Physics of Linear and Non Linear Optical Waveguides
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

Module - 03
Modes (Cont.)
Lecture - 31
Modes in an Optical Fiber (Contd.)

Hello student to the course of Physics of Linear and Non-Liner Optical Waveguides. Today, we have lecture number 31 and we will continue with the concept of Modes in Optical Fiber.

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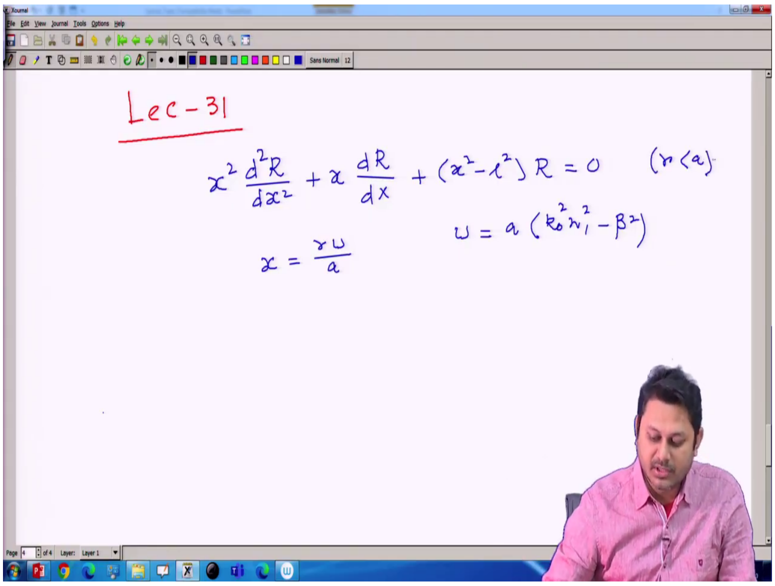

So, in the previous class, so we find that the wave equation can be written in terms of radial differential equation and azimuthal a one differential equation called azimuthal differential equation. And if I write radial equation, then it leads to a Bessel's kind of differential equation in this form that we derive.

So, let me write down once again the form of the equation, that was the equation and the variable x, if I go back to the class notes of the previous lecture, then here I defined x as r u divided by a.

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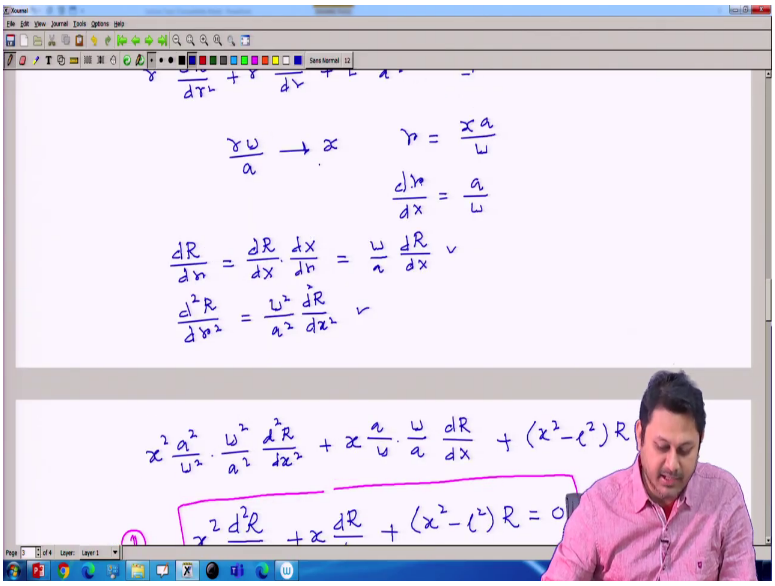

So, my x was r u divided by a and you should note that u was something like a k 0 square n 1 square minus beta square. So, this equation is for valid for when r is less than a. So, we are dealing with the region in the core region. So, this is this differential equation is valid in the core region when r is less than a.

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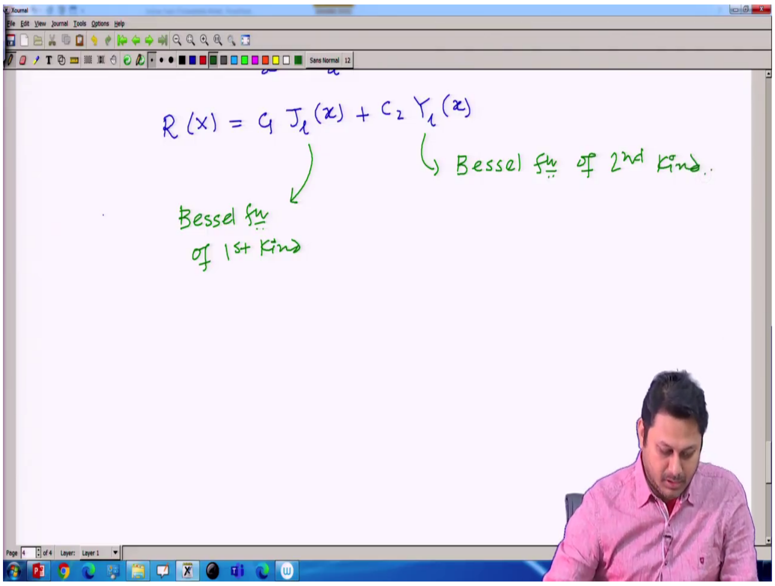

So, that the n was replaced by n 1. Well, we more straight forward solution for this kind of differential equation and if I write this solution, it should be something like some constant say C 1, then J 1 x plus C 2 Y 1 x. This J 1 is called the Bessel function of 1st kind and this is called the Bessel function of 2nd kind.

