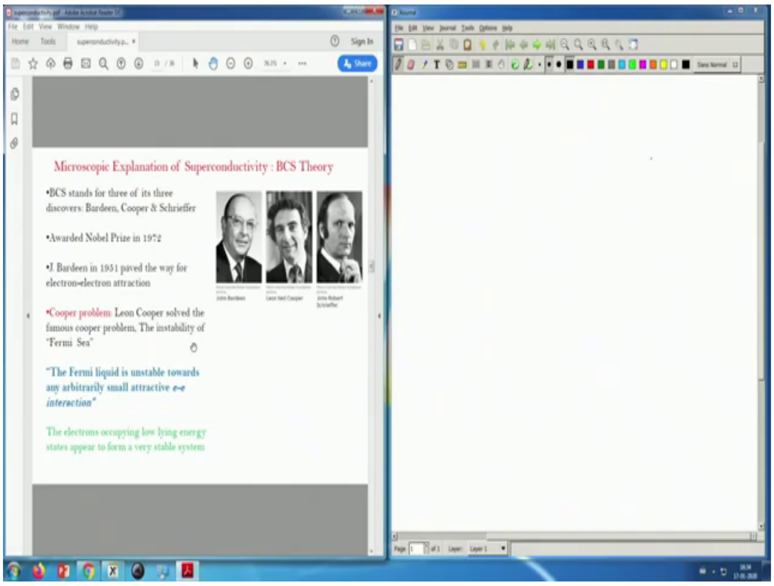
Electronic Theory of Solids
Prof. Arghya Taraphder
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture – 51
Instability of the Fermi Surface

Hello and welcome again. We have been discussing the theory of Superconductivity Microscopic Theory. This was as I said, formulated first by John Bardeen, Leon Cooper and Robert Schrieffer at Urbana Champaign in the US.

They were proposed a complete microscopic theory of superconductivity in a long paper. In physical review and that paper is a seminal paper that is an example of how a thorough calculation and detailed work needs to be done and of that high quality. So, that paper of course, won the Nobel Prize in 1972.

(Refer Slide Time: 01:31)



The whole idea of superconductivity as I mentioned was hinge was the fact that there is an attractive interaction between electrons, they must form a Cooper pair a pair, which later on came to be known as Cooper pair and that pair condenses to become a superconductor.

So, that is the line of thought that finally, culminated in formulating the theory of superconductivity. In this context I must mention that in the USSR, Landau and Ginsburg, were also formulating a theory of phase transition as a whole and superconductivity was one of their goals. They wrote down a phenomenological theory called Ginsberg gundog theory of phase transition close to critical point.

And, they were able to get many things out of that theory, which were experimentally verified in superconductors conductors. But they do not have did not have a microscopic understanding of the origin of this attractive interaction, how two electrons or large number of electrons belonging to a pharmacy interact by this attractive interaction to form a ground state, that is superconducting.

So, later on of course, these two theories were connected from microscopic theory, people who are able to derive the connection to Ginsburg Landau Theory. So, that is an aside Ginsburg Landau Theory will come back to us again, when we discuss electrodynamics of superconductor and the vortex lattice for example, in type two superconductors.

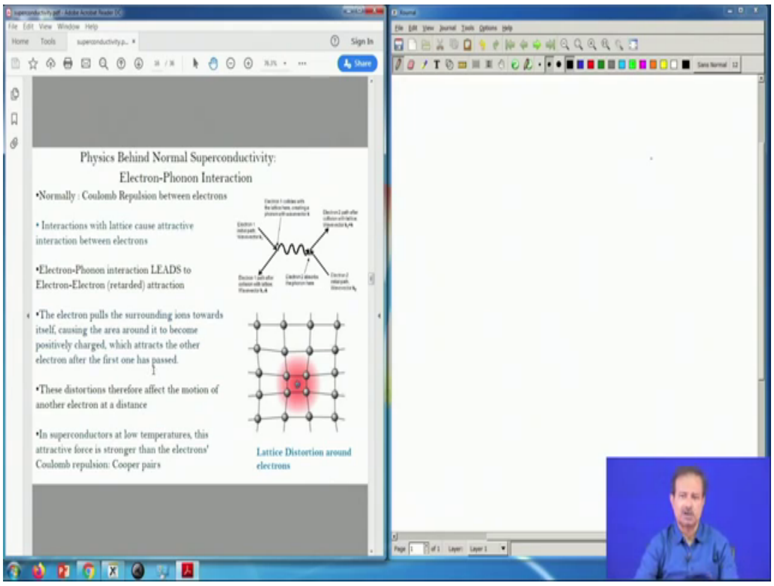
But, for the time being let us discuss the Microscopic Theory, which came to be known as BCS Theory. As a prelude to that there was a famous calculation, which is done by Leon Cooper and you can see that Leon Cooper is part of the team of three, who discovered this theory microscopic theory of superconductivity. And, Leon Cooper solved a problem, which is as I said before just take a pharmacy put two electrons above the Fermi level in. So, they are very pretty close to Fermi level, but just above because up to Fermi level everything is free occupied already nothing is free, no states are free.

So, then these two electrons interact via a retarded attractive interaction. And, that is I discussed the origin of retarded interaction. And, then under these conditions that these two electrons are interacting via this attractive interaction, Cooper was able to show that there is a bound state that forms, whose energy is negative with respect to the Fermi level.

So; that means, these two pairs form and their energy their energy is gained, because their total energy was for example, twice the Fermi level. Because both of them are pretty close to

Fermi level, but there are new energy under this attractive interaction was less than, these 2 EF, and; that means, that there is a gain in energy by exchange of phonons.

