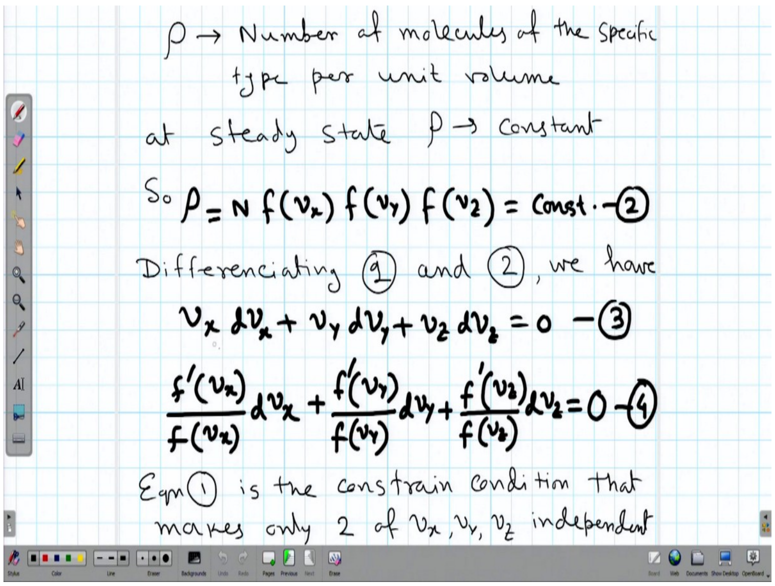
Thermal Physics
Prof. Debamalya Banerjee
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture –2
Maxwell's Law for Speed Distribution of Gas Molecules

(Refer Slide Time: 00:33)

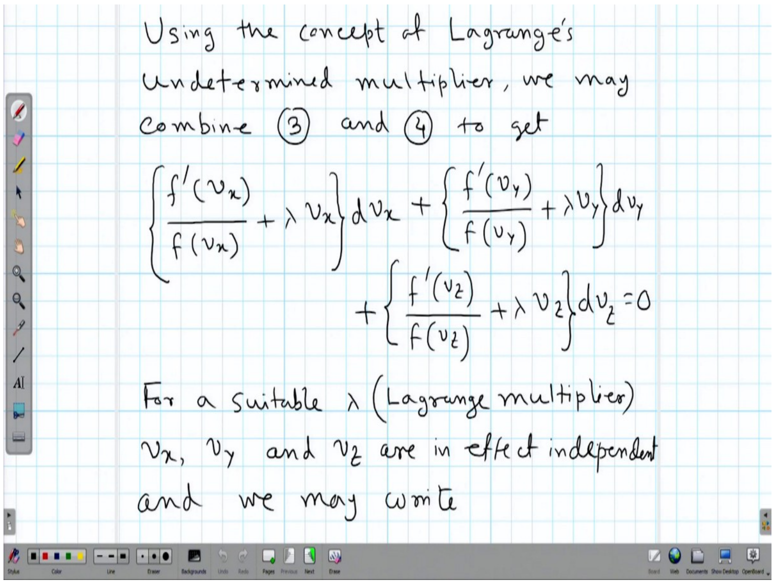


Welcome back. So, we have we will start from the position where we have left it in the last class. So, in last class if you recall there were two equations that is where we ended up we have the first equation which is the we got this idea taking a derivative of the velocity and then there is a second equation. Now the first relation essentially is originating from this constraint condition that we square v x squared plus v y squared plus v z square is equal to c squared which is taken as constant.

Because we are talking about a fixed felicity we want at present we want to see the distribution of components for a given velocity. Now and so because there is a constraint condition in differential form and there is a relation also in form of a differential I mean it is in differential form. All 3 coefficients of dv x dv y and dv z cannot be independent of each other because there is a constant condition as mentioned in the upper relation.

