Foundations of Classical Electrodynamics
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Lecture - 11
Curl Operator, Stokes Theorem

So, hello students to the foundation of classical electrodynamics course. So, today we will have lecture number 11. And today we are going to understand the curl operator and Stokes theorem.

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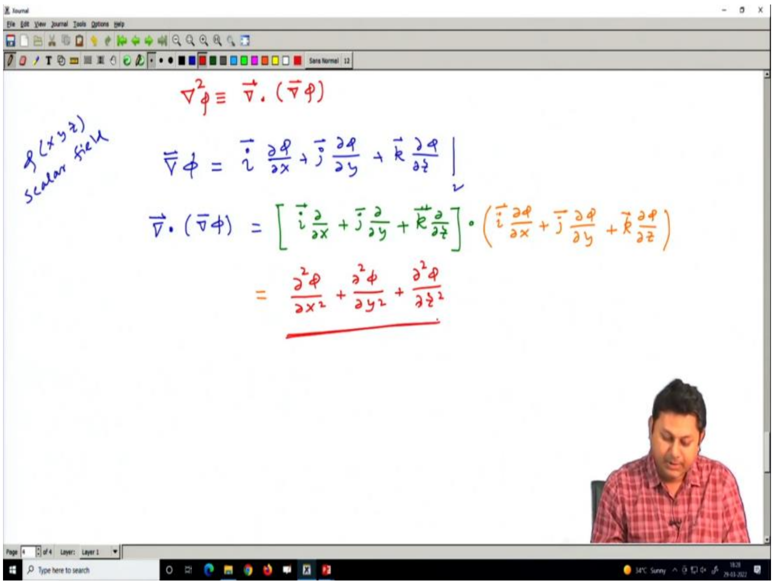

So, before going to the curl operator, so let me you know once again so, today is lecture number or class number 11. So, before going to the concept of curl of a vector let me remind what we did in the last class? So, in the last class in the very end part, we defined this operator and we mentioned that this is called a del square (∇^2) operator and it operates like this. I can have a gradient of a function φ and if it operates over φ, so this is the operation one should get.

So, this operator at the end of the day gives you the second order derivative. So, let us now quickly I give you as a homework but let me do it quickly. So, what is this? If I find out the you know the gradient of a scalar field in Cartesian coordinate let us consider this is a Cartesian coordinate. So, φ is a scalar field x y z and this is a scalar field. So, I am operating these over φ. So, what I will get? I will get i $\frac{\partial \varphi}{\partial x}$ + j $\frac{\partial \varphi}{\partial y}$ + k $\frac{\partial \varphi}{\partial z}$ this I will get.

Now, the next phase I want to find out these things, divergence of this quantity whatever I am getting here. Divergence of gradient of φ. So, I know the recipe when we say the divergence, so, I should have i $\frac{\partial}{\partial x}$ + j $\frac{\partial}{\partial y}$ + k $\frac{\partial}{\partial z}$ and then dot the rest of the term, which is i $\frac{\partial \varphi}{\partial x}$ + j $\frac{\partial \varphi}{\partial y}$ + k $\frac{\partial \varphi}{\partial z}$. Now eventually we will get that when I will be going to operate over i I mean this is a dot product so, I can have simply I can have this one. So, $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$. So, this is the operation we can have.

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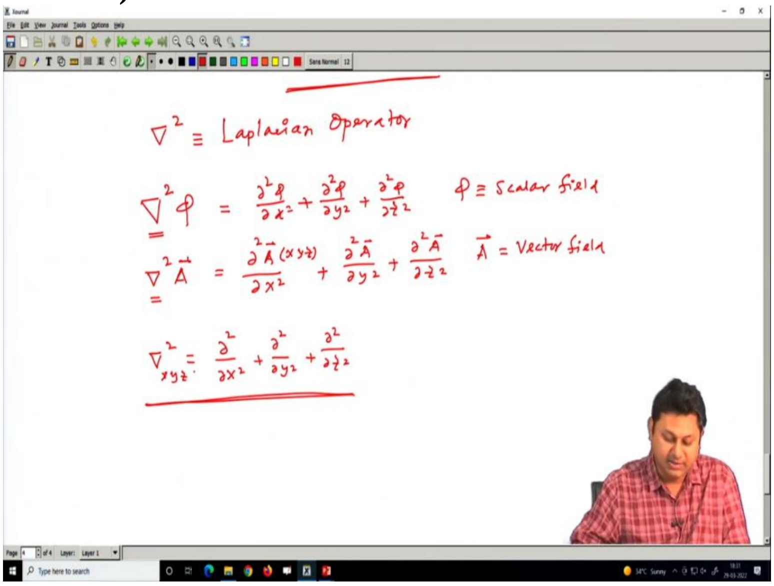

Now, this operator has a name and we call it this operator as Laplacian operator. This later we find that we have a profound implication of this operator in different cases, we will be going to get this fundamental operator and how it works we will be going to check, but at this moment also I like to you know mention here that this operator can operate over φ, which I know that it is giving like $\frac{\partial^2 \varphi}{\partial x^2}$ (where φ is a scalar field) + $\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$.