(Refer Slide Time: 05:23)



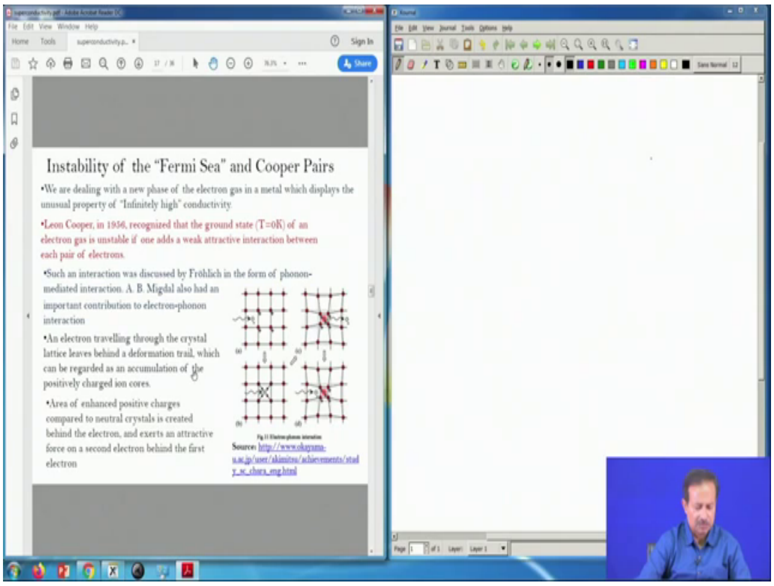
So, this so, this interaction as I as I discussed is via exchange of phonons like this. And, so it stabilizes these two electrons and then the natural conclusion from there on would be that what happens to the Fermi surface? Because, now if every electron finds a partner and pairs up and lowers his energy, then this will be a cascading effect.

And, almost all electrons here below the Fermi surface will try to go above the Fermi surface. And. then so those which are very close to the Fermi level for example, they will first go to the top by slightest excitation to above Fermi level and then use this attractive interaction to form a state, which is lower energy. And, so this will go on this will be just like cascade dominal effect, almost all the electrons will start doing this and; that means, the Fermi surface becomes unstable.

So, this is what the consequence is and this is this would be the many body consequence of such a calculation, which such a proof that Leon Cooper had come up with, but of course, there is a caveat that there is these all these electrons are also interacting with these pairs. So, the question is how to tackle this many body problem?

So, it is not a one step from that Leon's Leon Coopers calculation, which I will outline now it took a while to go from Coopers result to the BCS Theory.

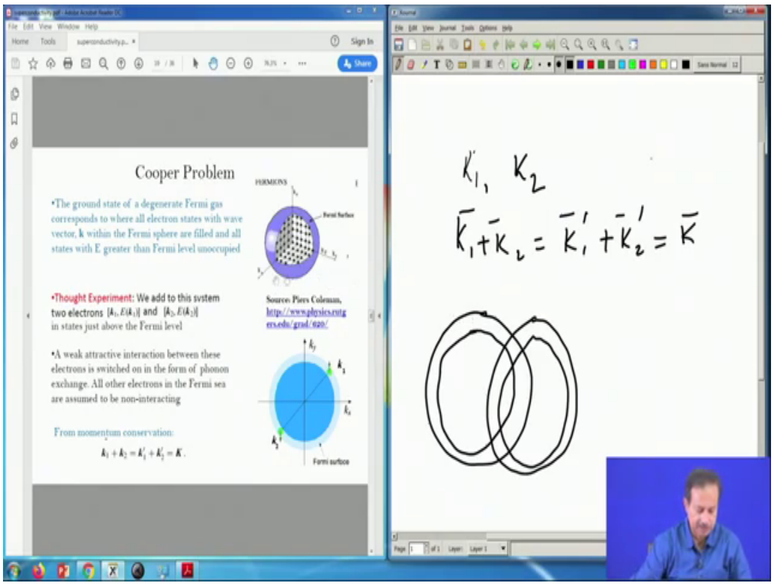
(Refer Slide Time: 07:09)



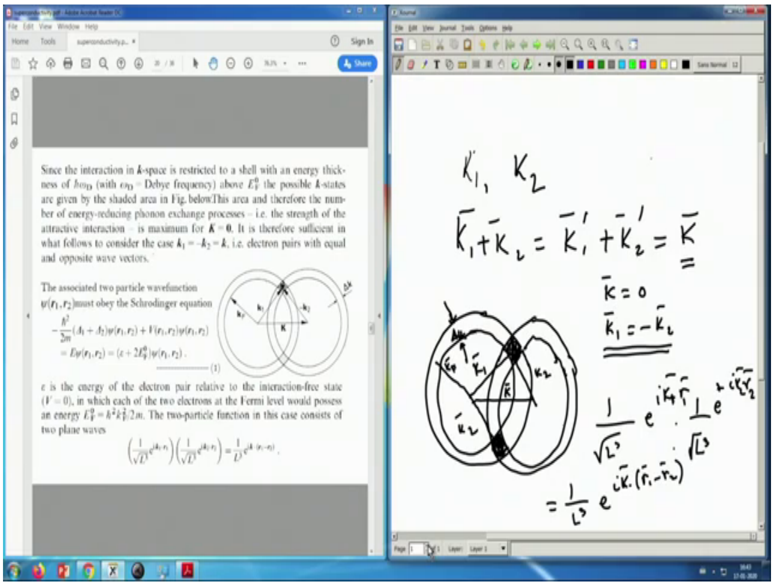
So, this I had discussed how this attractive interaction retarded in time, separated in time, works between two electrons and how screening helps in reducing the repulsive interaction within a short distance?