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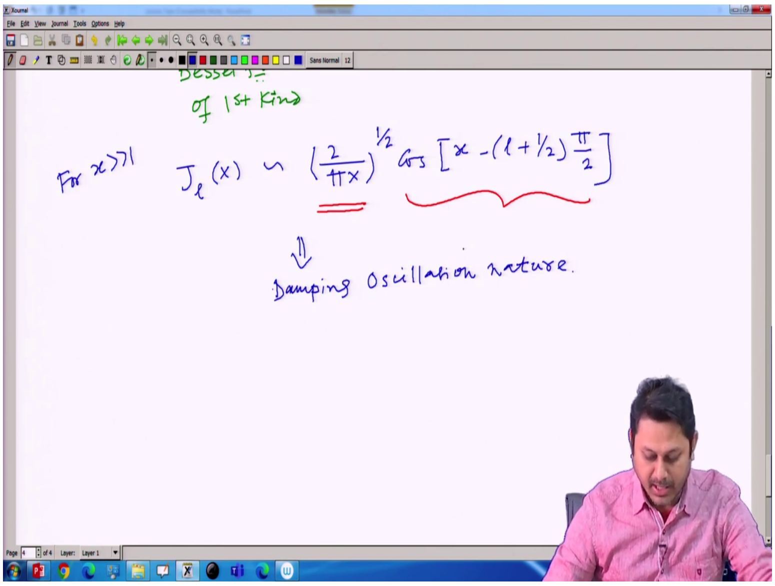

Now, if I quickly try to write J l x, one can approximate for x very very greater than. So, for x very very greater than 1, one can approximate these solutions like this; a approximate close form, I write.

In order to understand that these term is basically something which is a oscillatory term cos 1; this is a oscillatory term. On the other hand, this is a amplitude term which is reducing. So, we have some sort of oscillatory term with a reducing amplitude term. So, these two basically forms as a some sort of damping oscillations.

So, it forms some sort of damping oscillation nature. So, the Bessel's equation of the 1st kind, I find that it is having some kind of damping kind of oscillations and if I now, try to find out what is the form, then one can draw that.

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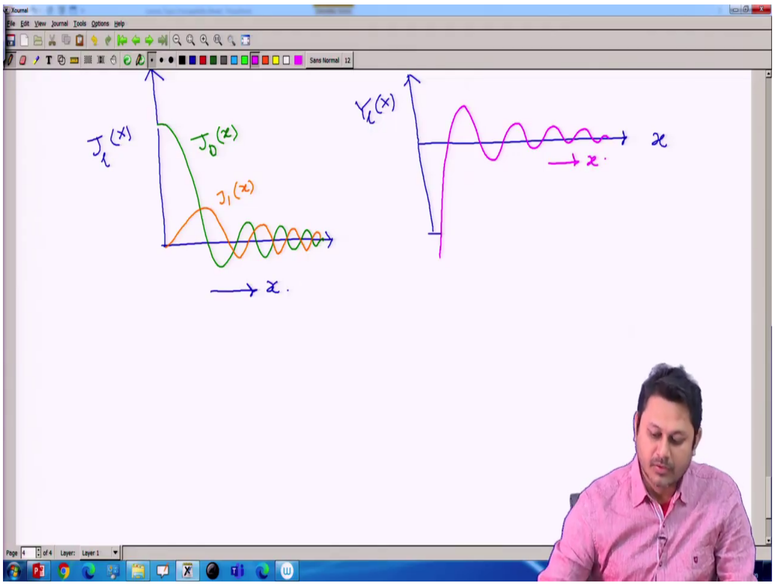

And let me try to draw that approximately. I request you to please find in your in the book or in the internet, how the function will look like. But let me draw to make you a idea that how the function will look like. If I draw that, for different l value, I am having a solution like this. Say this one for J say 1, sorry J 0 x. This curve for J 1 x and so on. In Y axis, I am plotting J l x and along this, I am plotting the variable x.

So, if x varies, then this is the form of the solution one can have. On the other hand, this is the first kind I am drawing. On the other hand, if I draw the Bessel's equation for the second kind, I should erase this because it has a singularity at x equal to 0 minus infinity to I just need to draw this one. Along this, I have x axis and I am plotting say Y l x.