So, the way in order to handle this we have to use Lagrange's undetermined multiple. Now if you are not familiar with how Lagrange is undetermined multiplier works I would request you to go back to your mathematics lectures or if you have taken a classical physics course already probably it is discussed when you were in the foundation of you know the basics of Lagrangian dynamics was discussed but mostly it is found in mathematics book.

(Refer Slide Time: 02:15)



So, we will switch to. Now for today's content we first start with the concept of Lagrange's undetermined multiplied I mean sorry not the concept but the application of Lagrange's undetermined multiplied. So, basically we add 3 and 4 at first we multiply equation 3 with a constant lambda. Now lambda is totally arbitrary because equation 3 is what essentially is zero.

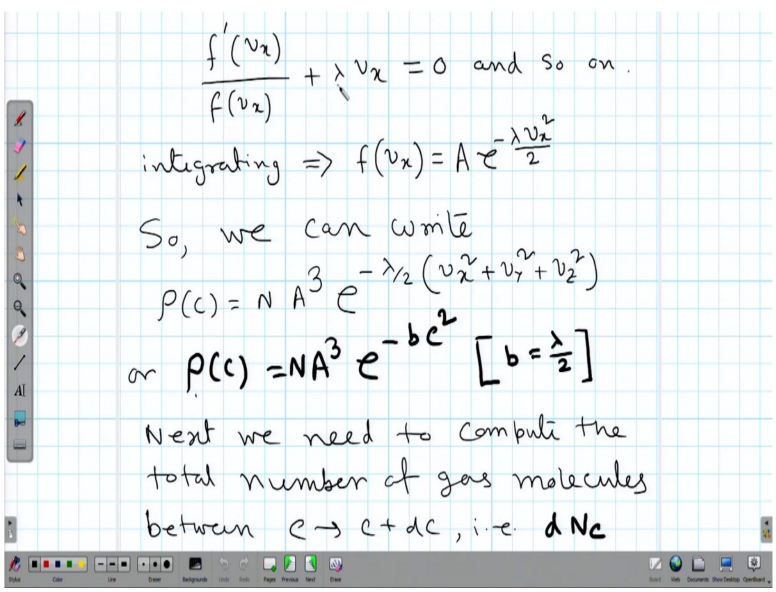
So, we can multiply anything with zero and that still remains zero and then we added with equation 4. So, basically we are adding something with zero. So, we are not violating any mathematical loss and once we, you know, separate the I mean we or rather we want to we will or rather once we you know put everything under dv I mean with the fact that dv x in one bracket and dv y in another bracket and devise it in another bracket we get this particular relation.

Now as we have already mentioned there are 3 variables and there is one constant conditions out of that 3 variable only 2 can be independent. Now because we have this Lagrange's undetermined multiplied in hand what we can do is also in other words in mathematical words if we have let us say okay let me quickly write this if it makes any sense. So, let us say we have a situation where we have c 1 ok sorry just a minute want to use c 1 dx 1 plus c 2 dx 2 plus c 3 dx 3 plus where is it plus c n dx n and if this is equal to zero.

In order to have all x 1 x 2 x 3 accent if all these coordinates are independent we can say that c 1 c 2 c 3 c n are individualized that is when we have every coordinate as independent. So, now go back let us go back to our situation we have something with dv x plus something with dv y plus something with dv z equal to zero and we know that dv x divide divided out of this 3 only 2 independent. So what we can do here is we can so out of this 3 coefficients only 2 will be zero sorry yeah 2 will be individually zero the third one will be not but we have this Lagrange's undetermined multiplied in hand.

So, by using what we can do is we can make the third relation that coefficient also equal to zero because this lambda let us assume that dv x the coefficients of dv x and dv y were already zero for dv z we have this lambda in hand and we can always choose a suitable value of lambda. So, that this whole thing becomes zero. Now once we do that that we are free to do that because lambda is anyway undermined.