So, that this attractive interaction is dominant over the length scale and time scale and that we are talking about ok.

(Refer Slide Time: 07:42)



(Refer Slide Time: 07:47)



So, let us now start discussing the Cooper problem, which is what we were discussing and as we said that the two electrons of momentum k 1 and k 2 both just above the Fermi surface, they interact via this in attractive interaction. And, K 1 plus K 2 equal to these scattered states k 1 plus k 2 prime k 1 prime plus k 2 prime the moment of the scattered state equal to K.

So, see any interaction in quantum mechanics can be thought of as a scattering process. So, this is a this is this process you start from k 1 and k 2 and you will end up with k 1 prime and k 2 prime, but you cannot conserve the momentum ok. So, this is what we are going to show how on a Fermi surface, how does it look like?

So, these are the two Fermi spheres we point we plotted and these are the annular over, which in the range h cross omega Debye. So, this delta k is over which the interaction works and the energy range will be h cross omega Debye across the Fermi surface. So, this is delta k sorry about this problem ok. So, and then there was this k 1 and k 2, these are the two moment k 1 and k 2.

So, one can draw a minus k 2 from starting from this center and so this will be the k 1 plus k 2 which is k, which is this one. And, we argued that the gain in energy through this exchange of phonon has to happen in this region where k 1 and k 2, k 1 and k 2 some coincide. So, k 1 minus so, this is these are the region where the two k 1 k 2 spheres, this is this is a circle, but this is in three dimensional sphere.

So, these two circles of k 1 and k 2, they coincide at this these regions of phase space in these regions of the Fermi for the states state space and then of course, this is to be maximized to get the maximum interaction between them. So, that the gain the maximum scattering, there are maximum number of scatterings that reduce the energy as I said.

So, energy reducing exchange processes are maximized, if the joint area the area. The shaded area can be increased as the shaded area increases there will be more and more scattering and so therefore, the energy will be gained. So, this is this is the whole idea ok.

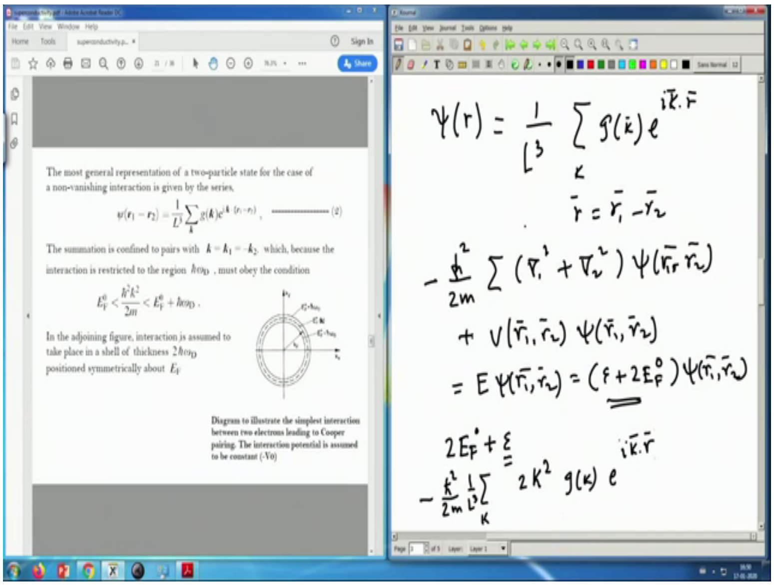
So, and of course, this is your k F. And, when is this shaded area maximum well it is maximum when k is equal to 0. So, k equal to 0 is the maximum. So, k 1 equal to minus k 2; that means, these two spheres or two circles here in this drawing were that is I have to draw two dimensionally, but these are spheres and these two basically coincide.

So, then; that means, this relation has to be satisfied for maximum gain in energy. So, maximum scattering puzzle area a maximum area for scattering ok, in this phase space. Now, that means, that of course, there are other regions also, there are other k values where the scattering will also happen, but what Leon Cooper decided is that we will look at only the maximum scattering regions, where the area is the maximum, the area of intersection is the maximum.

So, that is where he wanted to concentrate. And, because the amplitude is highest there of course, the rest will also add, but this is what one can one considers. And, that is because then they will then life becomes simpler also, because k 1 equal to minus k 2. There is only one momentum you have to take care in the in your theory which is k 1 for example ok.

So, they joined the two particle function in this case consists of a very simple choice. This is normalized in a box 1 by L cube, it give i k dot r and into 1 by 2 L cube e to the power minus i k 1 dot r and it plus i k to dot r ok. So, which is equal to 1 by L cube e to the power i k r 1. So, this is r 1 and this is r 2. So, i k dot r 1 minus r 2 for 2 electrons at r 1 and r 2. So, this is the combo joint wave function that one can write a 2 particle wave function.

(Refer Slide Time: 14:23)



So, let us now use this wave function and calculate the simplest possible case, where we write down the Schrodinger equation for the two under the potential and try to see if there is a bound state. Now, of course, this wave function is too simplistic one can have all possible combinations of such states that will be the most general representation. So, that is what says most general representation of a two particle state for the case of a non-vanishing interaction is given by this series.