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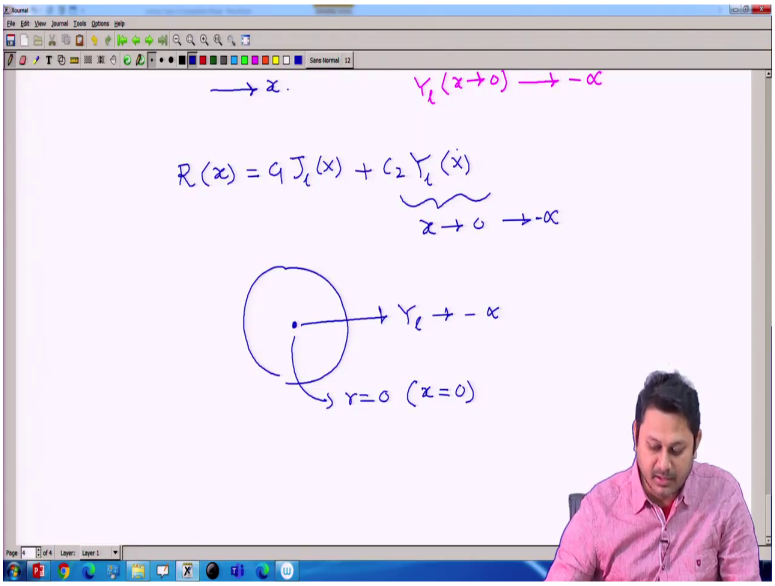

So, this Y l is having in nature like this. So, that means, this function. So, Y l when x tends to 0 goes to minus infinity. So, I have a singularity, I have a singularity or the function is blowing up at x tends to 0, that is a very important information.

In the solution, if you remember I had this one R x is equal to C 1 and then, the Bessel function J l x plus C 2 2nd kind x. But I find that this solution at x tends to 0 goes to infinity, minus of infinity which means if this is a fiber, x equal to 0 means this point.

So, this is r equal to 0 or x equal to 0 because r and x are related in this way only. So, this at this axis point, exactly at the axis of the core, I find the second function is blowing up. It is not blowing up, it is going to minus infinity rather. But I have a something going to minus infinity here. So, Y l tends to minus infinity. So, obviously, this should not be in the solutions because mod cannot go to minus infinity at x equal to 0.

So, that readily rule out the fact that the and the core region, the second kind of Bessel's functions should present.

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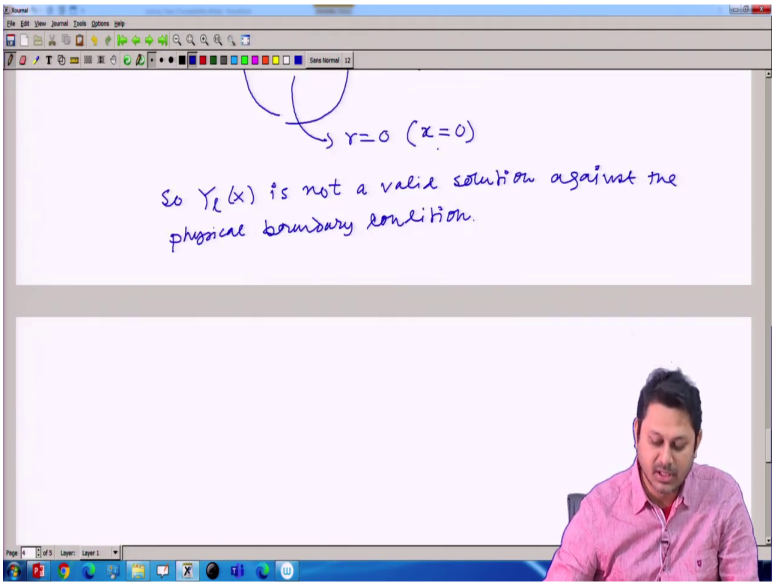

So, I can have only. So, I should write that. So, Y l x is not a valid solution, if I put the boundary conditions. So, this is some sort of boundary condition against the physical boundary condition. So, at x tends to 0; obviously, we should not have something which goes to minus infinity.

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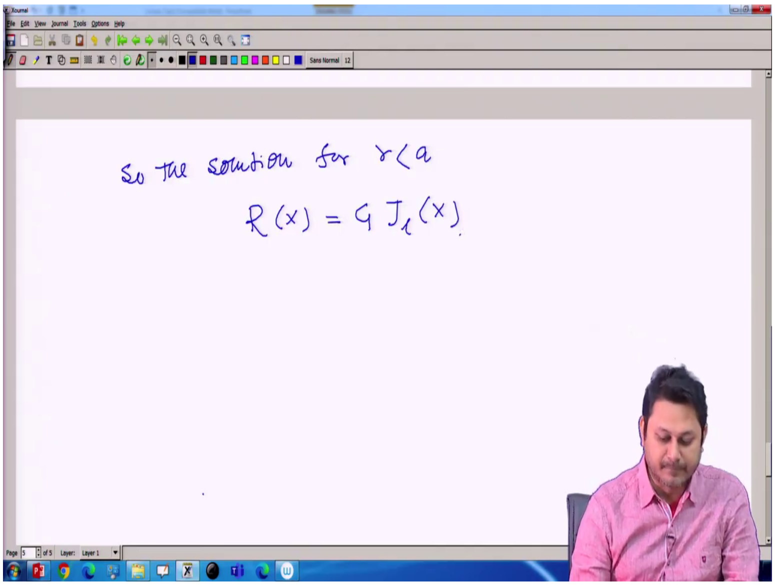

So, we will just eliminate this solution and I will have simply; so, the solution for x solution for in terms of r, r less than a is in terms of x, it is C 1 J l x, this is the solution.