(Refer Slide Time: 06:22)



We get 3 all 3 coefficients like this one and this one and this one individual equal to zero. So, we can write an equation for example for the first one we can write if prime vx of divided by vx +

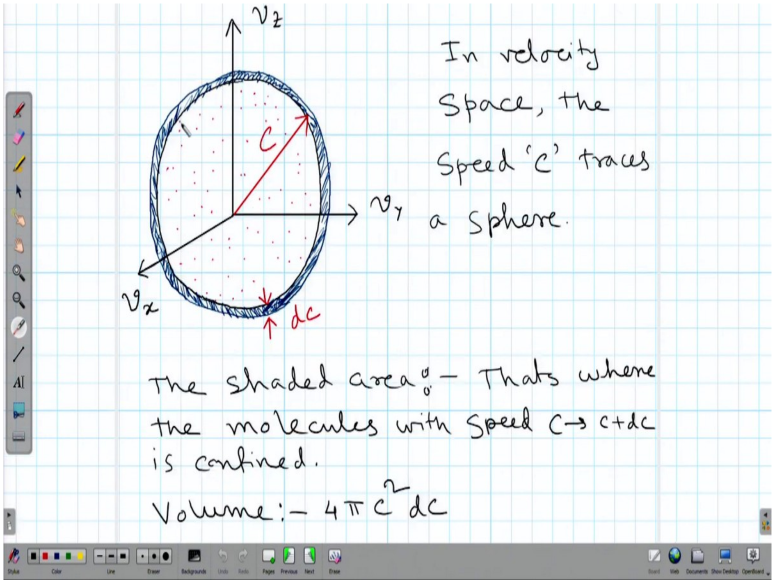
lambda v x is equal to zero and these relations are very easily integrable absolutely without any problem we can integrate those and we can get F of v x is equal to a to the power minus lambda v x squared weight right.

But and also we have to remember that the we have very similar expression for f of v y and f of v z because all 3 equations are identical all 3 coefficients of this equation are identical. So, if v x functional form f v x is known a visa is also known and we have rho c is equal to N A cubed it would be and please remember is the constant the same constant we are using all in all 3 case for all 3 functions. So, rho will be N times remembered rho was f v x f v y f v z times m right.

So, that product will be simply N A cubed to the power minus lambda by 2 v x squared plus vy squared plus vi z square. Now writing v x squared plus v y squared plus v z square is equal to c squared and defining v is equal to lambda by to a new constant we write rho c is equal to N A cube to the power minus b c square. Now next we want to compute these 2 constants A and b and finally we would like to go to we would like to compute d N c which is the total number of molecule in the gas assembly okay that has a speed between c to c plus sorry I will just bring it up a bit.

So, finally we want to compute d N c which is the number of gas molecules in the range c to c plus dc right. Now the question is how to compute A and b? Probably for this relation okay oh sorry let us first focus on d N c how to compute d N c using this relation.

(Refer Slide Time: 08:39)



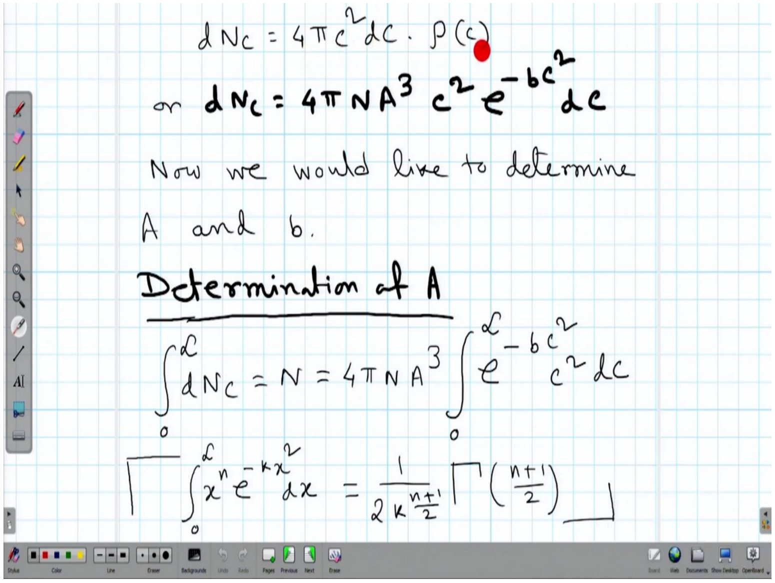
So, once again we assume that in velocity space the molecule moving on. Now if we want to compute the total number of molecule that has a speed c please remember it is not velocity anymore it is a speed. Speed is what speed is simply a magnitude. So, once we want to compute the speed we basically have sea okay I have given it I mean I have drawn it like a vector but it is frankly speaking it is not a vector is simply a scalar.

