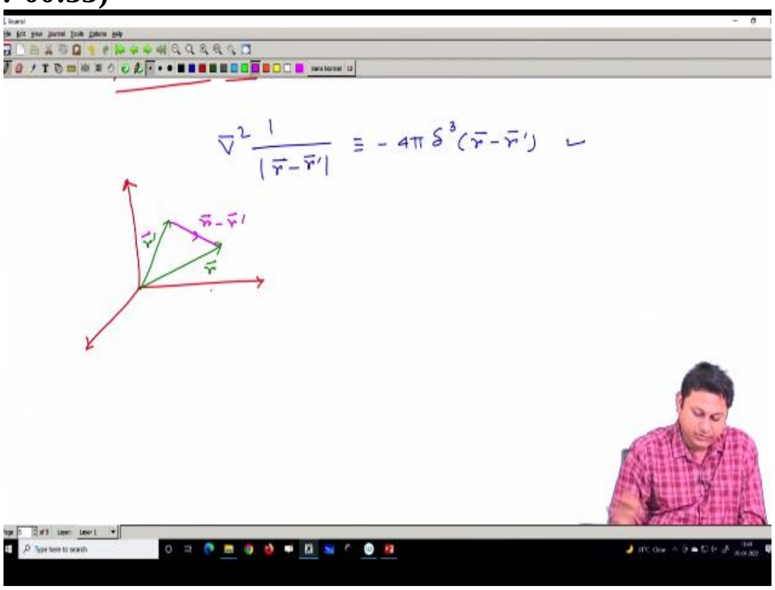
Foundation of Classical Electrodynamics
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology – Kharagpur

Lecture - 18
Helmholtz's Theorem

Hello students to the foundation of classical electrodynamics course. So, we are still in module 1 learning the mathematical preliminaries. So, today, we are going to understand the Helmholtz's theorem.

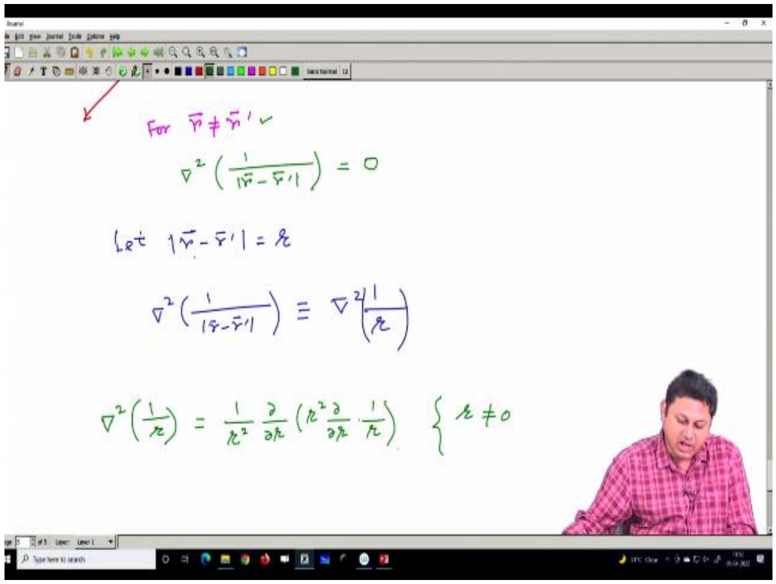
(Refer Slide Time: 00:33)



So, we have today class number 18. I am going to learn about the Helmholtz's theorem but before that we need to understand a very important identity. And that we are going to prove today and the identity is this. So, if I want to find out the Laplacian of a scalar field defined like this, this is a distance function, but the scalar value I am taking then that value is equivalent to minus of 4π and then delta function in 3 dimension $(\vec{r} - \vec{r}')$. This is a very important identity that regarding the delta function, we already understand delta function in last couple of classes.

So, we are going to exploit those identities to understand this expression, how these things are there. So, here if I now so this is my coordinate system say and in coordinate system, if I plot my r here and r' here, so, this is a point r and this is r'. So, this distance is simply $\vec{r} - \vec{r}'$ so, this is the system we are having. Now, you can see that for r not equal to r.

(Refer Slide Time: 02:46)

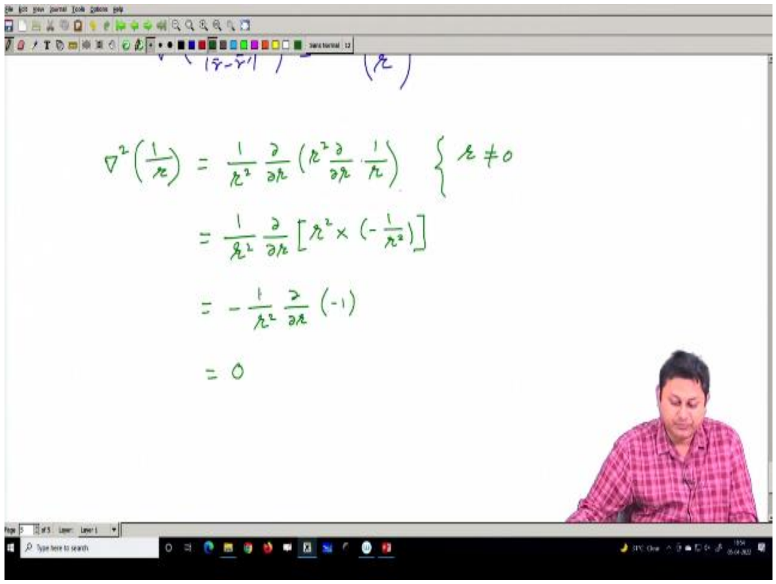


First we will do for r not equal to r' if I calculate this quantity simply comes out to be 0 under this condition how let us check. So, let $|\vec{r} - \vec{r}'|$ to be Л. So, then eventually I am calculating this quantity $|\vec{r} - \vec{r}'|$. So, this is equivalent to $\nabla^2 \left(\frac{1}{Л}\right)$ that I am calculating. Now with the brackets here, now, this quantity this Laplacian I am operating over a function $\frac{1}{Л}$. So, this Laplacian operator should be in spherical polar coordinate system.