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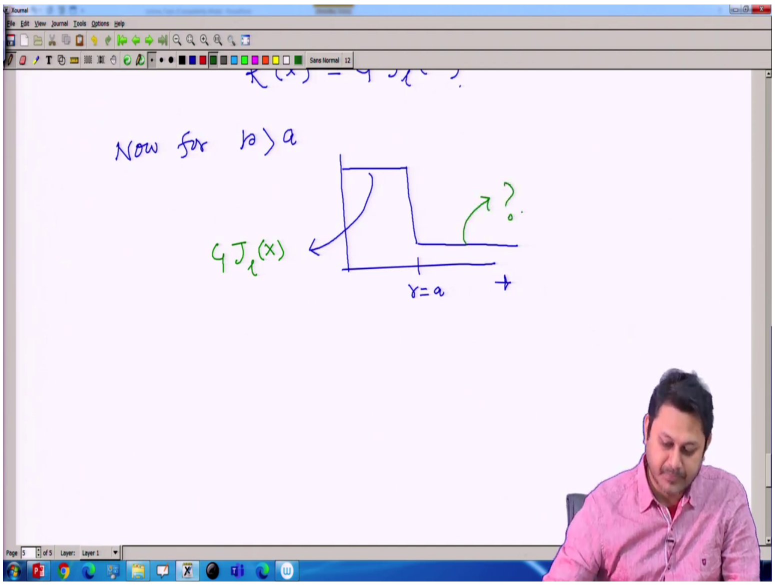

Well, now I need to find out the solution for other part also; r greater than a what happens? So, this is the structure. So, I am having a solution here. So, this is at r equal to a. So, I am having a solution here. The solution is having the form C 1 J l x. Now, what is the solution in this region? I need to find out this solution as well, the cladding region.

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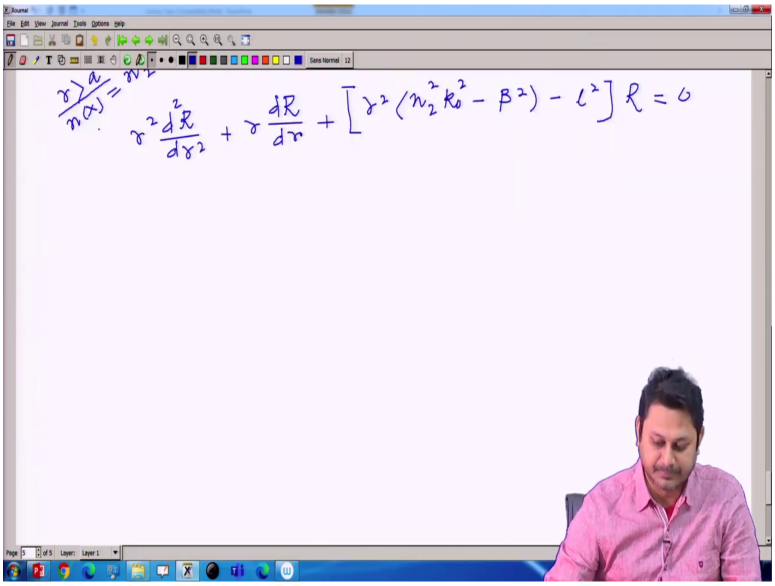

Well, I can also start from the main equation and this main equations, so now, I am dealing with the region when r is greater than a and mind it when r is greater than a, my n x should be equal to n 2. So, I can write this as this way. This is my original radial equations; I am writing down this original radial equation once more. But now, I should write it as n square n 2 square k 0 square minus beta square and then, l square R equal to 0. Previously, it was n 1 square.

Now, I replace this with a n 2 square because I am dealing this at the region when r is greater than a. Well, I should also consider for guided mode, we are having this condition.