So, all the molecules which will have a speed c will lie on the surface of this sphere and what do we want to compute? we want to compute the number of molecules in the range c to c plus dc because of course we cannot really calculate the exact number of particle molecules that are moving with c we always have to define a range T c small range okay. In the speed c to c plus dc essentially means that this number essentially means the number of molecules in the velocity space which will lie inside this thin shell.

You see this shell is mark the shell has a thickness dc so we have a sphere of a radius c which is a hollow sphere and then there is a spherical cell of thickness dc whatever molecules or whatever gas particles are inside this spherical cell they will have the speed between c to c plus dc. Now what is this volume thin volume this is simply 4 pi c square dc. I hope you are all familiar with this treatment of spherical cell if not just try to understand okay it is nothing very complicated just try to understand look at this picture over here and you will realize why the volume is 4 pi c square dc quality.

So, and we already have this expression for rho. So, we have density expression we have the expression for volume.

(Refer Slide Time: 10:47)



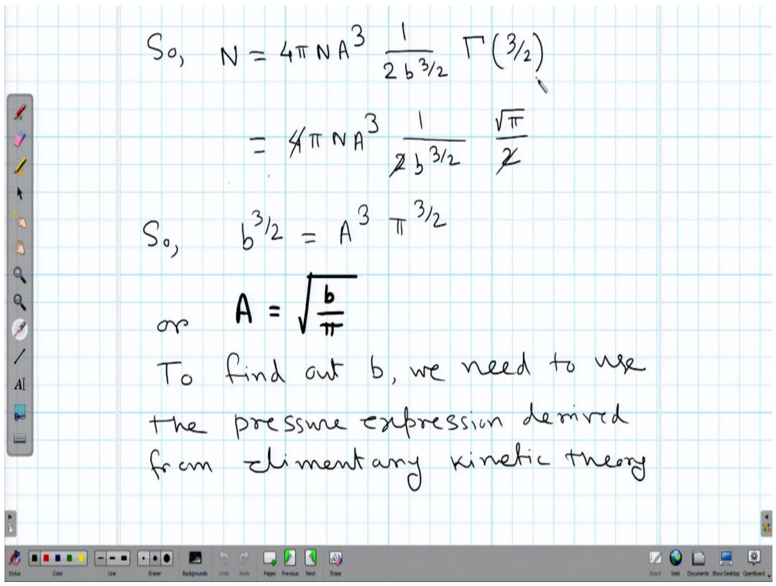
So, the final number is not nothing but d N c is equal to 4 pi c squared times rosy which is d N c equal to 4 pi N A cubed c squared e to the power minus p c squared. Now this is our expression for the famous Maxwell velocity distribution and next what we want to do is we want to determine the constants A and b. Now that determining one of the constant is I mean both the constants are easy one is very straightforward the one we are going to take a first.

All you have to understand that if we integrate this d N c for all possible velocity starting from zero to infinity we should get the total number of molecules or total number of particles gas particles present that is N. So, integration zero to infinity d N c is nothing but N and this integration you can write us 4 pi N A cubed integration zero to infinity e to the power minus b c squared c squared dc right.

So, just from this expression you can write. Now this class of integration is a special class of integration called the Gaussian integration which has this following form zero to infinity x to the power n e to the power minus k x squared dx is equal to 1 by 2 k to the power. So, I should actually put a bracket here to the power 2 n plus 1 gamma n plus one by 2 where gamma is a special class of function called the gamma function.