Because my function whatever the function I am having, which is $\frac{1}{Л}$ is in spherical polar coordinate. So, and also the θ φ is not there so the derivative with respect to θ φ will not going to play any role here. So, if I just only expand these ∇^2 operator in r θ φ coordinate the r part of that it simply gives us $\frac{1}{Л^2} \frac{\partial}{\partial Л}$ and then $Л^2 \frac{\partial}{\partial Л}$ and that we are going to operate over $\frac{1}{Л}$ that is the operation we will have now here one thing you should note that the Л is not equal to 0.

Because that is the condition, which we had here because Л is $\vec{r} - \vec{r}'$ and r is not equal to r'. So, if you make a $|\vec{r} - \vec{r}'|$ then that value should be in a non-vanishing quantity.

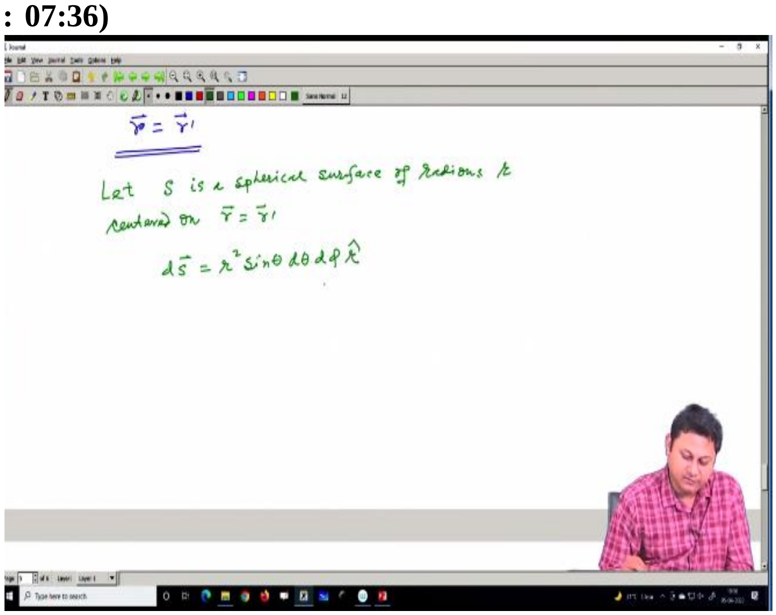
(Refer Slide Time: 06:03)



So, then the rest thing is mechanical so, we have $\frac{1}{Л^2}$ and we take the coordinate system in such a way that $\vec{r} - \vec{r}\,'$ is that means, I am putting my coordinate system over this r' so, that $\vec{r} - \vec{r}\,'$ became you know the radius of this sphere over which we are calculating. So, then it is $\frac{\partial}{\partial Л}$ and we are having $Л^2$ here multiplied by -1 by $Л^2$ because I am having a partial derivative $\frac{1}{Л}$ partial derivative with respect to Л of $\frac{1}{Л}$ so, it should be $-\frac{1}{Л^2}$.

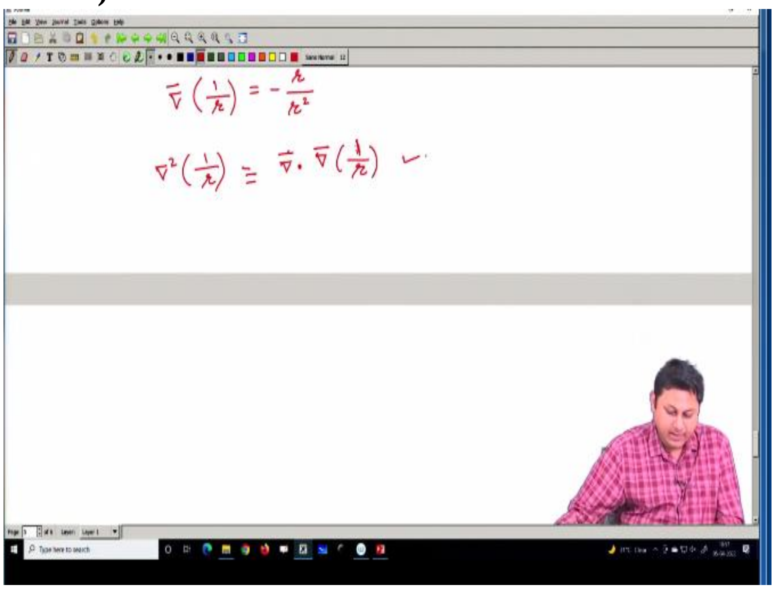
So, then we simply have $-\frac{1}{Л^2}\frac{\partial}{\partial Л}$ and this will cancel and we have -1 here which is a constant. So, this simply gives us 0. So, now, we will be going to check the other thing that is what happened.

