

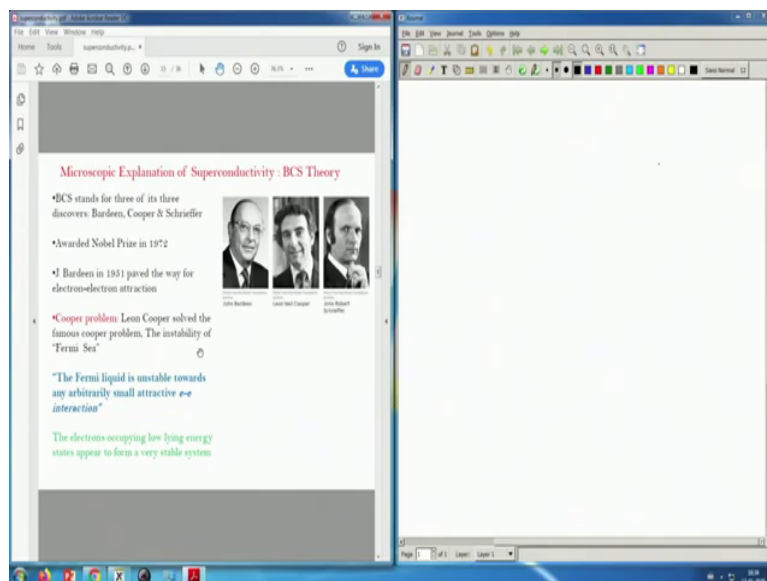
Electronic Theory of Solids
Prof. Arghya Taraphder
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture – 51
Instability of the Fermi Surface

Hello and welcome again. We have been discussing the theory of Superconductivity Microscopic Theory. This was as I said, formulated first by John Bardeen, Leon Cooper and Robert Schrieffer at Urbana Champaign in the US.

They were proposed a complete microscopic theory of superconductivity in a long paper. In physical review and that paper is a seminal paper that is an example of how a thorough calculation and detailed work needs to be done and of that high quality. So, that paper of course, won the Nobel Prize in 1972.

(Refer Slide Time: 01:31)



The whole idea of superconductivity as I mentioned was hinge was the fact that there is an attractive interaction between electrons, they must form a Cooper pair a pair, which later on came to be known as Cooper pair and that pair condenses to become a superconductor.

So, that is the line of thought that finally, culminated in formulating the theory of superconductivity. In this context I must mention that in the USSR, Landau and Ginsburg, were also formulating a theory of phase transition as a whole and superconductivity was one of their goals. They wrote down a phenomenological theory called Ginsburg Landau theory of phase transition close to critical point.

And, they were able to get many things out of that theory, which were experimentally verified in superconductors. But they do not have a microscopic understanding of the origin of this attractive interaction, how two electrons or large number of electrons belonging to a Fermi sea interact by this attractive interaction to form a ground state, that is superconducting.

So, later on of course, these two theories were connected from microscopic theory, people who are able to derive the connection to Ginsburg Landau Theory. So, that is an aside Ginsburg Landau Theory will come back to us again, when we discuss electrodynamics of superconductor and the vortex lattice for example, in type two superconductors.

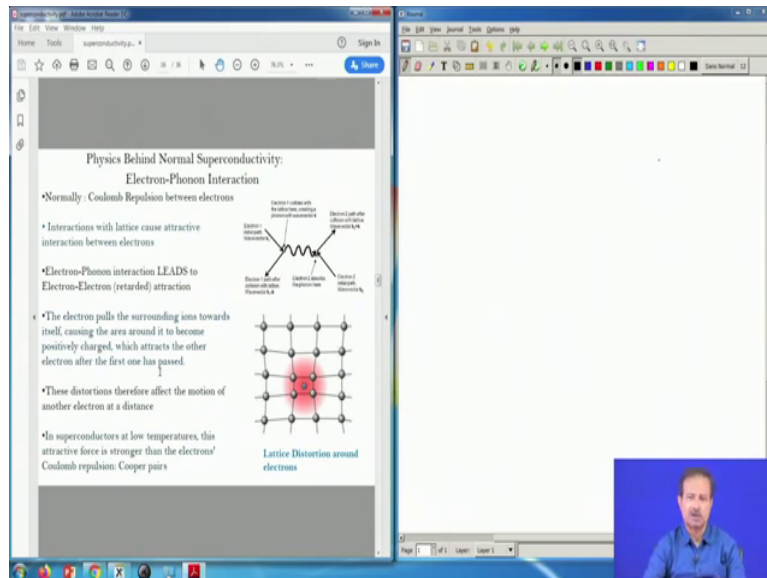
But, for the time being let us discuss the Microscopic Theory, which came to be known as BCS Theory. As a prelude to that there was a famous calculation, which is done by Leon Cooper and you can see that Leon Cooper is part of the team of three, who discovered this theory microscopic theory of superconductivity. And, Leon Cooper solved a problem, which is as I said before just take a Fermi sea put two electrons above the Fermi level in. So, they are very pretty close to Fermi level, but just above because up to Fermi level everything is free occupied already nothing is free, no states are free.