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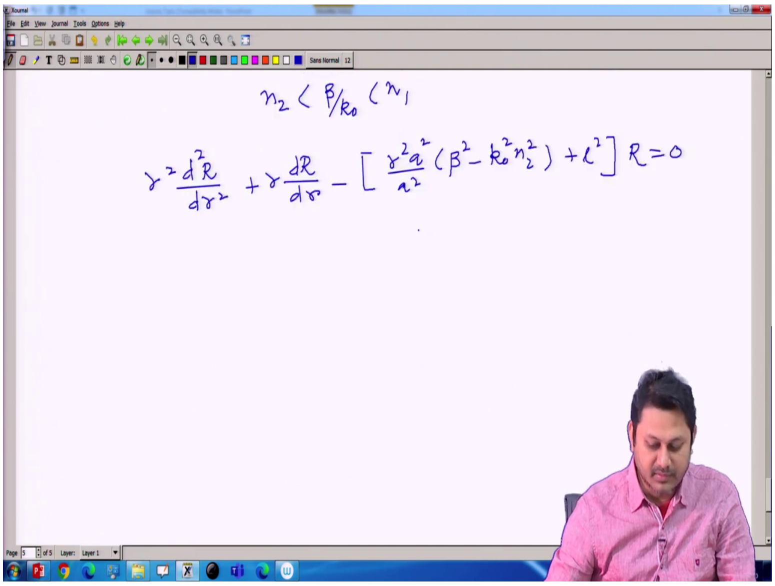

So, make it positive. I need to switch it and after doing after switching this, I can write it as the procedure is exactly same that we have done in the previous case, I am do not writing all the intermediate steps. But it is quite understood. I can make a small manipulation here. It should be r square a square. So, I put r square a square divided by a square, this we have already done before.

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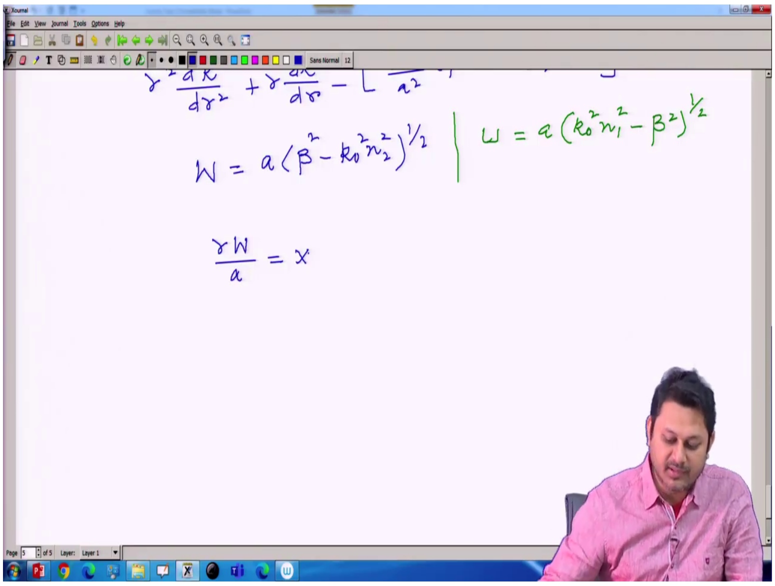

And now, I put a new variable new parameter called W which is a multiplied by beta square minus k 0 square n 2 square whole to the power half exactly like U. If I write U here. So, U was a, but it was k 0 square n 1 square minus beta square whole to the power half. I am having a similar kind of expression, but here it is n 2, related to n 2 and it is beta 2 minus k 0 square n.

So, that this quantity is positive that is all. Next, I will write r a as x and if I do the same procedure that we have done in the previous case, I can now have a differential equation of the form this.

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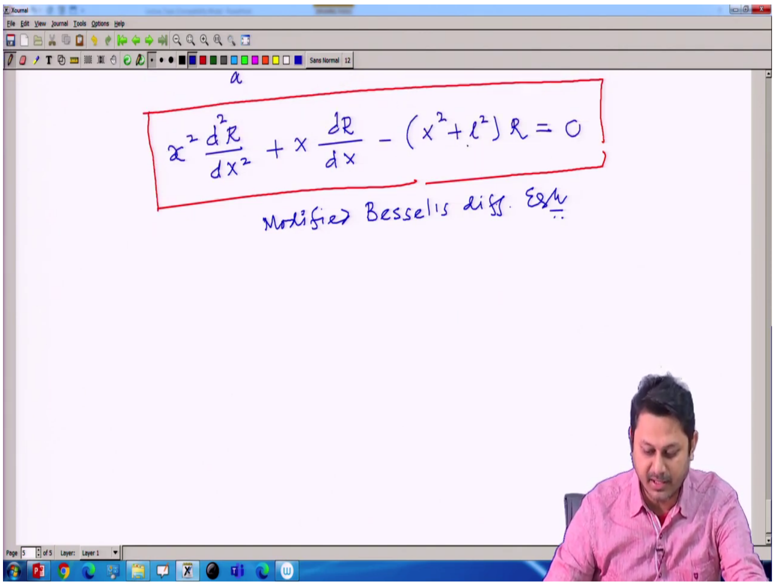

It should be x square; sorry, here I have minus sign and I am having x square plus l square R equal to 0. This is also a to some extent known differential equation and we called it Modified Bessel's differential equation. This is a modified Bessel's differential equation, only thing that is going to change is this negative signed here and this plus sign.

If I go back to the differential equation that we have for Bessel, it was plus sign here and this is minus sign and now, here again, we are having this sign as minus a similar expression only thing is that this sign is minus and here we are a plus.