(Refer Slide Time: 07:36)



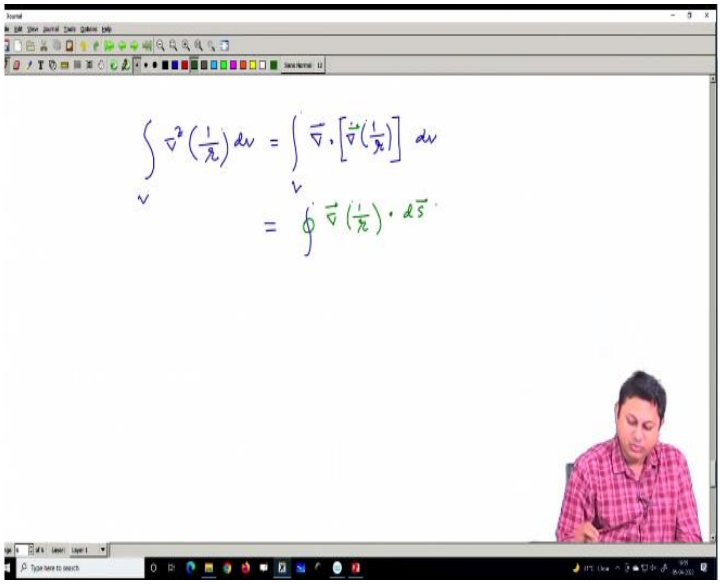
When r = r' that is the point so, I am just having this r over this r' so, that whatever the value we are having is 0. So, at r at r' what we get that we are going to do here next. So, for this case we can consider. So, let S is a spherical surface of radius r this is another radius I am using the same so, centered on r = r'. So, I have a spherical surface, which center is at r = r' with radius r. Now, if I want to calculate the ds element, the surface element it should be simply $r^2 \sin\theta$ and then $d\theta$ and then $d\varphi$ and then $\hat{r}$. So, this is the surface element we know.

(Refer Slide Time: 10:09)



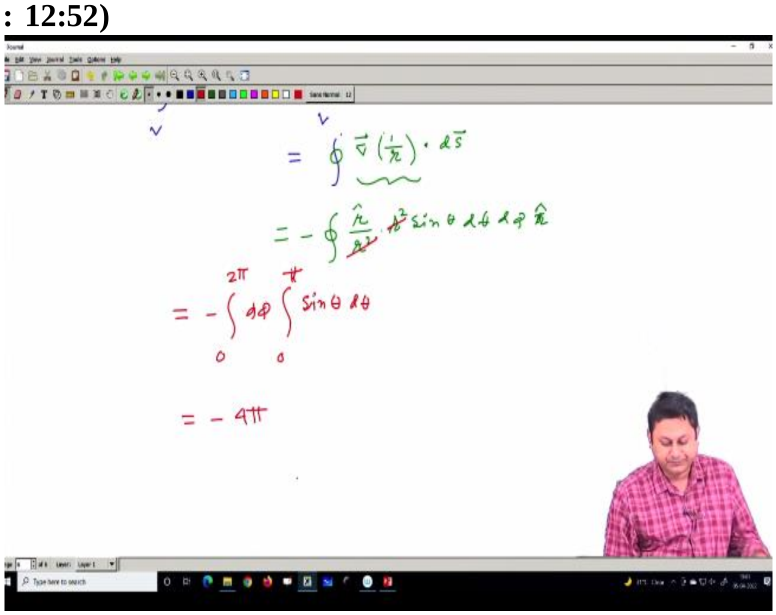
Now, if I calculate this quantity the divergence of $\frac{1}{\mathcal{r}}$ it simply gives me $-\frac{\hat{\mathcal{r}}}{\mathcal{r}^2}$. Now, why I am doing this, because I want to calculate this quantity this is a Laplacian and I am going to calculate this quantity. This quantity is equivalent to the $\nabla \cdot \nabla\ (\frac{1}{\mathcal{r}})$ that is that we know the previous class we already mentioned.

(Refer Slide Time: 11:15)



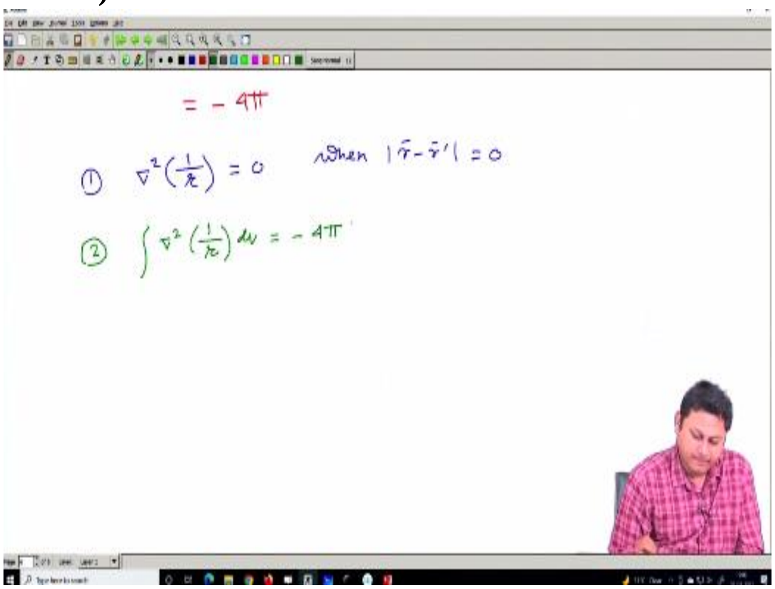
Now, if I calculate the volume integral in this way because I am taking a surface element here so, I can calculate the volume element of this quantity, which is $\frac{1}{\text{Л}}$ and dv then that is according to the way we derive it here it should be simply the $\nabla \cdot \nabla\ (\frac{1}{\text{Л}})$ and then dv. Now, exploiting this is over volume exploiting this divergence theorem I can write it as the closed surface integral of this quantity put the vector sign here put the vector sign here of this quantity over this ds that we calculated earlier.