Here φ is a scalar field but this operator can also operate over a vector field and in that case it is simply $d^2\vec{A}$, which is a function of x, y, z and I am having x^2 plus y^2, z^2. Now, this is a scalar operator mind it unlike the del operator ($\vec{\nabla}$), this is not a vector operator. This is a scalar operator and eventually this operator give us $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

So, that means it can operate over a scalar field and it can operate over a vector field. When it operates over a scalar field and result will be a scalar, when it operates over a vector field the end result will be vector that is all. So, this is roughly you know the way this Laplacian operator operates later we will discuss more about this operator. What is the form of this operator in other coordinate system very important mind it this is in Cartesian coordinate system.

So, this is very simple, but for other coordinate system like cylindrical coordinate system and spherical coordinate system, it is not that easy. So, we need to check this and we will be going to find out.

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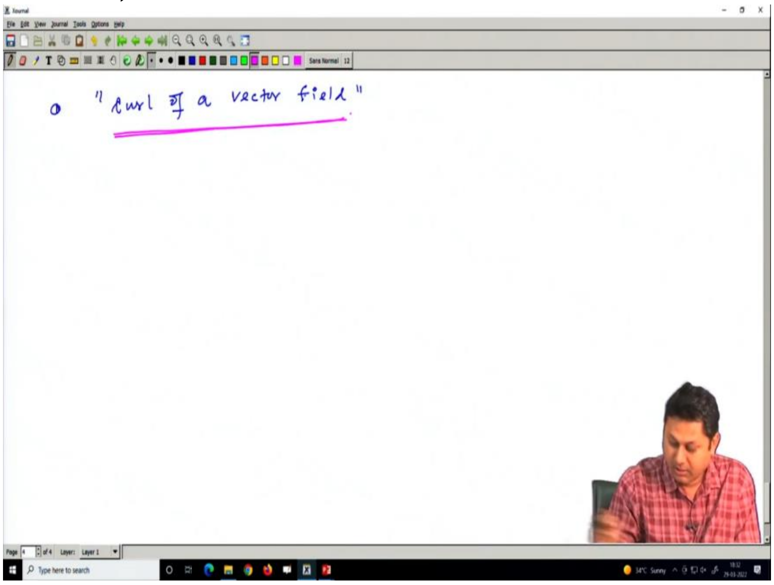

So, now the today's thing that we quickly like to understand is the curl of a vector function or vector field, so, curl of a vector field that is the next topic we did like to explore.

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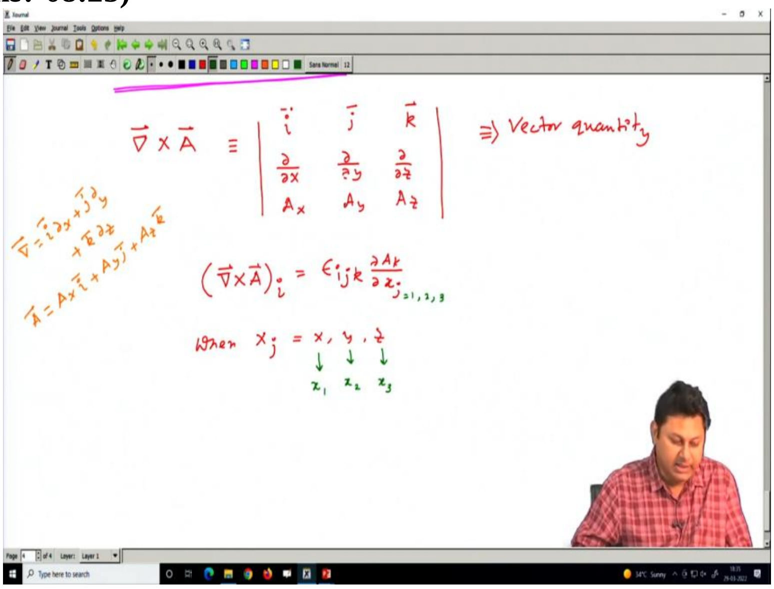

Let me first define how curl is operated? So, curl is an operator it operates over a vector field, this is a vector operator and that over a vector field and it operates like this like a curl. So, here I like to put unit vector i, j and k, here it will be the operator $\frac{\partial}{\partial x}\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ and finally, I have the components of the vector field A_x, A_y, A_z and this should be a vector quantity.