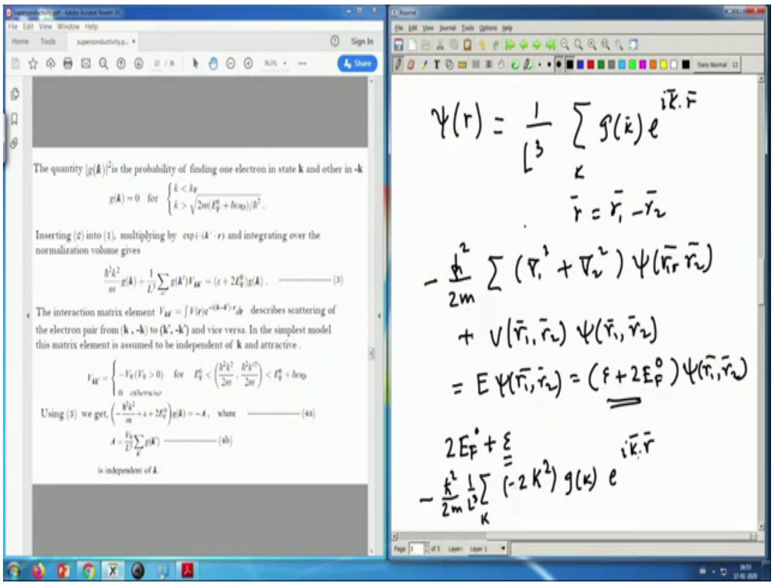
So, one can write down psi of r equal to one by L cube sum over k sum g k weight factor if your i k dot r, where r is equal to r 1 minus r 2 ok.

So, the summation of this is confined to the this k equal to k 1 equal to minus k 2. So, that is that is already there. Now, that restriction means also in terms of energy, that you are confining yourself within a and within an annulus, just as I drew in the other case, in the momentum space. So, this is also in the momentum space, but the corresponding energies should be within E F plus h cross omega d and e f minus h cross omega d in that range.

So, that is what as I mentioned the interaction is retarded; retarded means it is separated in time, which means it acts only on a narrow region in the energy space. And, how much, what is that energy space, well it has to be connected to the phonon energies. And, that is exactly that is the lowest, that is the low energy scale, here for because the Fermi level is of course, a very large scale compared to that h cross omega Debye is much smaller.

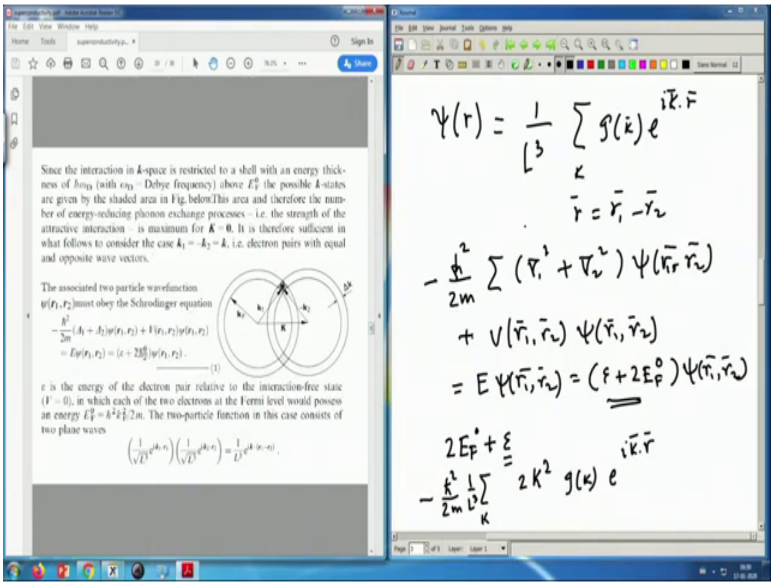
And, that is that because phonon is responsible for causing this interaction attractive interaction, it is natural, it can actually be shown easily that d it is attractive, over a region which is in energy space it is above to h cross omega Debye ok.

(Refer Slide Time: 17:01)



And, g k is the probability of g k square of course, is the probability of finding one electron in state k and other electron in state minus k ok. So, then of course, you also know that g k this k has to be restricted in this region it is a 0, and in this region again it is 0 because it is within that small region.

(Refer Slide Time: 17:40)



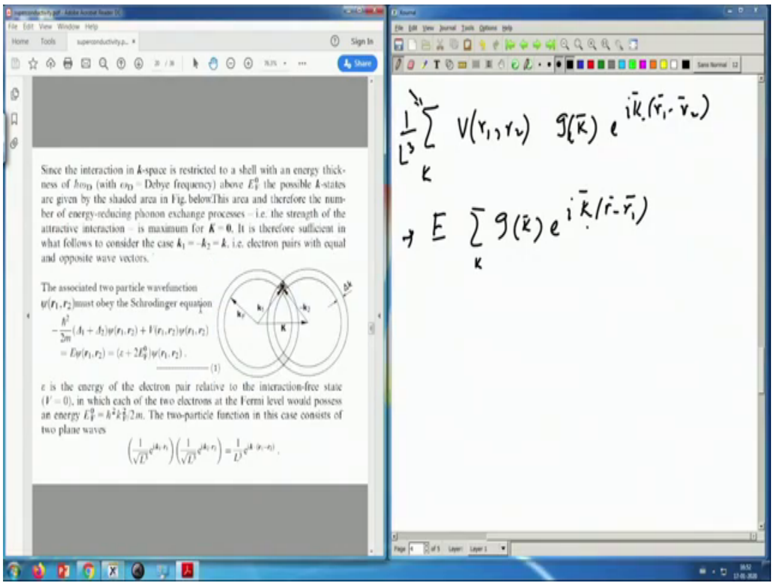
Now, just use the Schrodinger equation which is what is written here? So, it is minus h cross square by so, it is we are doing a 2 particle problem, h cross square by twice m, delta 1 square plus delta 2 square psi r 1 minus r 2 or I can write as they right, you can also write I mean it is of course, a function of r one minus r 2 as we have seen. So, you can also write it as r 1 comma r 2 which is the way it is written here, but anyway.