So, then these two electrons interact via a retarded attractive interaction. And, that is I discussed the origin of retarded interaction. And, then under these conditions that these two electrons are interacting via this attractive interaction, Cooper was able to show that there is a bound state that forms, whose energy is negative with respect to the Fermi level.

So; that means, these two pairs form and their energy their energy is gained, because their total energy was for example, twice the Fermi level. Because both of them are pretty close to

Fermi level, but there are new energy under this attractive interaction was less than, these 2 EF, and; that means, that there is a gain in energy by exchange of phonons.

(Refer Slide Time: 05:23)



So, this so, this interaction as I as I discussed is via exchange of phonons like this. And, so it stabilizes these two electrons and then the natural conclusion from there on would be that what happens to the Fermi surface? Because, now if every electron finds a partner and pairs up and lowers his energy, then this will be a cascading effect.

And, almost all electrons here below the Fermi surface will try to go above the Fermi surface. And, then so those which are very close to the Fermi level for example, they will first go to the top by slightest excitation to above Fermi level and then use this attractive interaction to form a state, which is lower energy. And, so this will go on this will be just like cascade dominal effect, almost all the electrons will start doing this and; that means, the Fermi surface becomes unstable.

So, this is what the consequence is and this is this would be the many body consequence of such a calculation, which such a proof that Leon Cooper had come up with, but of course, there is a caveat that there is these all these electrons are also interacting with these pairs. So, the question is how to tackle this many body problem?

So, it is not a one step from that Leon's Leon Coopers calculation, which I will outline now it took a while to go from Coopers result to the BCS Theory.

(Refer Slide Time: 07:09)

The image shows a screenshot of a presentation slide titled "Instability of the Fermi Sea and Cooper Pairs". The slide contains the following text:

- We are dealing with a new phase of the electron gas in a metal which displays the unusual property of "Infinitely high" conductivity.
- Leon Cooper, in 1956, recognized that the ground state ($T=0K$) of an electron gas is unstable if one adds a weak attractive interaction between each pair of electrons.
- Such an interaction was discussed by Fröhlich in the form of phonon-mediated interaction. A. B. Migdal also had an important contribution to electron-phonon interaction.
- An electron travelling through the crystal lattice leaves behind a deformation trail, which can be regarded as an accumulation of the positively charged ion cores.
- Area of enhanced positive charges compared to neutral crystals is created behind the electron, and exerts an attractive force on a second electron behind the first electron.

There are two diagrams illustrating the phonon-mediated interaction. The first diagram shows a lattice of ions with an electron moving through it, creating a trail of positive charges. The second diagram shows two electrons interacting with the lattice, with the first electron creating a trail of positive charges that attracts the second electron. A source link is provided at the bottom: http://www.uky.edu/~ljp/chem/lectures/instal/2-cooper_mig.html. The slide is displayed in a browser window, and a small video inset of the presenter is visible in the bottom right corner.

So, this I had discussed how this attractive interaction retarded in time, separated in time, works between two electrons and how screening helps in reducing the repulsive interaction within a short distance?

So, that this attractive interaction is dominant over the length scale and time scale and that we are talking about ok.

(Refer Slide Time: 07:42)

The slide on the left is titled "Cooper Problem" and contains the following text:

- The ground state of a degenerate Fermi gas corresponds to where all electron states with wave vector, \mathbf{k} within the Fermi sphere are filled and all states with E greater than Fermi level unoccupied
- Thought Experiment:** We add to this system two electrons $|\mathbf{k}_1, \uparrow, \downarrow\rangle$ and $|\mathbf{k}_2, \uparrow, \downarrow\rangle$ in states just above the Fermi level
- A weak attractive interaction between these electrons is switched on in the form of phonon exchange. All other electrons in the Fermi sea are assumed to be non-interacting
- From momentum conservation: $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}'_1 + \mathbf{k}'_2 = \mathbf{K}$.

The slide includes a diagram of a Fermi sphere and a diagram of two overlapping Fermi surfaces. The whiteboard on the right has handwritten notes:

k_1, k_2
 $\bar{k}_1 + \bar{k}_2 = \bar{k}'_1 + \bar{k}'_2 = \bar{K}$

Below the equations are two overlapping circles representing Fermi surfaces. A small video inset of a man is visible in the bottom right corner.

(Refer Slide Time: 07:47)

The slide on the left contains the following text:

- Since the interaction in \mathbf{k} -space is restricted to a shell with an energy thickness of $\hbar\omega_D$ (with $\omega_D =$ Debye frequency) above E_F the possible \mathbf{k} -states are given by the shaded area in Fig. below. This area and therefore the number of energy-reducing phonon exchange processes - i.e. the strength of the attractive interaction - is maximum for $\mathbf{K} = 0$. It is therefore sufficient in what follows to consider the case $\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k}$, i.e. electron pairs with equal and opposite wave vectors.
- The associated two particle wavefunction $\psi(\mathbf{r}_1, \mathbf{r}_2)$ must obey the Schrödinger equation

$$-\frac{\hbar^2}{2m} (\nabla_1^2 + \nabla_2^2) \psi(\mathbf{r}_1, \mathbf{r}_2) + V(\mathbf{r}_1, \mathbf{r}_2) \psi(\mathbf{r}_1, \mathbf{r}_2) = E \psi(\mathbf{r}_1, \mathbf{r}_2) \quad (1)$$