(Refer Slide Time: 12:52)



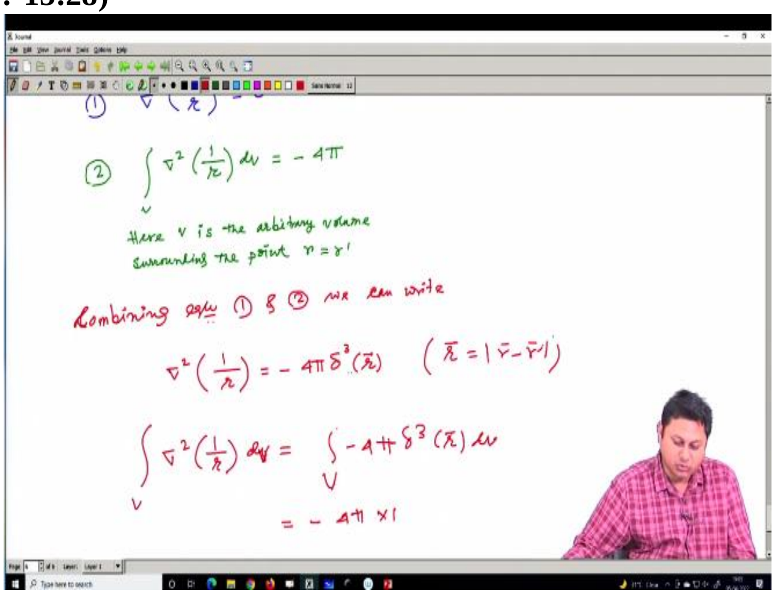
So, if I make a divergence I know what is the value. This value is minus of still I have a close integral $\hat{\text{Л}}$ Л^2 and then the ds is simply $\text{Л}^2 \sin\theta\ d\theta\ d\varphi$ and this $\hat{\text{Л}}$ this is Л. Now, this Л^2 Л^2 will cancel out and eventually I get minus of integration 0 to 2π for $d\varphi$ and then integration 0 to π for $\sin\theta\, d\theta$ and that quantity is simply -4π this is not equal to 0 so -4π.

(Refer Slide Time: 14:25)



So, now, I have 2 conditions one is this so one condition I have is this one $\nabla^2(\frac{1}{r}) = 0$ when $r - r' = 0$ and another condition is this $\int \nabla^2(\frac{1}{r})\, dv$ is equal to -4π.

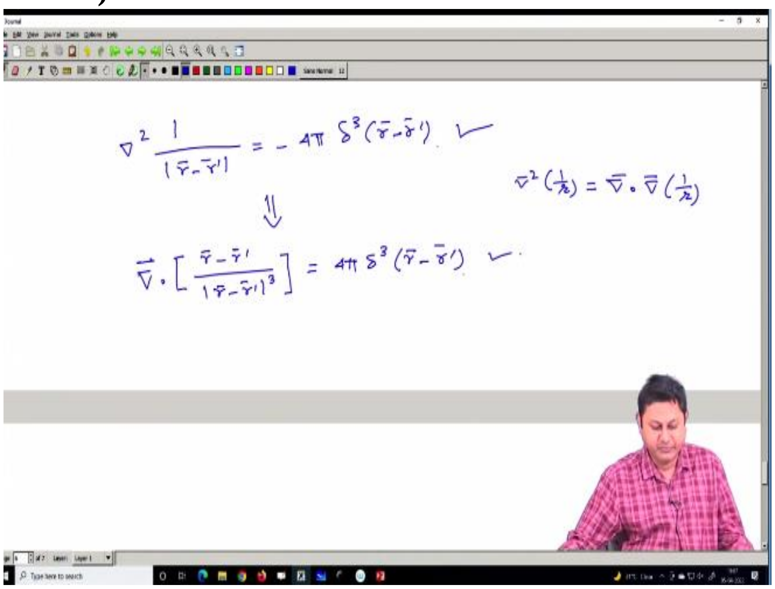
(Refer Slide Time: 15:28)



So, where v is the over this volume integral here you should note that v is the arbitrary volume surrounding the point $r = r'$. So, v is the arbitrary volume that is surrounding the point $r = r'$. So, now combining this 1 and 2 we can write a very nice equation. So, combining equation 1 and equation 2 we can write that $\nabla^2(\frac{1}{\text{Л}})$ = -4π and I can associate this delta function δ^3 should be Л this where r' why I put this delta function now, from equation 2 I can see that.

So, if I use equation 2 then we have $\frac{1}{\text{Л}}$ dv and that quantity simply can be written as v –4π because this quantity I write it here and then delta function over this Л dv and this simply give us -4π and this entire integration should be 1. So, I will get -4π, which is equation 2.