So, let me just put that r 1 r 1 comma 2 plus this interaction V of r 1 r 2 psi r 1 r 2 equal to sum E psi r 1 r 2, which let us take as epsilon plus 2 E F naught psi r 1 r 2, why does one do that? Ok. So, let us see why one does this, the reason is that suppose the two electrons were exactly at the Fermi level then the total energy would be E 2 F 0. And, now here we have taken them slightly above the Fermi level so; that means, there is an energy very small amount of energy epsilon.

Now, at the end of the day if it turns out that this Epsilon is becoming negative; that means, there is a solution which actually stabilizes it this kind of it two particle state. So, these two particles form a bound state. And, that is why 1 is taking this and the endgame is to show that this epsilon has a negative value. And, that is what we will do and that is what Leon Cooper also did. He showed that under certain conditions the conditions allow outline again this epsilon is indeed possible to be negative ok.

So, let us now do one thing, that we multiply this equation which is this equation by. So, let us just write down for example, first this g k e to the power i k r in this equation. So, this will be minus h cross square by twice m sum over K delta 1 square plus delta 2 square we will operate 1 to the power i k dot r. So, it will give you 2 times h 2 times k square here times g k ok. And, then of course, this E to the power i k dot r 1 by L cube and all that ok. Then there is this V term so, that V term is here this V term V r 1 r 2.

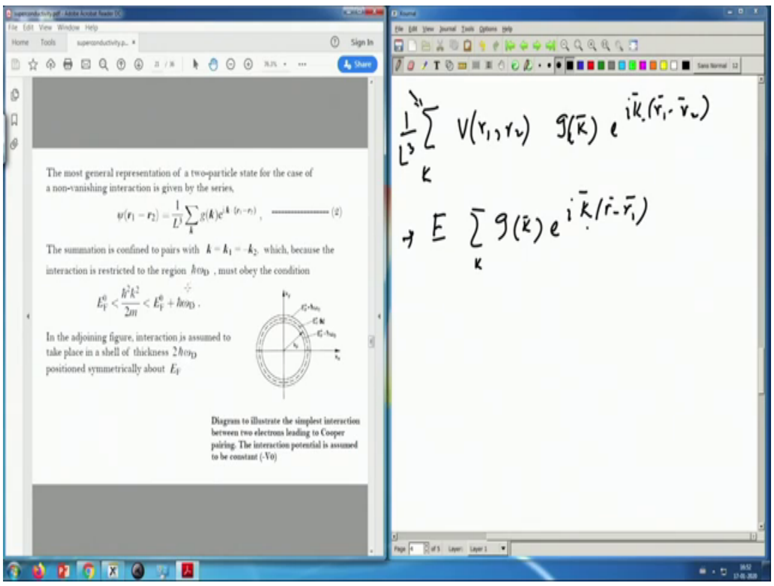
(Refer Slide Time: 21:25)



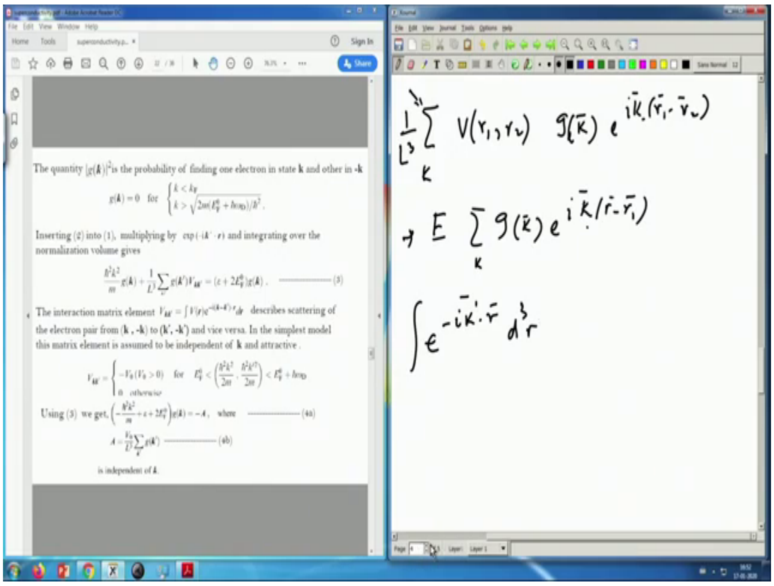
So, let us just put it in V r 1 r 2 sum over k g k e to the power i k r 1 minus r 2 and of course, the k sums are always restricted within that window, but ok. So, that is there and there is 1 by L cube.

So, that is the V term giving and the e term gives me E times sum over k g k e to the power i k r 1 minus r 2. So, these are the two terms in addition to the other one that I have ok.

(Refer Slide Time: 22:24)



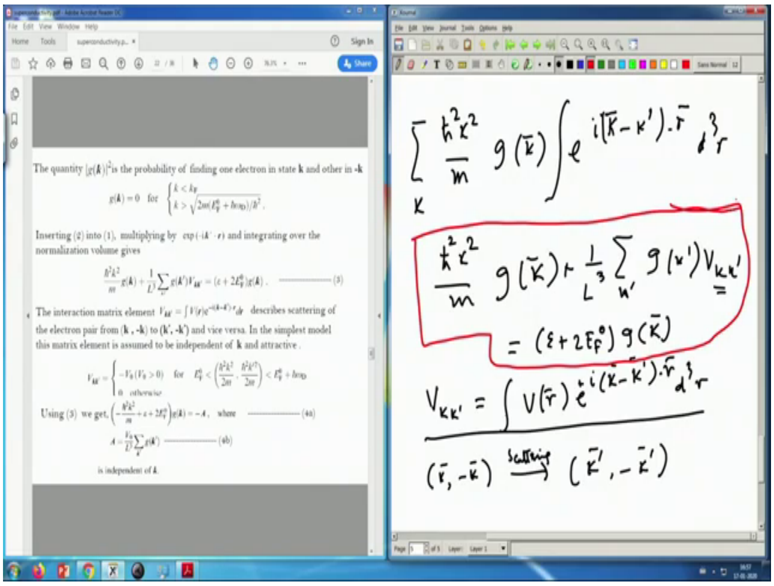
(Refer Slide Time: 22:32)



Now, what one does is that one uses a trick, so that you can get rid of this sum over k and so that is what is written here, if you multiply both sides by e to the power minus i k prime dot r and then integrate over all r ok.