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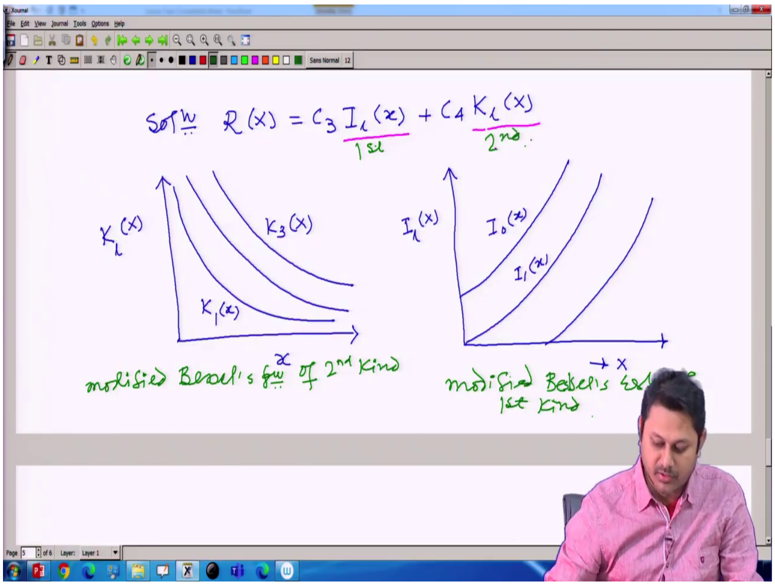

So, for this case also we have certain solutions. So, the solutions is known actually and if I write this solution, it should be say another constant C 3 and it is represented traditional it is represented in this way plus C 4 K of l x. Now, again we can plot and put the boundary condition exactly the procedure that we have done in the previous case and if I now start plotting, so let us check how the function will look like.

So, now, in the first case, I am plotting. By the way, this is called the modified Bessel equation. This is called the modified Bessel's Bessel equation and this is the modified Bessel equation of 1st kind and this is the modified Bessel equation of the 2nd kind. This is the modified Bessel equation of the 1st kind and this is the 2nd kind; modified Bessel equation of the 2nd kind.

So, I am going to plot that because I need to know how the function behaves as a with change x and for K, I am having a family of curves like that for different l values. Say this is for K 1 x, K 2 x, this is for K 3 x and so on. So, this is the way this function is going to behave with respect to x. On the other hand, for I, if I plot I l as a function of x, this is changing x. It will be something like this. Say this is for I 0 x, I 1 x and so on.

So, again, I find something interesting, we find something interesting that for I, when it goes to x tends to infinity, then this function is blowing up. So, it goes to infinity. So, this is for modified Bessel’s function of 1st kind; this is for this is 2nd kind actually and this is the modified Bessel equation of 1st kind.

It seems that I when I write I write, so this is the 2nd kind. So, I have a mistake here. So, this is the 2nd kind and this is the 1st kind. So, the K is seems to be the 2nd; I, this is correct. So, this is 1st. So, I am writing the correct one; 1st kind and 2nd kind. Sorry, it is correct ok.

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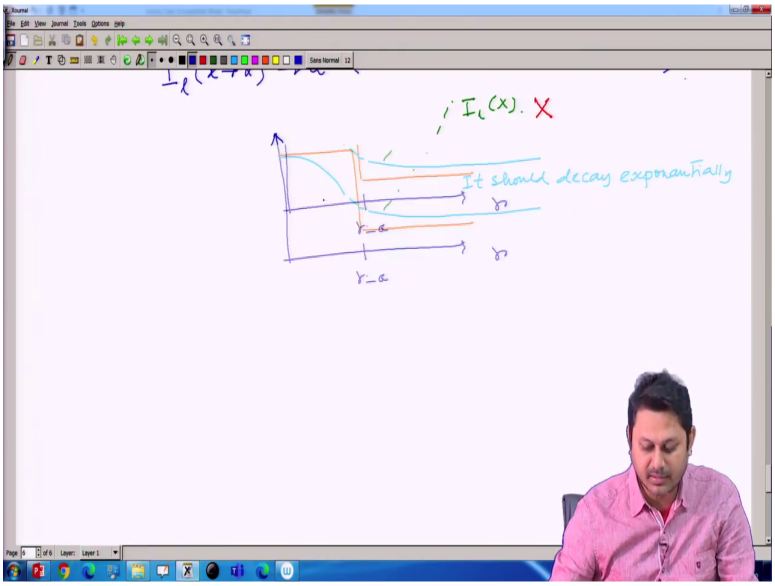

So, if I, so what I find that I l x tends to infinity goes to infinity, positive infinity. So, which is again not acceptable, which is not acceptable, which is not acceptable solution. Why it is that? So, let me try to understand this. So, the fiber structure if I draw quickly, it is that. So, obviously, I should have some kind of sinusoidal solution here, some kind of variation.