Unlike the divergence, the end result of the curl is a vector quantity. Now, here I mean you should quickly you know write what is my $\vec{\nabla}$ and this is i ∂_x+ j ∂_y+ k ∂_z. This is the shorthand notation of the partial operator partial derivative and my A is simply A_x i + A_y j + A_z k and I put it here like a A × B form. Now, what is the ith component? We know how to deal with this. So, and this is called curl of the vector field A and ith component will be simply ε_{ijk} and then we have $\frac{\partial A_k}{\partial x_j}$.

Now, here when x_j stands for x, y and z with you know indices 1, 2, 3. So, that means, what I am trying to say is this is x_1, this is x_2 and this is x_3, this is the way we define j = 1, 2 and 3. So, this is the way one can operate this curl operator over A and we will get this vector quantity. Now, the question is what is you know geometrical interpretation of that and this operator basically tells us that how curly the vector field is. So, I like to show here once again.

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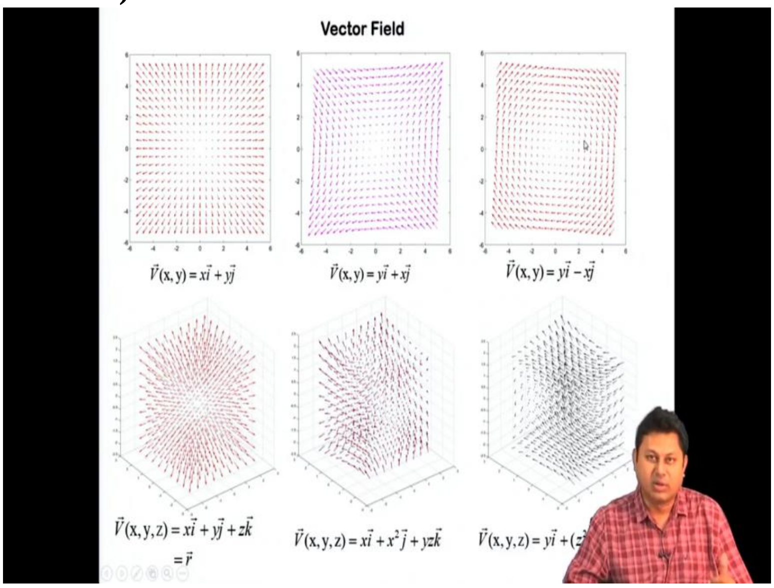

This figure of this vector field that I mentioned in the last class that here you can see that a vector field this last one V(x, y) = yi + xj, you can see apart from other I mean from these 2 x the first one and second one, it is having a natural curliness and this natural curliness, you can measure with this curl operator, this curl operator tells us that how curly these vector fields are, for a given point here if you calculate you will find a nonzero value.

And that tells us that this vector field is curly and what is the amount of curliness gives us by the magnitude because when you calculate the curl of a vector, so it eventually gives you a vector quantity. So, that means you have a magnitude and direction and then the direction will give you because it is the curliness. So, it is a vector quantity. So, it gives you the direction as well as the magnitude and this magnitude will tell us that how curly the vector is. We will come to this in detail maybe later.

But let me whatever we have, so let me write it down that curl of A ($\vec{\nabla} \times \vec{A}$) geometrically or if I want to understand physically what it is doing over a vector field.

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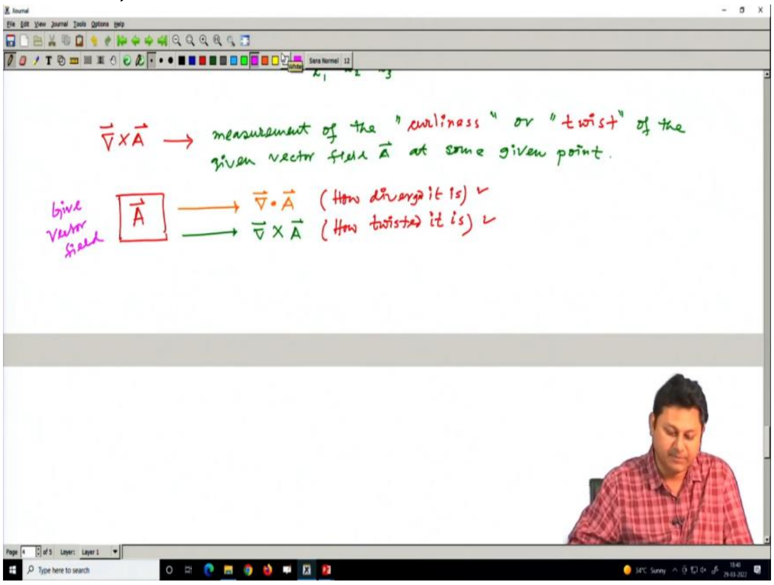