So, I am not going into the details of this but once we compare we see this N here is equal to 2. So, N is 2 and minus kx squared is minus vc squared okay. So, if we can make A 1 to 1 correspondence the integration this integration gives 1 over 2 between the power 3 half gamma 3 half there are standard tables for gamma function you can always look up.

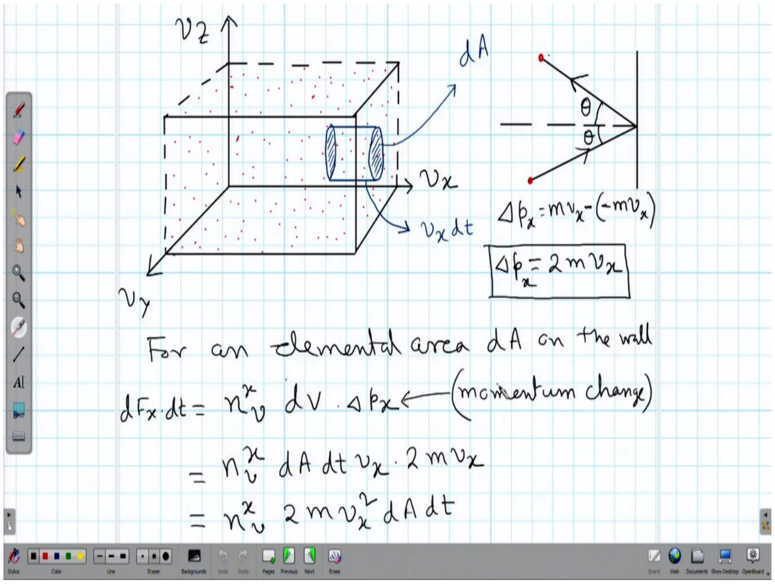
(Refer Slide Time: 13:00)



Gamma fraction is allowed gamma is allowed only thing is gamma of minus integer is not allowed any other real and imaginary function a real and imaginary number can have I mean a value in the common function table. So, 3 half is there you can compute that you can have a look at that tables which are freely available on internet and the value is root pi by 2 simplifying we get this relation that A is equal to root of b by pi. Now to find out b, now that means what instead of 2 constants we have A and b here we can express A in terms of b.

So, all we have to do is we have to find out b from by some by some other means and for that we need to use the elementary treatment of pressure from kind of using kinetic theory.

(Refer Slide Time: 13:57)



Now let us go back to our first picture which we have drawn we have drawn a similar figure a cube or you know parallel parallelogram that is containing the gas and here we just representing this in the velocity space or actually you should not actually we should make it x y and z not v x v y V z sorry there is a small mistake. So, it will be simply x y and z. Now strictly speaking this is not exactly the oh yeah that is fine we have already drawn the right handed coordinate this is fine right.

So, yeah I have written v x here ideally it should be x in this direction y in this direction but just for the sake of simplicity I know it is wrong but let me write why here and x here as I have already written all the expressions in terms of v x knowing that it is wrong it is not exactly the right handed coordinate system we are familiar with this is the left handed coordinate system but that is okay I mean it is just a representation not very; we cannot we can afford to do this take this liberty.

Now just focus on this elemental area dA on the surface let us say this is the axis although frankly speaking it might be why but does not matter. So, let us say this is the elemental area. Now any particle this marked in red that is hitting wall and going back. Now this phenomena is responsible for pressure. Please understand this pressure in a gas container is primarily because the molecules are being bombarded on the boundary wall.

Now let us assume this particle is moving in this direction with an angle thicker and it will require in another in this direction with another making this angle theta. So, altogether there will be a deviation of 2 theta. So, but because these are elastic collisions that is one of the basic assumption of kinetic theory the total change in momentum will be only along the x direction and this will be delta p x is equal to m v x.

