinate system easy. So, we have dx i + dy j. So, I can put the value of H here which is $\int_l (xy^2\hat{\imath} + x^2y\,\hat{\jmath})$ • $(dx\,\hat{\imath} + dy\,\hat{\jmath})$. So, this quantity if I know fine it should be $xy^2$ dx integration one integration and another integration will be from P to Q this is also from P to Q and another integration is same this is $x^2y$ dy.

Now, please note that over this line we have a relation and this relation is this, this relationship is already given. So, this is having a relationship and that is $x = y^2$. So, this is the way x and y is related over this line, so, we will want to use that. So, I have say $x = y^2$ and then $dx = 2y\,dy$, then

I will be going to use this here in this expression. So, I need to find out integration P to Q the line integration $\vec{H} \bullet \vec{dl}$ and here, if I convert everything in terms of x.

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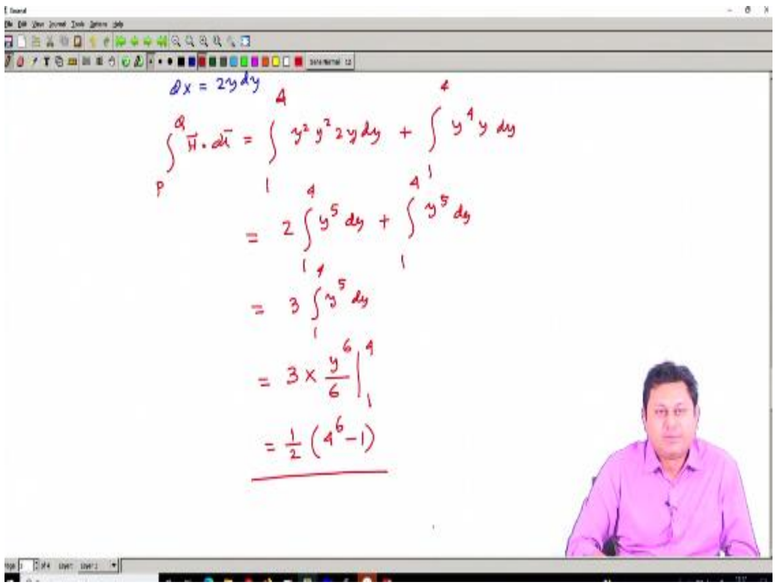


Then my limit should be simply 1 to 4. So, 1 to 4 no, if I convert everything, so, because x is 16 so, y is 4, so, my limit is 1 to 4, if I convert everything in terms of y, you can do that everything by converting x it is up to you. So, the first is x I just write $y^2$ and $y^2$ is already there another $y^2$ dx is 2y dy that is the first integral everything in terms of x, whatever the second one 1 to 4 and $x^2$, I simply write y to the power 4 and y I should write y dy is already there.

So, I have here this quantity 2 integration $\int_1^4 y^5$dy + $\int_1^4 y^5$dy. So, this is simply 3 $\int_1^4 y^5$dy. So, we have 3 × $\frac{y^6}{6}$ with the limit say 1 to 4. So, this is $\frac{1}{2}$ and then I have $(4^6 - 1)$. So, this is the value of the integration one can have, so, I do not have much time today to you know do the other problems. So, let us meet in the next class and discuss more about the problem.

So, the next class tutorial will be continued. And we will go on to solve a few problems there. With that note, I would like to conclude here. Thank you very much for your attention and see you in the next class.