So, $\vec{\nabla} \times \vec{A}$ this quantity is basically a measurement of the curliness or another word twist how twisted it is. Twist of the given vector field A at some given point. So, that means in the bottom I can write, so if a vector field is given to you. So, 2 ways you can make it treatment, there are 2 ways you can calculate how divergent it is by calculating the divergence of this vector. And if you find there is for a given point you can calculate this value is nonzero or not.

If you find this a nonzero value that means, this vector field has a divergence at this given point or you can also find out how curly it is for a given point at a given point with this operation. So, A is a given vector field and with these 2 operation one is called the divergence and another is called the curl. You can measure how the vector is spread out, how much it is spreading out, how diverge it is or how twisted it is.

These 2 measurements you can do. Later we will do more problems and then maybe you can understand, so, this is a given vector field and you can measure these 2 quantities as soon as it is given you can measure these 2 quantities and the divergence and curl are 2 very important properties of a vector field that we will be going to use later.

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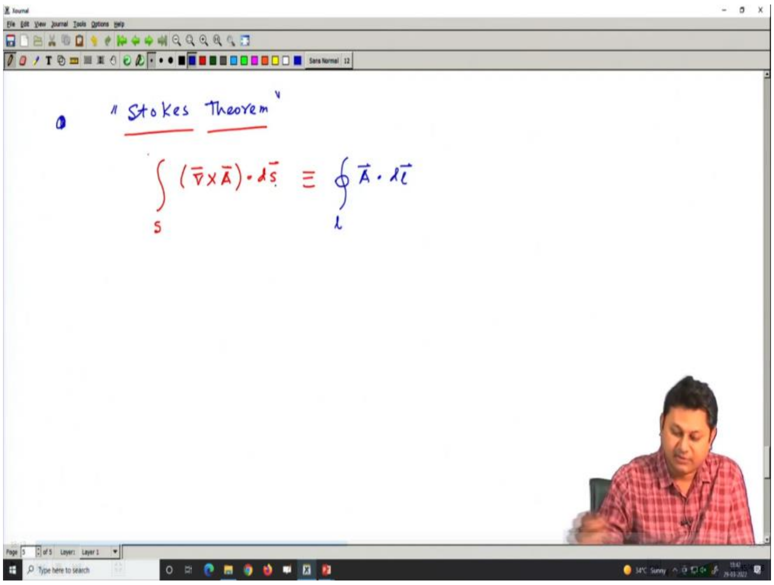

And these theorems are also very important. So, now we are going to introduce another theorem, which is again very important and this theorem is called the Stokes theorem. The mathematical statement of the Stokes theorem, let us try to understand let me write down first the mathematical statement and it says that if I am calculating a surface integral of a curl of a vector field over a surface. That quantity is eventually the same value we are calculating if I am having a closed line integral and then this vector field over this given line and what is the line? The line which is encircling the surface.

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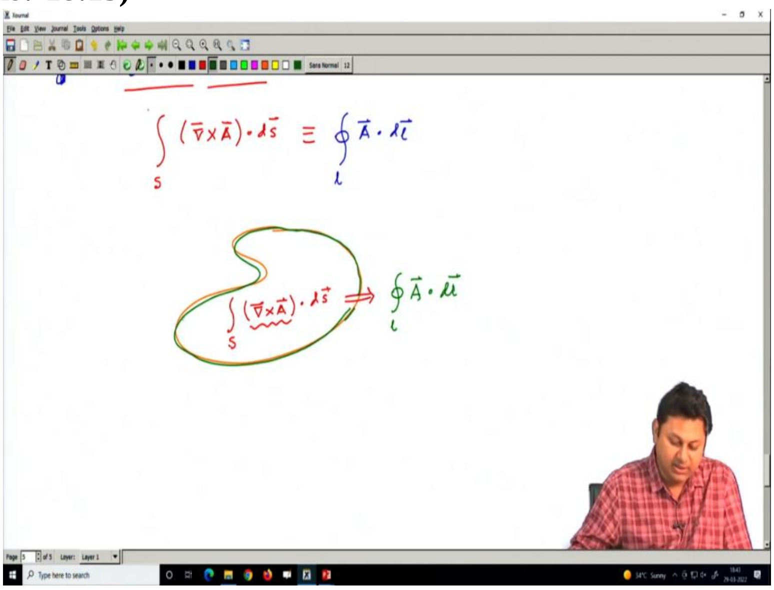

That means, if this is my surface suppose, let me draw it suppose this is an arbitrary surface and I am calculating the $\vec{\nabla}\times\vec{A}$, I calculate and this is a vector quantity and then I calculate the flux of this quantity over the surface, this is the surface integral. So, flux is nothing but this is a vector quantity and this vector quantity if I make a dot product with S and then integrates, so, what we are getting is the flux.