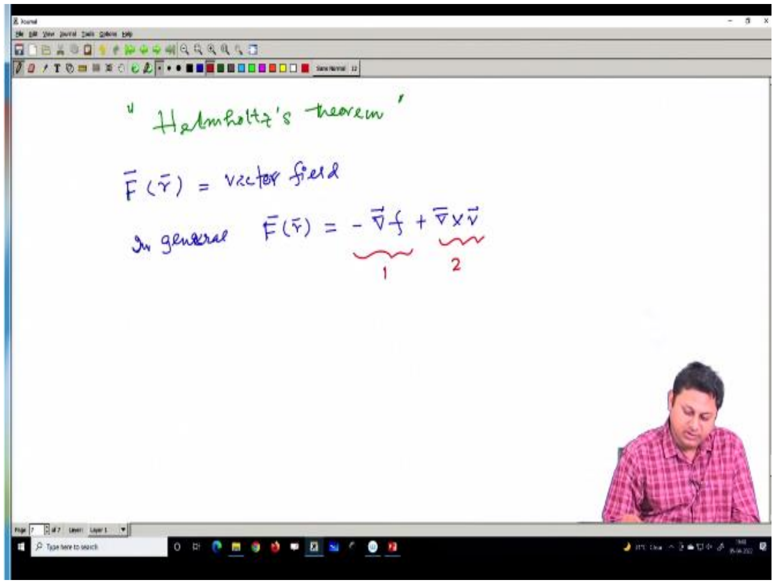
(Refer Slide Time: 18:34)



So, that means I am having a very important expression here and that is $\nabla^2 \frac{1}{|\vec{r}-\vec{r}\prime|}$ is equivalent to -4π δ^3 ($\vec{r}$- $\vec{r}$'). And also from that we can write another thing that from here which is consistent that del if I make a divergence of this quantity $\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}\prime|^3}$ then that quantity should be 4π then δ^3 ($\vec{r}$- $\vec{r}$'). So, this is consistent because you just need to you know expand these things. So, take it as a homework so, please try to find this correlation.

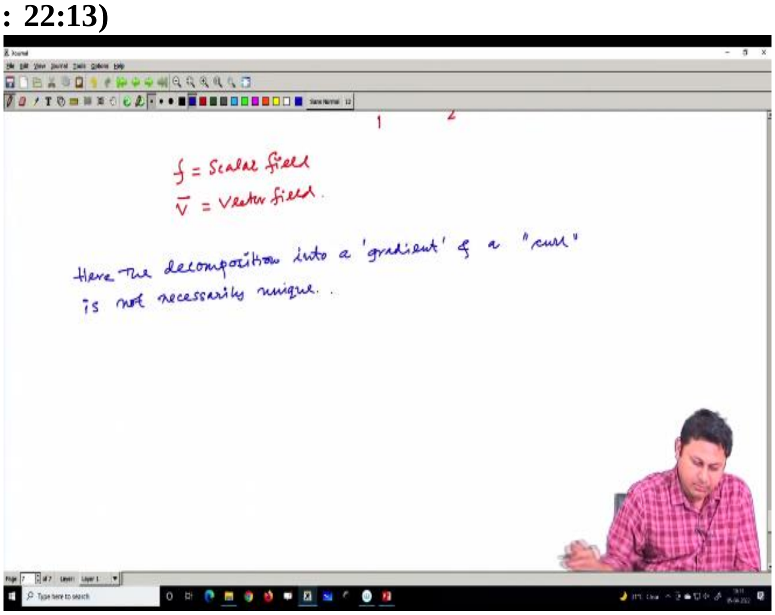
What do you need to do where I am giving you the hints? So, ∇^2 ($\frac{1}{\text{Л}}$) is simply ∇ dot this $\frac{1}{\text{Л}}$. So, you can expand these things in this way and then the rest of the thing is followed you just need to calculate this $\nabla(\frac{1}{\text{Л}})$, where Л is $\vec{r}$- $\vec{r}$' and these things will simply follow.

(Refer Slide Time: 20:25)



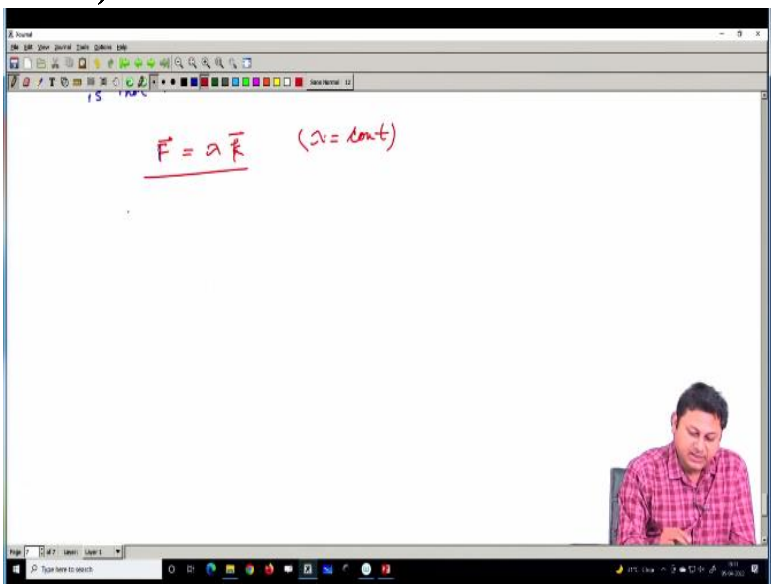
Now, we are in a position to discuss about this Helmholtz's theorem. Before going to the detailing of the Helmholtz's theorem let us you know write this so, if I say let us consider $\vec{F}(\vec{r})$ is a vector field, which is a function of position. Now, in general this vector field $\vec{F}(\vec{r})$ can be represented as this way this can be represented as a gradient of a scalar function plus curl of a vector field that means, a vector field can be decomposed into 2 parts one is this one and another is this one. So, I can decompose this vector field into 2 parts.