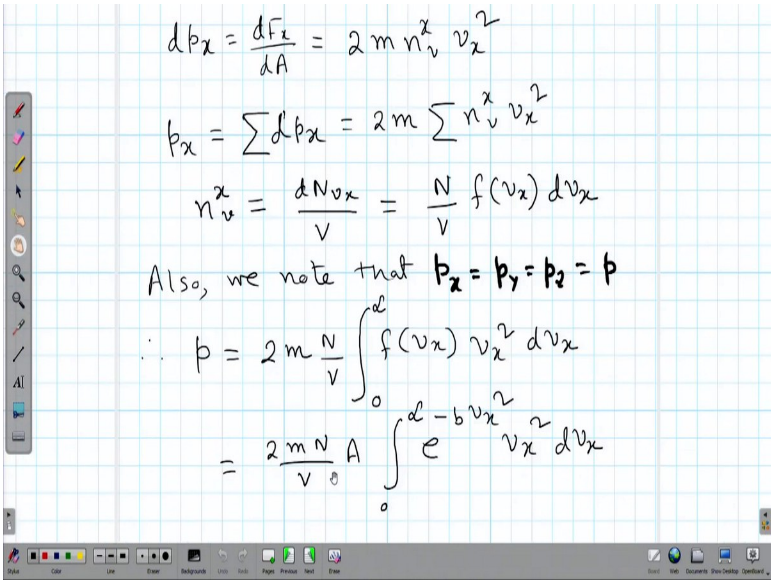
Let us say it has velocity components v x v y v z only the x component will change its direction because these are elastic collisions right. Now if we try to find out the total force or impulse of force in a time interval short time interval dt on this small area dA that will be given by dF x times dt that is the impulse and it will be given as n superscript x v dV delta p x. Now the first term is the number density.

I will explain that little in with some little more detail dV is the volume of the cylinder which has a site of one of the sites the cross section area dA and the length of dx v x dt because if we are considering a short time interval dt only the gas molecules these are within this distance can strike the wall anything that is lying outside this distance will not be able to reach the wall in the short interval. So, this explains a lot I mean sorry this explains why this is why this dv can be written as dA times dt times v x.

Delta p x is already the momentum change is 2m v x please do not mix it up with the pressure because unfortunately for pressure and momentum we have to write the same we are family we write p small p. So, I am just trying to put a delta p x and please note that this is momentum nothing to do with pressure as of now okay. So, anyway we put the value of delta p x as 2m v x. Now coming back 2m v x what is that? This is the total number or number density that means the total number of molecules in this assembly with the speed with the with the x component of velocity v x divided by the volume.

So, in brief n v x is dN v x by V, right dN v x is something that we have already defined. So, I have just substituted for dN v x in the next slide.

(Refer Slide Time: 19:08)



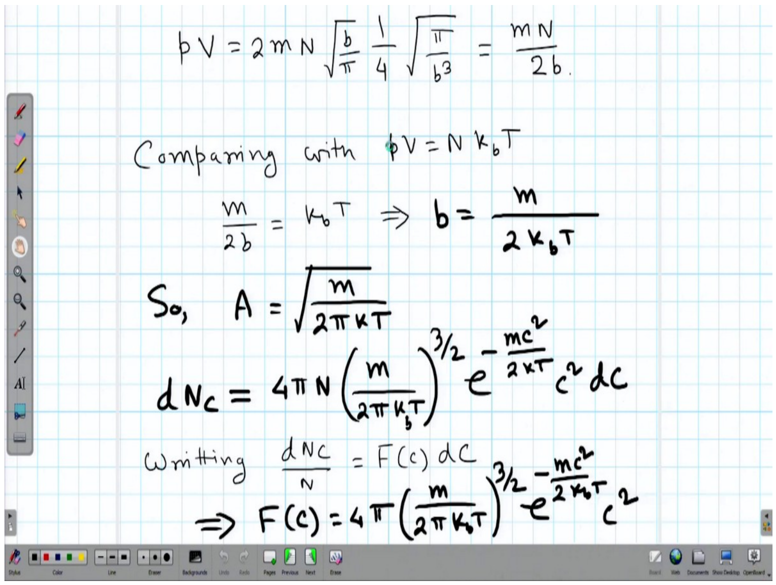
You well in a moment. So, now we want to compute the pressure what is pressure? Pressure is now you see this dt sorry this dt from this side and dt from this side cancels out nicely and finally we have the pressure expression which is d F x by dA that is force per unit area which is 2 m n v x v x square. Now and this is only for this, this is this elemental pressure dp x on this area right. Now if we want to compute the; but anyway does not matter because we are taking the derivative. So, pressure at this point pressure on this surface and pressure on this entire face will be exactly the same.