So, the total flux if I calculate for this given surface that value is equivalent to if I only calculate the line integral the line is this one it is encircling the entire surface, this is the line integral. So, this close line integral and if I simply calculate the given vector field over this line element, so that is the statement. So, first I am doing the curl and then make a surface integration.

Whatever the value I am getting, this is a scalar quantity, you can see that then I integrate the line integral of the line, the line I choose in such a way that it is encircling the entire surface that is all, that is the mathematical statement.

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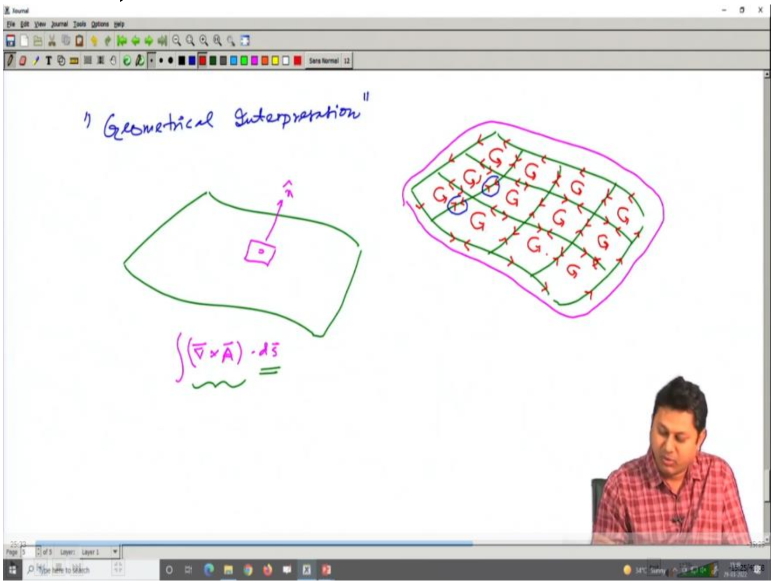

Geometrically if I want to understand before going to a rigorous proof, it is quite nice. And so, this is a geometrical interpretation of the statement if I want to find out. So, in geometrical interpretation, I can see that, let me have a surface like this, this is a say surface element I am having and over this surface element over the surface, I am having a surface element, a tiny surface here. And I calculate, n is say along this direction, this is a unit vector of the surface and I am calculating that quantity. This quantity integral of this $\vec{\nabla} \times \vec{A}$, whatever and dot ds.

Now, if I now make a subsection of this entire surface here, so let me draw the surface once again, same surface I am drawing here. Now, I am integrating with small surface area, this is a small surface area. So that means eventually I can divide this entire surface to different sections. These are the small ds that will add and make a complete surface and for each surface, I am calculating this quantity for $\vec{\nabla} \times \vec{A}$ is basically gives me the curliness of the vector field.

So, eventually what I am getting is I am getting this kind of curliness here, here and that we are measuring. Each section, I am having a curl, so, curl is basically gives me the twist, how twist it is. So, I am calculating these twist at each ds surface so everything is ds. Now, if you look carefully, you will see that if I, so there is a cancellation, so if I go to the boundary with this twist, I will have this, these are the boundaries.

And it is going in this direction, this direction, for this case, it is going this direction, this direction, this direction, this direction. For this case, it is going this direction, this direction, this direction, this direction and for this case, it is going this direction, this direction, this direction, this direction and this direction, for and here, we have this direction, this direction, this direction, this direction, in this case, this direction, this direction, this direction, this direction.

In this case, it is this direction, this direction, this direction, this direction, in the lower case. It is this direction, this direction, this direction, this direction, this direction, I am just drawing the direction accordingly. And for this case, it is this direction, this this one, this one. Now, you can see that when you calculate the curl of entire this mat, and all this ds you calculate and then integrate it over. The final outcome is only this portion here this is the final outcome because this is the line, which is not cancelling out by the curliness, can you see that.

All these terms that is sitting here intermediate links they are cancelling, I am having here also this except the boundary all the intermediate region if cancelling out when you calculate the $\vec{\nabla} \times \vec{A}$, which is the measurement of curliness over the small regions.