(Refer Slide Time: 22:13)



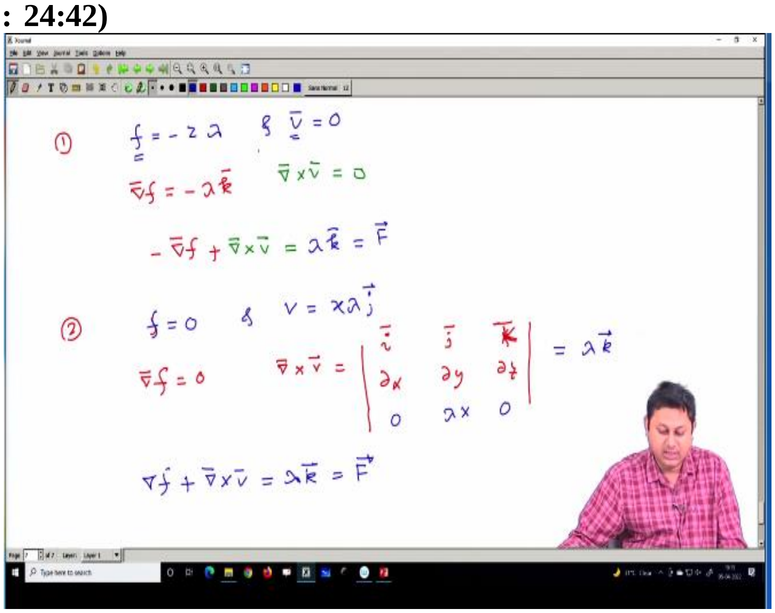
And where f is a scalar field and v is a vector field. Now, the question is the choice of so, if $\vec{F}(\vec{r})$ is given if this is a given vector field the question is can I construct f and v? The answer is yes we can construct f and v in such a way that the gradient of the scalar field and the curl of the vector field the combination of these 2 can give rise to whatever the vector field is given to me. So, I can construct this vector field by choosing properly f and v, but the question is f and v is unique can I choose uniquely so, that so let us take. So, the answer is it is not necessarily unique. So, I should write it here the decomposition into a gradient and a curl is not necessarily unique.

(Refer Slide Time: 24:15)



An example can be given for like F suppose simply constant vector so, λ here is constant so this is it is given. So, I can construct my F in this way.

(Refer Slide Time: 24:42)



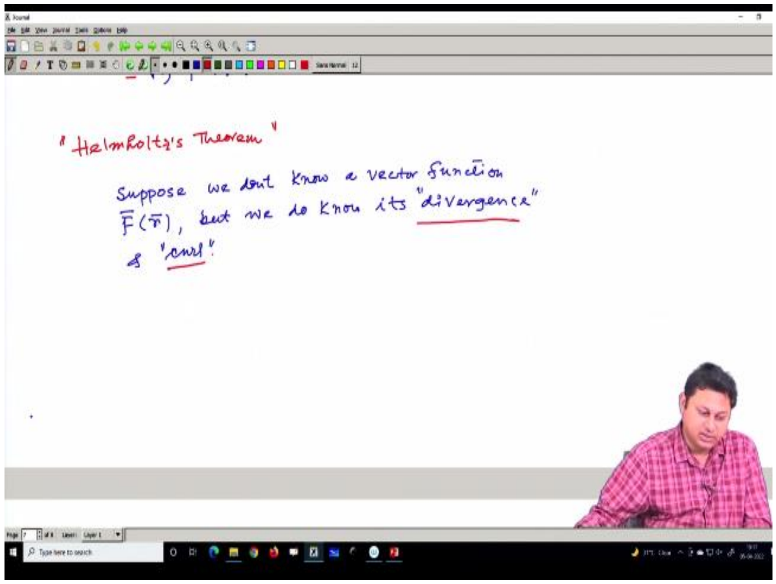
So, I can construct f like –zλ and v has 0 this is my choice one with this can I get my f let us do it. So, if I want to find out this quantity what we get $\vec{\nabla}$f of this it should be simply -λ$\hat{k}$. If I want to find out the curl of this quantity what we get $\vec{\nabla} \times \vec{v}$ it will simply give me 0. So that means this with a negative sign plus this simply give me λ maybe I can put in a different color give me $\lambda\hat{k}$, which is my given function F that is the given function.

So, I can construct f and v suitably such that this decomposition works now, this is my choice 1. Now, I can make another choice my choice 2. So, let us make another choice. The second choice is simply I put f = 0 and v = $x\lambda\hat{\imath}$, this is my choice f = 0 should they produce the same vector field. So, f is 0 simply gives me the $\vec{\nabla}$f is simply 0 what about this, what is the $\vec{\nabla} \times \vec{v}$ now? The $\vec{\nabla} \times \vec{v}$ I need to calculate it is i j and then k ∂_x ∂_y ∂_z and the x component this is not i they should be j so, it should be 0 and then λx and 0.

If you execute that you will find it should be simply $\lambda\hat{k}$. So that means, if I add these 2 quantities for this given f and v I will get back $\lambda\hat{k}$ like previous way, which is the given function F. So, for the given function F I can construct 2 different f and v and both cases this decomposition whatever the decomposition I am saying I should put a negative sign and this negative sign is a convention I must say here this negative sign here the way we put to define a vector field in terms of a gradient of a scalar function and curl of a vector function.