But here, after that I will going to expect that it should go, it should decay exponentially. So, it should decay exponentially. But for I l, it is not the case. For I l, it is going like this. So, this is for I l x which is not acceptable. So, which is not acceptable, because I do not want a field to be going to infinity when I go to when r is going to infinity is not an acceptable kind of solution.

This is at r equal to a. So, I am going to expect a sinusoidal solution here in the co region and cladding region, I should have some kind of exponential decay solution. So, the modified

Bessel of the 2nd kind is following these things. So, obviously, I will I l if not a valid solution. So, with the boundary condition I can eliminate that.

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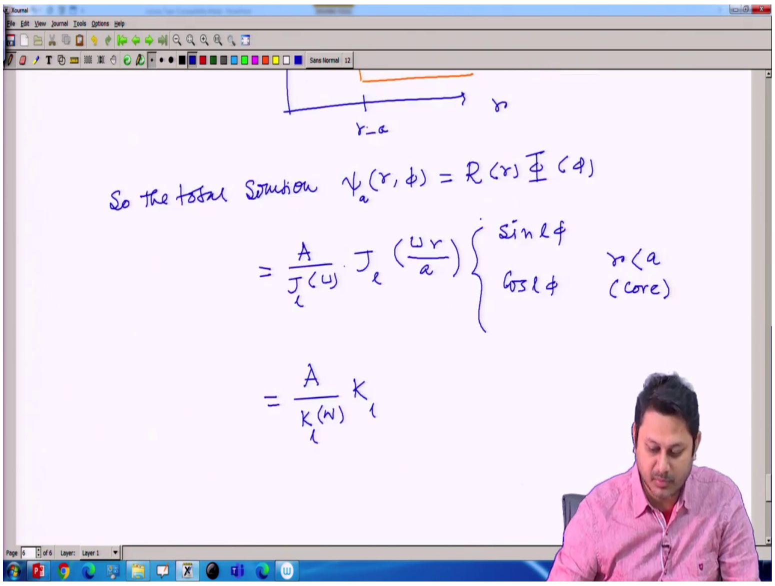

So, my solutions if I now write the solutions, finally so, the total solution if I write which is psi r. So, this is the a, r phi is equal to R r and big phi phi. So, now, all the solutions in my hand for different regions. I can write it as A J of l U multiplied by J of l U r by a and I can have a sinusoidal component here either cos l phi or sin l phi. This is for x or r is than rather r is less than a, the core region.

In the cladding region, I will going to have a solution with the form like this because I terms is the modified Bessel function of the 1st kind is no longer valid here. So, only the 2nd kind is valid.

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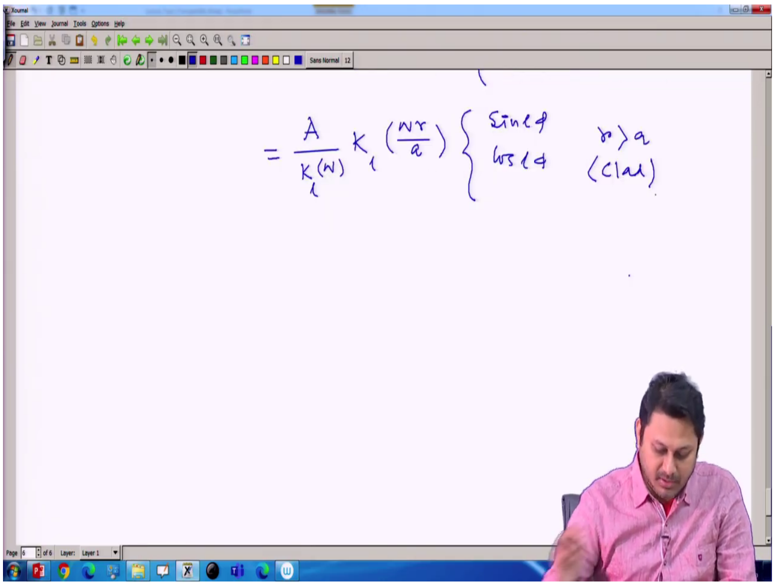

So, I can write it as this. This is for r greater than a which is clad. Why this amplitude is written in this way?

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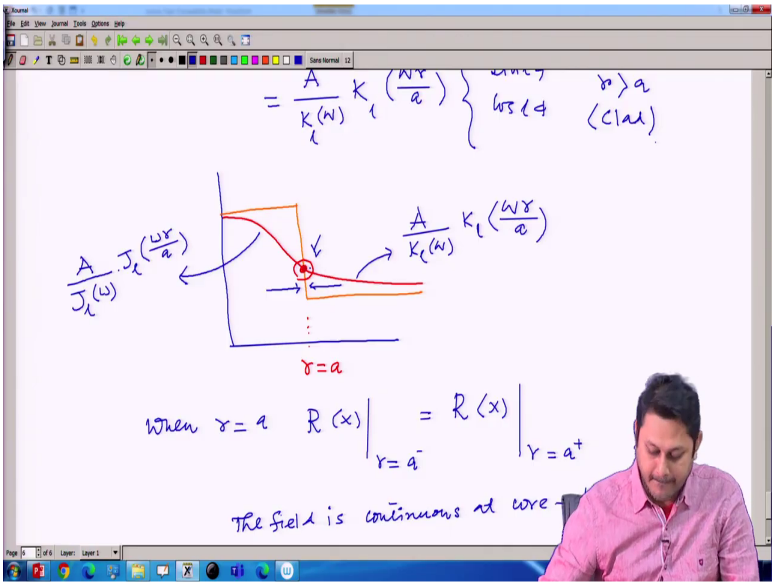