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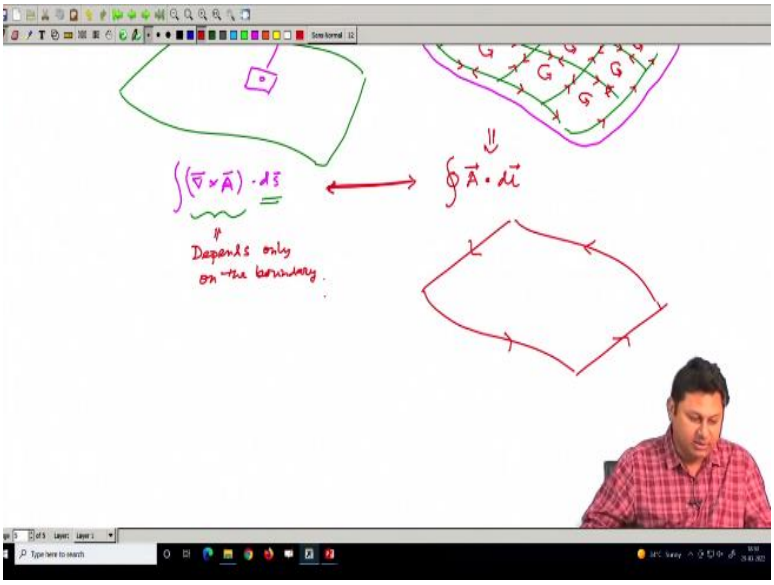

And at the end of the day this thing is equivalent to the calculation of the entire vector field the line integral of this vector field over this closed line. So, then these 2 things are seems to be correlated like this. So, these are the geometrical understanding so, these things depend only on the boundary. So, eventually I can calculate only the line integral on the boundary of the surface and that will give you the same result if you calculate the curliness.

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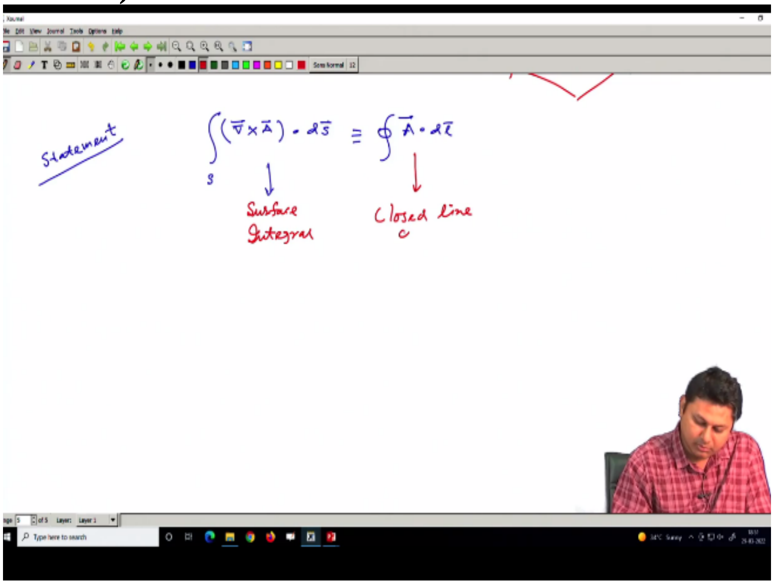

Now, we will be going to prove this rigorously, we will have a rigorous proof. So, the statement of the Stokes theorem once again let me write that the surface integral of $\vec{\nabla} \times \vec{A}$ over a surface is equivalent to the line integral of this close line integral where the line is encircling the entire surface. So, this is a surface integral and this is line integrals, closed line integrals. Let me prove it like the previous way we can also prove that and let me do it here.

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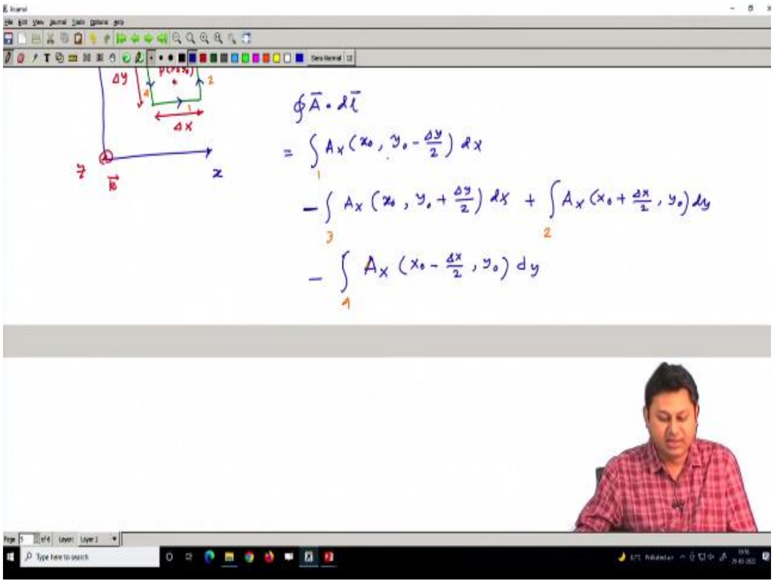