So, let me show you what happens to one of these terms. For example, this one, this one had g k e to the power i so, this had minus h cross square.

(Refer Slide Time: 23:10)



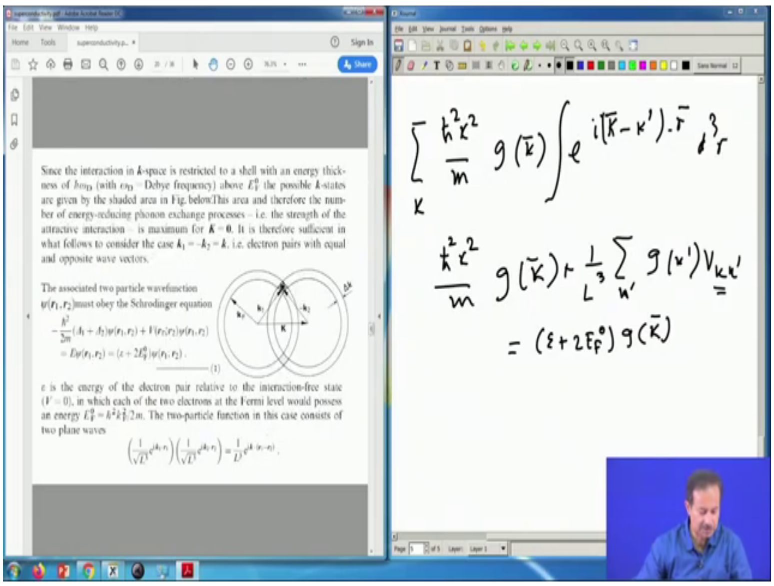
So, this becomes plus because this is a there is a minus that comes in sorry there is minus 2 K square, because e to the power i k del square of i k will give you e to the power i k dot r del square operator will give you minus k square twice, that will give you minus 2 K square.

So, the result is h cross square k square by m sum over k g k e to the power i k minus k prime dot r integral over r ok. So, that is what you will get from the first term. For example, now this integral gives me a delta function and that delta function just reduces one of these integrals this, this whole thing becomes just 1 k the k summation vanishes. So, that is this term h cross square, k square by m of g of k.

So, this is k equal to k prime. So, I can replace it write as k prime here, but since k i mean I can choose that k prime as my k. So, it does not really matter this equation instead of writing for k prime I can write the equation for k. So, that is what I am doing.

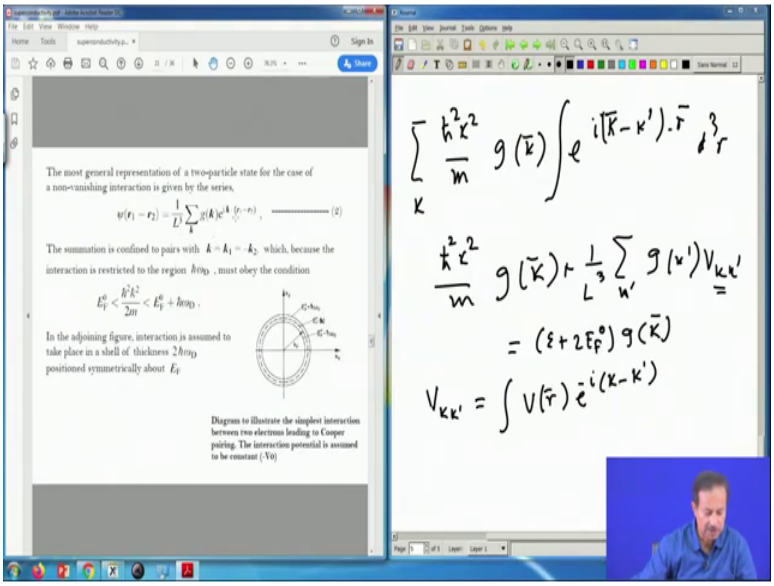
But, the summation vanishes because k is now fixed to k prime. And, that k prime I am calling as k here plus 1 by L cube g of k prime V k prime sum over k prime equal to epsilon plus 2 E F naught g of k ok. So, this V k k prime is basically just there the remember that integral, that we had so, let me just show it here yeah.

(Refer Slide Time: 25:39)



This is the one where we put this sum over k this one.