But also but we have to remember that this is only for those molecules for which the velocity has a x component of v x what about the others other molecules? There are many different speed ranges different velocity ranges. So, we have to submit up over all possible velocities in this manner. So, it will be p the total pressure p x which is equal to 2 m sum over n v x v x. Now we substitute for this n v sub superscript x is equal to dN v x by v which is nothing but N by v F v x dv x.

Let us go back quickly where is it right yeah remember this expression from the last class dN v x is equal to N f v x dv x similarly dN v y and dN v v z was defined. So, we are just using this expression and finally also we have to note that p x is equal to p y is equal to p z that means the pressure in all directions are equivalent that is only one pressure right. So, it does not matter whether we measure it along x axis y axis z axis.

So, the final pressure we instead of p x we can write p this is nothing but 2 by m sorry 2 m N divided by v and integration zero to infinity f v x v x square dv x.

(Refer Slide Time: 21:51)



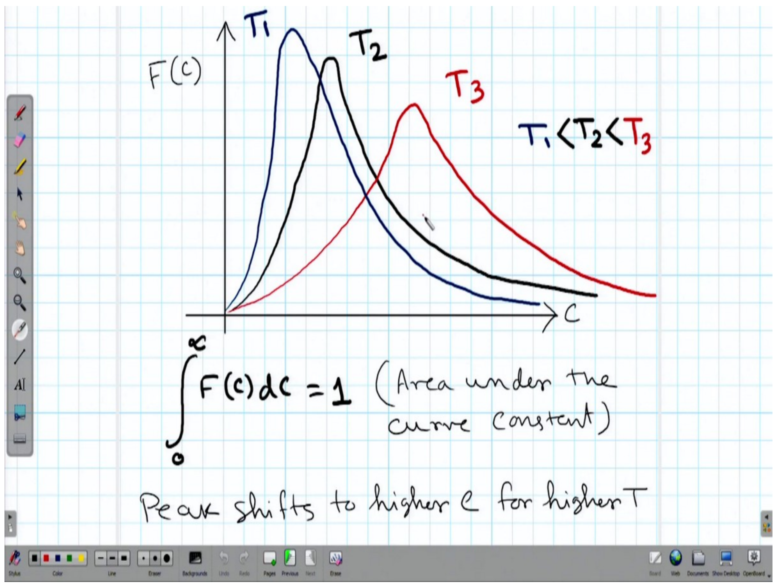
Now if we compute this integral once again we get this Gaussian type of integral and this integral I mean we have already computed this what is the value here we know the value is root pi by 4 b to the power 3 half right. So, we just substitute for this. So, root by b cube one over 4 and we get m N by 2 b. Now and instead of p we just what do we do? We just take this v from here and multiply with p. So, finally we compare it with the ideal gas equation p v equal to n k b T where n is the total number of molecule we simply get b is equal to m by 2 k b T.

Now we have A in terms of we already know A is equal to root over of pi by T. So, we have b is equal to simply easy sorry it was b by pi or b by pi yeah b by pi right. So, A is equal to root over m by 2 pi and the final expression Maxwell's velocity distribution expression is this d N c is equal to 4 pi N m by 2 pi k T whole to the power 3 half's it will be power minus mc squared 2 k T square. Now we can write this as d N c by N which is nothing but you know density number density of molecules in the speed range c to c + dc.