So, this is the coordinate system I am having say this is my x and this is my y and we can have a small block here surface element so that this is my Δx and this length is Δy. I can have a point here the centre P, which has a coordinate x_0, y_0 like the before and this is the direction of z and this is along the k direction that for this surface that is the direction of the surface. So, I can write it here. So, ds is the magnitude of ds here and with unit vector k. Now let me calculate $(\vec{A} \bullet \overrightarrow{dl})$.

Now, this $\oint \vec{A} \bullet \overrightarrow{dl}$ If I calculate, I can write it in this 4 lines, one line is here another line is here. And this is the way I will go because there is a line integral I need to have a direction as well. So, this is integration of like before I have A_x, then x_0 and let us start with this one, this upper one. Let us start with this lower one. So, it is y_0 and then $-\frac{\Delta y}{2}$ so that is the x component of the vector field along this direction, because I am making a line integral mind it and then dx, that is for this line.

So, let me write it 1, 2, 3, 4. So, this is my 1, this is my 2, this is my 3 and this is my 4, these 4 lines I need to integrate. So, this is line 1. Then I add the next one and that is for say, let us do that for 3. So, this is for line 3, there is a special reason I take 1 and 3, and 2 or 4 because they are in opposite direction that is why. So, I have A_x here, previously the way we had and then x_0 and now I have $y_0 + \frac{\Delta y}{2}$, because in 3, this is y_0 plus and dx.

But at the same point, you need to be careful that here the line is this direction and here the line is in opposite direction. So, that is why this sign should be a negative sign. Here we have a negative sign. In the similar way, if I add another 2, say plus integration of say line 2, for line 2, I can have A_x and now $x_0 + \frac{\Delta x}{2}$ and y_0 and then dy. This is in the anti-clockwise direction. So, I am moving this direction, the other direction if I consider there will be a negative sign.

And that integral will be simply integral that is for fourth line we will be A_x ($x_0 - \frac{\Delta x}{2}$, y_0) and again, changing parameter is over dy. Now, the similar way that we had in the previous case, the expanding these as a Taylor series, I already did it, I am not going to do that once again. So, we will get simply this quantity.

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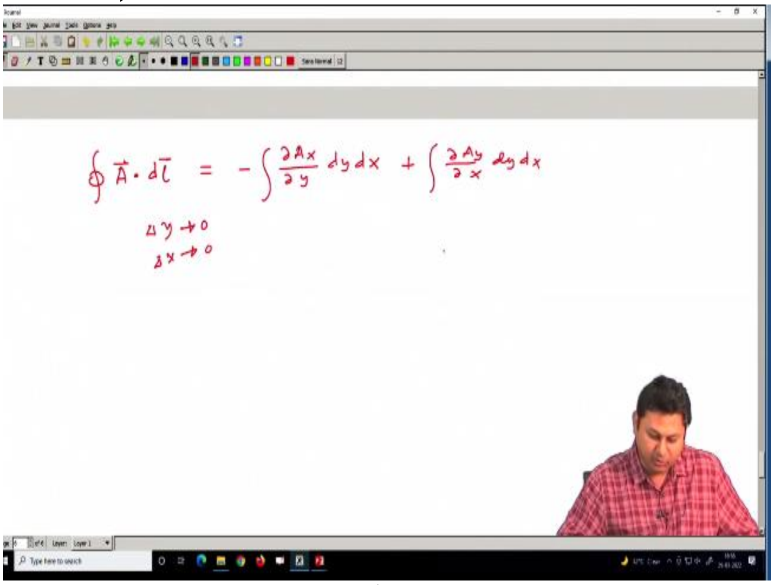

So, left-hand side you are calculating $\oint \vec{A} \bullet \overrightarrow{dl}$ and what you are getting in the right-hand side is this. So, if I just extend this Taylor series the expansion, you will simply get $-\int \frac{\partial A_x}{\partial y}\, dy\, dx$ with the condition that Δy tends to 0, Δx tends to 0 for tiny, another term you are also getting that is $+\int \frac{\partial A_y}{\partial x}\, dy\, dx$, what you need to do is just expanding if you expand this quantity, so let me do for first term the second term will be very straightforward. So, I am doing here.