(Refer Slide Time: 25:46)

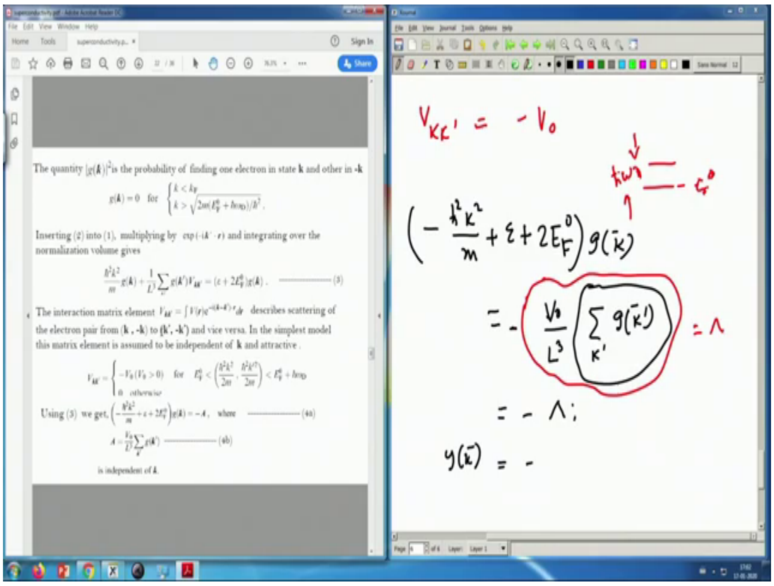


So, then you this is multiplied by V r 1 r 2 and then I multiply again by e to the power minus i k prime r 1 minus r 2 and then sum over integral over r. So, that whole thing is now called g of just put it in and you will see that, this is, this is what g of V of k k prime is V of k k prime is just this integral V of r is r 1 minus r 2 e to the power i k minus k prime.

So, let me just check the signs i k r 1 minus r 2. So, I have plus i k minus k prime dot r d 3 of r. So, this is my V of k k prime. So, this is r independent, r is integrated over it only depends on k and k prime. So, this V k k prime basically describes scattering of the electron pair from k minus k. So, from k minus k state to k prime minus k prime.

So, this scattering is described by the potential ok fine. So, now, now comes this approximation of course, this is a complicated equation there is a sum over k k prime here. So, you cannot solve it so easily. So, this is my equation this I have to solve and then there is a k prime sum. So, each k is connected to all other k each k prime is connected to all other ks. So, I cannot solve it analytically here ok. So, what one does is that one now motivates physically.

(Refer Slide Time: 28:09)



So, that physical motivation is already there we already know that the V k k prime is negative for a region where V 0 is positive. For where these k and k prime are such that E F 0 and E F plus h cross omega Debye within this window the corresponding energy is h cross square k square by twice m and h cross square k prime square by twice r within that region.

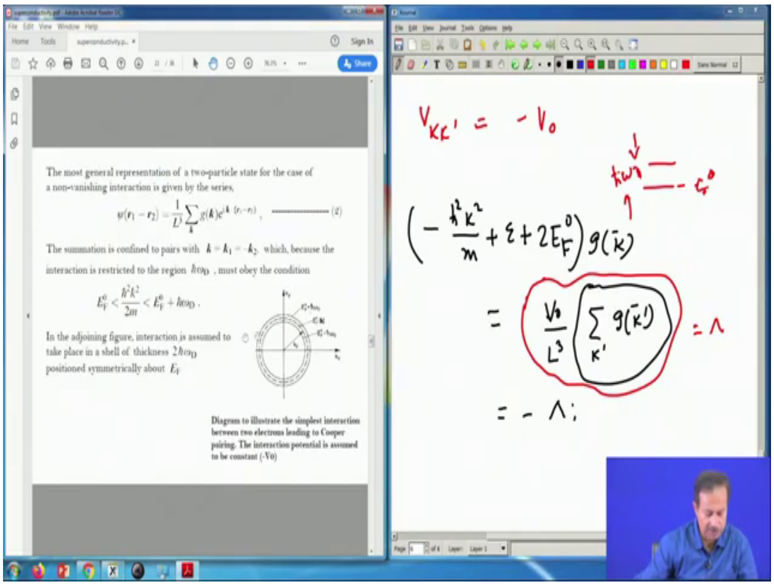
So, they these two energies of the two electrons are just above the Fermi level, but within the range of h cross omega Debye . So, just above Fermi level E F 0 in a region, which is just omega Debye of h cross omega Debye above it. So, the both the energies of these original electrons the electrons that are participating that are scattering are here.

So, with this scattering is basically the two electrons at k minus k going to k prime minus k prime. So, and in all these for all these k and k primes the energies must be within this region and that is what this condition here tells you ok. So, if you do that then of course, your life becomes much simpler because now V k k prime is no longer dependent on k k prime.

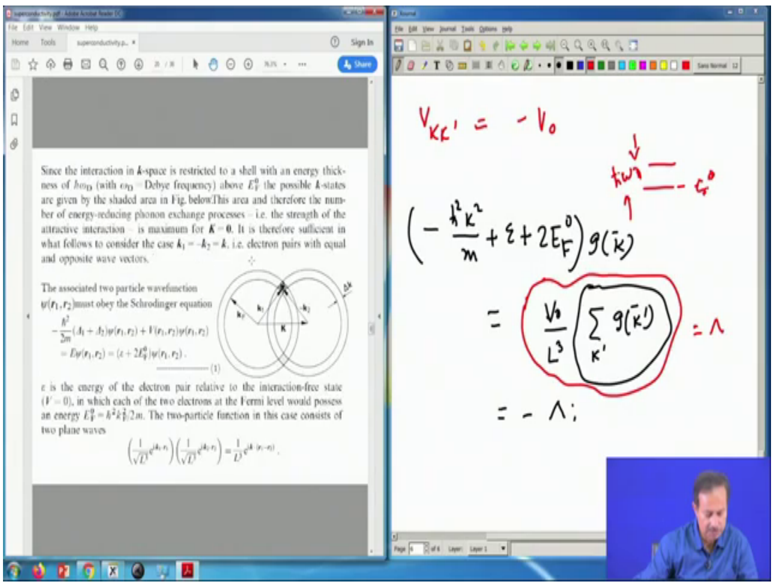
So, immediately a huge simplification occurs what you get is minus h cross square k square by m plus epsilon plus 2 E F 0 is into g of k is equal to V 0 by L cube sum over k prime g k prime ok. So, this really is an enormous simplification, because look at this right hand side this object here is now k independent, because all k's are summed here ok. So, this is just a constant, I can just write it as sum minus lambda it is just a number ok. So, this is just a k independent object is just a number sitting here.