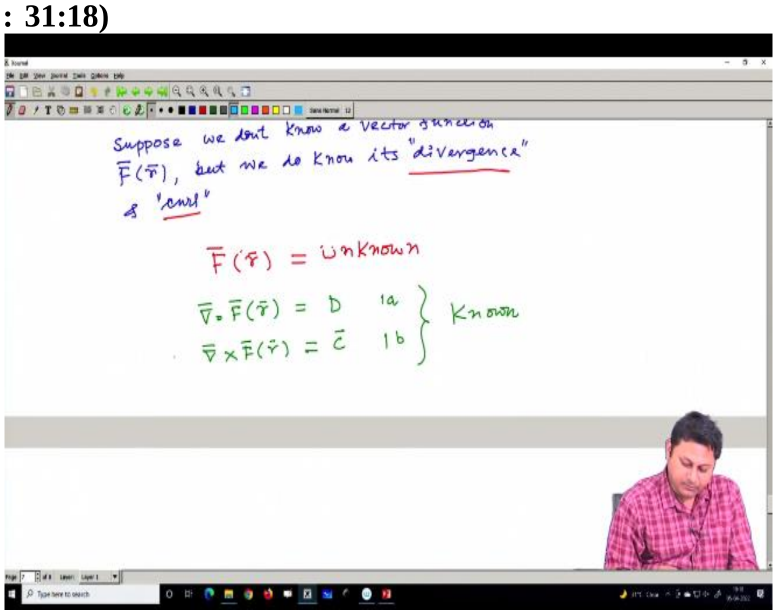
And this negative sign is a convention to put. There is no special reason to put this negative sign, but later we will find that this leads to some sort of potential that we will going to I mean normally there is a for potential we put a negative sign and that this carry through. So, we can see that 2 different values can give these the same.

(Refer Slide Time: 29:30)



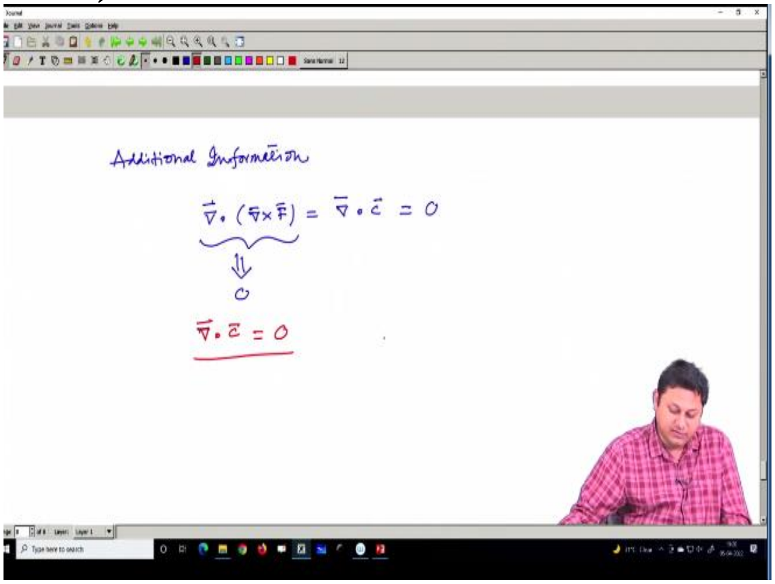
Now, let us go to the Helmholtz's theorem the statement rather so, again I am writing here so suppose we do not know a vector function $\vec{F}(\vec{r})$, the explicit form of this vector function is not known, but we do know its divergence and curl this is important the divergence and curl of this vector function is known. So, let me write it here clearly.

(Refer Slide Time: 31:18)



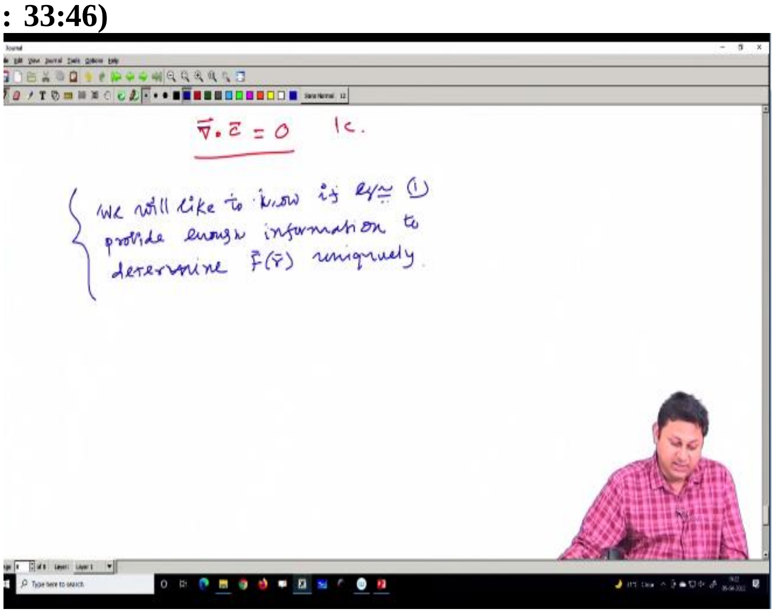
So, what it is saying that? It is saying that this function as a whole is unknown but, these things is known as a divergence of this quantity whatever the unknown functional I have the divergence value say equation 1a and the curl of that quantity that vector field is separately known say $\vec{C}$ 1b this 2 is known. Note that once we know this equation it readily gives us another information and that information is this the additional information we are having here.

(Refer Slide Time: 32:21)



The additional information is since equation 1b is known, if I make a divergence of that $\vec{\nabla}\times\vec{F}$ because $\vec{C}$ is known that means it is dot $\vec{C}$. So, this has to be 0, because divergence of curl of anything always 0. So, I have an additional information here that $\vec{\nabla}\bullet\vec{C}$ is 0. So, I have 3 information in my hand one is the $\vec{\nabla}\bullet\vec{F}$ is D this is known, the $\vec{\nabla}\times\vec{F}$ is $\vec{C}$ this is known and from this equation 1b I have an additional information that is here. So, let me write it 1c.