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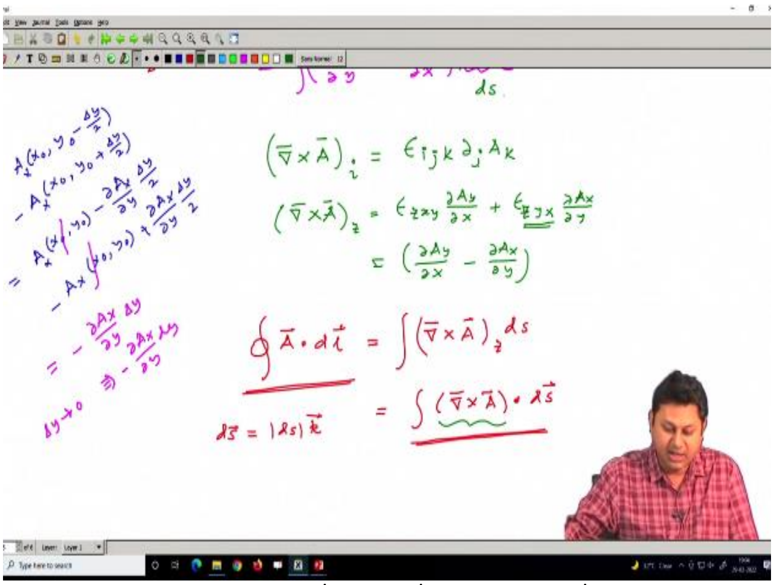

So, you are having A x_0 the first term $y_0 + \frac{\Delta y}{2}$ $y_0 - \frac{\Delta y}{2}$. So, ($y_0 - \frac{\Delta y}{2}$) this is one term and then you have $-A$ (x_0, $y_0 + \frac{\Delta y}{2}$), this quantity is A (x_0, $y_0 - \partial$A), this is A_x so this is x $\frac{\partial A_x}{\partial y}$ and this $\frac{\Delta y}{2}$ and then - $-A$ (x_0, y_0), there should be negative sign again. So, previously there is a negative sign and now we have a minus and this will be a plus sign, + $\frac{\partial A_x}{\partial y}$ $\frac{\Delta y}{2}$. So, this term eventually gives me because these things will cancel out like the before.

And eventually, I am getting minus of here $\frac{\partial A_x}{\partial y}$ dy and now if I put Δy tends to 0, so this things lead me $-\frac{\partial A_x}{\partial y}$ dy, which I already put here. So, $-\frac{\partial}{\partial y}$ dy, dx was already there, in a similar way you can get this. Now, if I combine these 2 I can write it as $\frac{\partial A_x}{\partial y}$ $-\frac{\partial A_y}{\partial x}$ and dx dy. Mind this dx dy is nothing but ds. Now, if I go back to this curl thing $\vec{\nabla} \times \vec{A}$ ith element is ε_{ijk} ∂_j A_k that is the way we define.

So, what is the z component of these things if I want to find? z component will be simply ε_{zxy} $\frac{\partial A_y}{\partial x}$ and then the non-vanishing term will be plus ε_{zyx} and then $\frac{\partial A_x}{\partial y}$, so these things are with I mean if I write it is simply $\frac{\partial A_y}{\partial x}$ $-\frac{\partial A_x}{\partial y}$ this minus sign is because of this term because it is anti-cyclic. So, that quantity if this is z component so, I can find that for I mean this is eventually tells me that if I calculate the line integral the $\oint \vec{A} \bullet \vec{dl}$.

In the right-hand side, I am getting $\vec{\nabla} \times \vec{A}$ with its z component and ds. Now ds is along you know if I just simply put down this vector sign you need to understand that ds is $|ds|$ and then k vector. So, exploiting that relation and this is z component also we have a k vector for k dot k should one. So, simply we can have that $(\vec{\nabla} \times \vec{A}) \bullet d\vec{s}$ in vectorial form. So, you can see that is the statement we are looking for and we prove this.

The left-hand side, we are having the closed line integral and in the right-hand side we have a surface integral where we are calculating the curl of the vector field and make a surface integral that should be equivalent to the line integral while the line is enclosing the surface we prove this with this simple technique and then you should make a look that all of these proofs are very straightforward, I believe, you understand that and the next class we will do more about some identity using this curl and if important identities are there.

So, today's class is very important because we understand how the curl is operating over a vector field and what is the physical significance of that. So, with that note, I like to conclude today's class. So, thank you for your attention and see you in the next class.