Because of the fact that you can see that this is the structure the fiber and whatever the mode we have here at this point, they should be continuity, which is at r equal to a point, there should be smooth transition of the mode. So, that means, when say r equal to a, so we have R x r tends to say a minus a minus is equal to R x r equal to a plus; a minus a plus means I am approaching from this side and I am approaching from this side.

So, at this point, the radial part there is a continuity and if I have a continuity, so that means, the field is continuous at core-clad boundary.

The field is continuous at core-clad boundary and if you put this here in the solution, you can readily find that when r is equal to a, then this J l U and J l U will going to cancel out and we have a at this point and in the similar way from this side, this equation when r is equal to a this K l W and K l W will cancel out and again, we have A here.

So, in this region, I have a solution which is A divided by J of l U, then J of l U r by a, this is the solution here and this region the solution is A divided by K l W and then K l W r divided by a which is decaying. At this point, at this point both the solutions are both the solutions give in a same value which is a. Well, another boundary condition is also in our hand and that is the derivative.

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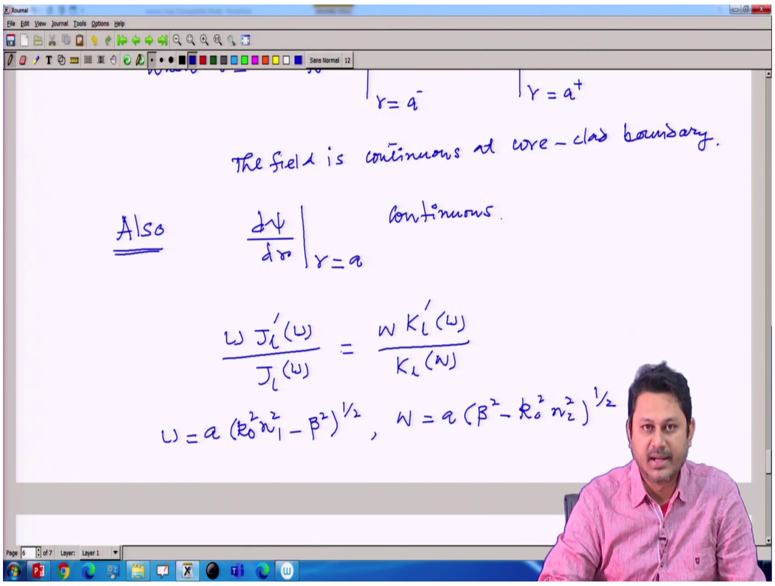

So, also the derivative is continuous. So, this is a very very important boundary condition that d psi if the psi is a field d r at r equal to a is continuous. So, that means, so that means, I am having U J l prime U divided by J l U is equal to W K l prime U divided by K l W.

This prime suggest that it is a derivative with respect to it is derivative with respect to r and if I do that, then I will going to have this equation; where U is a multiplied by k 0 square n 1 square minus beta square whole to the power of and W is a beta square minus k 0 square n 2 square whole to the power half.

Now, if you look very carefully, if you check it carefully, then you will find that this procedure is exactly the same procedure we are using for slab wave guides. This is exactly this procedure wise this is the same. First, we calculate the field at the core region, then we calculate the field at the cladding region, then put the boundary condition and the first boundary condition gives that this is a continuous the field is continuous and second boundary conditions suggest that there is a the derivative is also continuous.

And when you put the second boundary condition, we should have a transcendental equation in our hand. This is basically a transcendental equation we are having because this is a function of U is the function of beta; W is a function of beta. So, both the cases in left hand side and both and right and side, we have a function of beta and we have another function of beta and they these two are same.

And now, that means, we are having some kind of transcendental equation and we need to solve this transcendental equation graphically to find out what is the solution of the beta and that is one of the goal because in the last calculation, when we calculate the modes in planar waveguides, then also we calculated the beta value and we find that there is some discrete values of beta which basically gives the propagation constants of different modes.

Here, will also going to find the same thing; only thing only difference is in that case the solutions looks at least the transcendental equation look little bit easier psi tan psi equal to say V square minus I square whole to the power half. If you remember, but here we are having a similar kind of equation transcendental equation, but it is a little bit complicated because it is related to some Bessel’s functions and also, the derivative of the Bessel functions.

So, in the next class, we will use some kind of recursion relation to make it simple and try to find out how the solution is look like. So, with that note, I will like to conclude.

Thank you for your attention and see you in the next class.