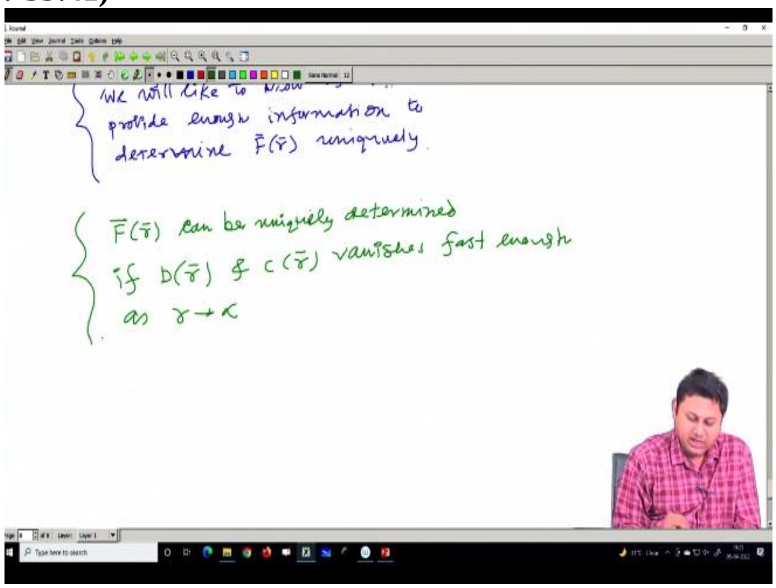
(Refer Slide Time: 33:46)



Now, next thing I am not going to prove that as I mentioned in my so, we will like to know if equation 1, 1 means 1 a b c all these 3 equation provide enough information to determine $\vec{F}(\vec{r})$

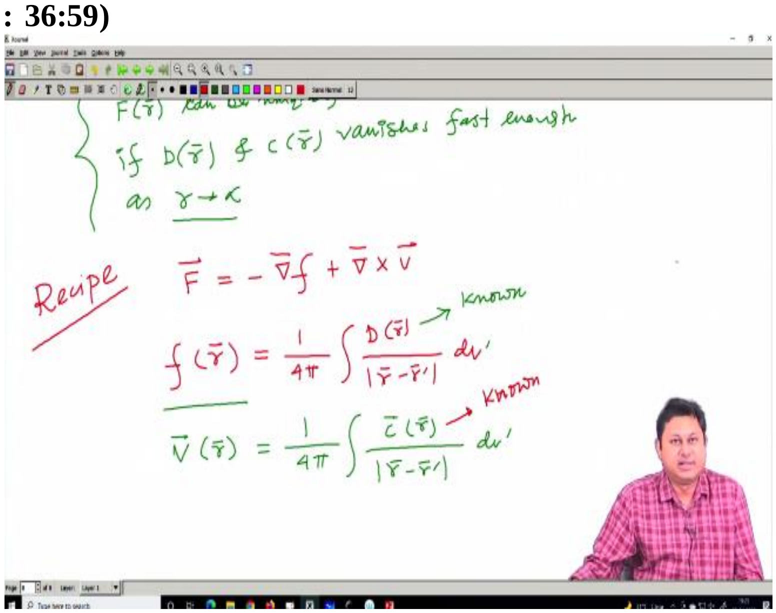
uniquely so, the point is this is known the divergence and curl of this function is known. Now, the question is it enough to you know to construct $\vec{F}(\vec{r})$ uniquely.

(Refer Slide Time: 35:41)



And the Helmholtz's theorem is suggesting that $\vec{F}(\vec{r})$ can be uniquely determined if this D, which is a function of r the divergence function and $\vec{C}$, which is a function of r vanishes fast enough as r tends to infinity, so this is the condition we have. So, this condition as I mentioned it is difficult to prove because this is a lengthy proof. I will show you first thing.

(Refer Slide Time: 36:59)



So, then I am going to explain so, how you find this, the recipe is to determine $\vec{F}(\vec{r})$? So, F I can divide it in this form, as I mentioned this plus I can decompose in this way, any given vector function I can define in this way. Now I can construct my f so that is interesting part I can construct my f exploiting the value of D, which is known, and the result should be this f($\vec{r}$) should be $\frac{1}{4\pi}$ then integration this D is known, which is a function of r divided by (r – r')dv, this is the way I can define my f($\vec{r}$) because this quantity is known.

So, that means I can execute this integration and I will figure out what is my f. Similar way my v, which is the vector field, which is here can be constructed by knowing the curl value and this is simply $\vec{C}(\vec{r})$ and r – r' they should have dv' there should be prime because this integration is over r' and here also this is known so, that means, I can construct my F try to understand this. So, I can construct my function F this I can construct this is unknown, but D and C are known.

Once D and C are known I can construct exploiting this known C and D I can construct f and v and this f and v I can put it here in this equation to figure out my F. So, this is a very important theorem because later we will be going to use this directly when we deal with electric field and magnetic field. So, today I do not have much time to discuss in the next class again, we will be going to discuss these Helmholtz's theorem quickly.

And then we will, going to finish our mathematical preliminary portion that is the module 1 and then we will go to jump module 2 where we start to electrostatics in vacuum. So, with that note, I would like to conclude here, see you in the next class. As I mentioned in the next class, again, we will come back to this Helmholtz's theorem part and discuss few things so that your understanding is very much clear. And then we jump to the electrostatic part. Thank you. See you in the next class.