And we write this as F c dc where Fc is this function. Now this is an interesting function we have one term which is constant in terms of speed but the coefficient the this term which is constant in terms of speed but changes with temperature and we have this term and this term which this term which is dependent on both speed and temperature and we have this term which is only speed.

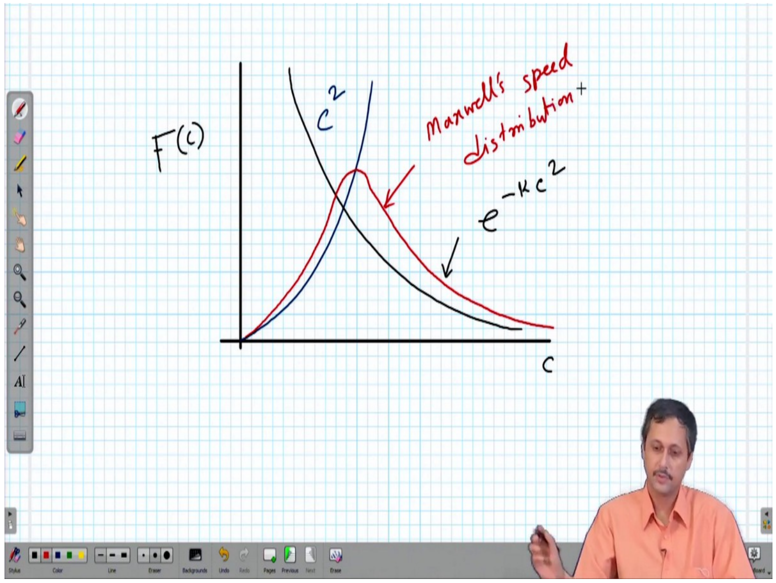
Now look at this term as the sorry look at this term as the speed increases this is going up as c squared and look at this term as the speed increases this is decreasing exponentially. So, for now if we take temperature as a constant quantity this term will be dominant at high values of c and this term will be dominant for small values of c and look at this black curve here.

(Refer Slide Time: 24:24)



So, it will be something like this okay let me quickly go back to this our class rough document skate and what we can do is I can try to draw this quickly for you.

(Refer Slide Time: 24:40)



So, this is my xi this is my c. So, in F c we have one term which goes as c squared right ok. We have one term that is that goes as c square and we have one term that falls off as okay no not like this maybe like this yeah fall off as e to the power minus c squared I mean some constant tank c squared keeping the temperature constant some constant times. So, the overall range should be an overall summation should be something like.

At some point it should turn back and start going right. So, this is my Maxwell's speed distribution right. So, now in this part what I have tried is I tried to draw these 3 lines corresponding to 3 different temperatures. When I draw the previous one I did not I mean I consider temperature to be constant right. But then when temperature increases see what happens this coefficient changes the I mean this factor also changes.

So, overall there is a shift. Now you think of why with increasing temperature they should be going right and why with decreasing temperature it should be going left but I think if you just give it a thought you can find it out. So, I have given I have drawn it for 3 different temperatures T 1, T 2, T 3 but also the note that not only it shifts to the right or to the left but please remember the integration zero to infinity F c dc is equal to one it has to be because what is F c? F c is nothing but F c d c is nothing but d N c by N.

So, integration Fc dc is mean from zero to infinity means integrating the d N c from zero to infinity and dividing it by N which will be N by N which will be equal to one. So, the area under the curve should be constant okay. So, if the peak shifts to the right it should also smear out a bit. So, that the idea remains constant similarly if it shifts to the left it should be little sharper the decrease should be little sharper.

So, that the area under the curve remains the same. Now for today's lecture we will end here in the next class we will try to define 3 quantities or 3 different speeds which is the mean speed the root mean square speed and the most probable speed for this velocity distribution okay. So, goodbye for now.