On the left hand side of course, I still have g of k. So, the lambda is equal lambda is this entire object, this entire object is called lambda ok. See, there is a minus V 0. So, I have to be careful. So, this was g k k prime.

(Refer Slide Time: 31:36)



(Refer Slide Time: 31:37)

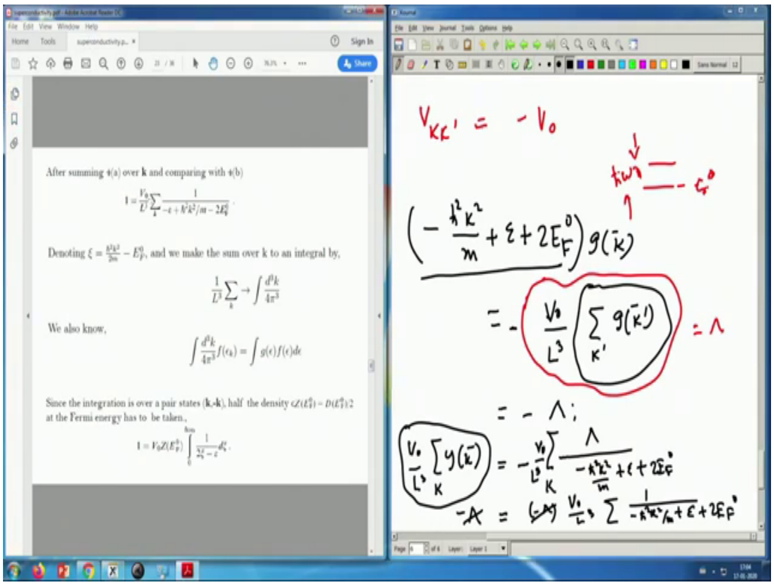


Let me just see how it, whether there is a minus sign epsilon is V k k prime ok. So, this is h cross square k square by m this thing equal to V k k prime that is correct.

Student: (Refer Time: 31:57).

So, this was also minus yeah sign here ok. So, my remember h cross square k square by m has been brought to the right. And, so this is ok. So, that is that is perfectly all right. So, in this condition the this whole thing in the left hand side is just now a lambda.

(Refer Slide Time: 32:26)

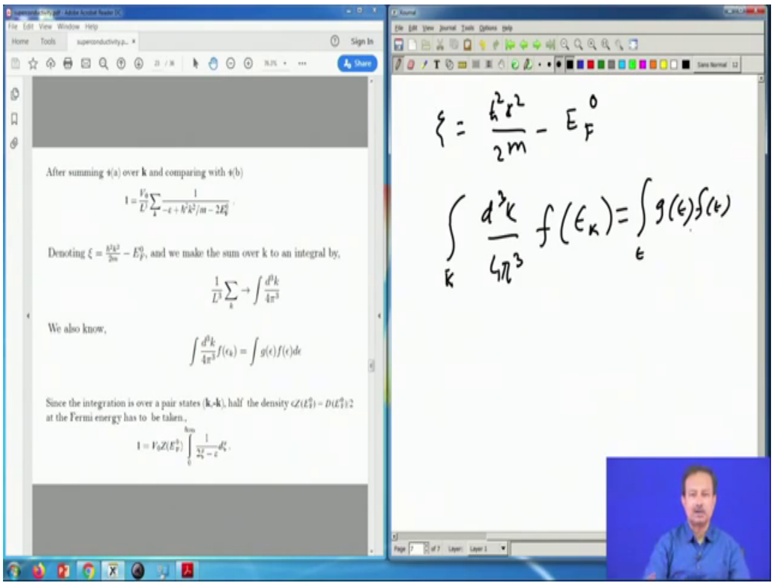


So, let us just go one more step that is independent of k. Now, what one can do is that one just writes g of k equal to ok. So, let me just check the once more, how they defined it? Ok. Ok. Fine g of k is then what I do is that I multiply it by V 0 by lambda cube and sum over k. And, this is equal to lambda by sum over k lambda by h cross square k square by m plus epsilon plus 2 E F 0 ok.

So, all I have done is that I have first written g k equal to minus lambda divided by this thing and then I summed over this both sides by multiplying by V 0 by lambda L cube ok. Now, you can see that this thing is again minus lambda. So, this is becoming lambda minus lambda and that cancels with this minus lambda here.

And, so, this left hand side is my minus lambda something the right hand side is V 0 by L cube sum over all this 1 by h minus h cross square k square by m plus c plus this epsilon plus 2 E F 0 and then there was this minus lambda. So, these two cancel I get the equation that is written here. So, that is that is this equation ok. And, this is the equation we have to solve.

(Refer Slide Time: 34:50)



Now, this we redefine just a definition one uses a definition as xi equal to h cross square k square by m minus E F 0 ok. So, that is one is defining with the 2 here ok. So, that is that 2 is so this is like a single particle energy minus the Fermi level. So, original electronic energy measured from the Fermi level. So, that is what this is?

So, do you define this and then your equation becomes then you use the standard the standard relations that sum over k can be converted to an integral over k, because there is a large number of k's as we did in the free electron theories. And, then of course, there is this other relation that we use that, any integral like this of a function, which is a function of epsilon k only can be converted to a density of states integral over energy. This is an integral over k whereas; this integral is our energy. So, this is doable.

So, with this we will now start solving this Cooper problem and see what happens.