E is the energy of the electron pair relative to the interaction-free state ($V = 0$), in which each of the two electrons at the Fermi level would possess an energy $E_F^2 = \hbar^2 k_F^2 / 2m$. The two-particle function in this case consists of two plane waves

$$\left(\frac{1}{\sqrt{2}} e^{i\mathbf{k}_1 \cdot \mathbf{r}_1} \right) \left(\frac{1}{\sqrt{2}} e^{i\mathbf{k}_2 \cdot \mathbf{r}_2} \right) = \frac{1}{2} e^{i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)}$$

The whiteboard on the right has handwritten notes:

k_1, k_2
 $\bar{k}_1 + \bar{k}_2 = \bar{k}'_1 + \bar{k}'_2 = \bar{K}$
 $\bar{K} = 0$
 $\bar{k}_1 = -\bar{k}_2$

Below the equations are two overlapping circles representing Fermi surfaces. A small video inset of a man is visible in the bottom right corner.

So, let us now start discussing the Cooper problem, which is what we were discussing and as we said that the two electrons of momentum k_1 and k_2 both just above the Fermi surface, they interact via this in attractive interaction. And, $k_1 + k_2$ equal to these scattered states $k_1 + k_2$ prime k_1 prime plus k_2 prime the moment of the scattered state equal to K .

So, see any interaction in quantum mechanics can be thought of as a scattering process. So, this is a this is this process you start from k_1 and k_2 and you will end up with k_1' and k_2' , but you cannot conserve the momentum ok. So, this is what we are going to show how on a Fermi surface, how does it look like?

So, these are the two Fermi spheres we point we plotted and these are the annular over, which in the range $\hbar \text{ cross } \omega_{\text{Debye}}$. So, this Δk is over which the interaction works and the energy range will be $\hbar \text{ cross } \omega_{\text{Debye}}$ across the Fermi surface. So, this is Δk sorry about this problem ok. So, and then there was this k_1 and k_2 , these are the two moment k_1 and k_2 .

So, one can draw a minus k_2 from starting from this center and so this will be the k_1 plus k_2 which is k , which is this one. And, we argued that the gain in energy through this exchange of phonon has to happen in this region where k_1 and k_2 , k_1 and k_2 some coincide. So, k_1 minus so, this is these are the region where the two k_1 k_2 spheres, this is this is a circle, but this is in three dimensional sphere.

So, these two circles of k_1 and k_2 , they coincide at this these regions of phase space in these regions of the Fermi for the states state space and then of course, this is to be maximized to get the maximum interaction between them. So, that the gain the maximum scattering, there are maximum number of scatterings that reduce the energy as I said.

So, energy reducing exchange processes are maximized, if the joint area the area. The shaded area can be increased as the shaded area increases there will be more and more scattering and so therefore, the energy will be gained. So, this is this is the whole idea ok.

So, and of course, this is your k_F . And, when is this shaded area maximum well it is maximum when k is equal to 0. So, k equal to 0 is the maximum. So, k_1 equal to minus k_2 ; that means, these two spheres or two circles here in this drawing were that is I have to draw two dimensionally, but these are spheres and these two basically coincide.

So, then; that means, this relation has to be satisfied for maximum gain in energy. So, maximum scattering puzzle area a maximum area for scattering ok, in this phase space. Now, that means, that of course, there are other regions also, there are other k values where the

scattering will also happen, but what Leon Cooper decided is that we will look at only the maximum scattering regions, where the area is the maximum, the area of intersection is the maximum.

So, that is where he wanted to concentrate. And, because the amplitude is highest there of course, the rest will also add, but this is what one can consider. And, that is because then they will then life becomes simpler also, because k_1 equal to minus k_2 . There is only one momentum you have to take care in the in your theory which is k_1 for example ok.

So, they joined the two particle function in this case consists of a very simple choice. This is normalized in a box 1 by L cube, it give $i \mathbf{k} \cdot \mathbf{r}$ and into 1 by $2 L$ cube e to the power minus $i \mathbf{k}_1 \cdot \mathbf{r}_1$ and it plus $i \mathbf{k}_2 \cdot \mathbf{r}_2$ ok. So, which is equal to 1 by L cube e to the power $i \mathbf{k} \cdot \mathbf{r}$. So, this is \mathbf{r}_1 and this is \mathbf{r}_2 . So, $i \mathbf{k} \cdot \mathbf{r}_1$ minus $i \mathbf{k} \cdot \mathbf{r}_2$ for 2 electrons at \mathbf{r}_1 and \mathbf{r}_2 . So, this is the combo joint wave function that one can write a 2 particle wave function.

(Refer Slide Time: 14:23)

The slide on the left contains the following text:

The most general representation of a two-particle state for the case of a non-vanishing interaction is given by the series,

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{L^3} \sum_{\mathbf{k}} g(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)} \quad (1)$$

The summation is confined to pairs with $\mathbf{k} = \mathbf{k}_1 = -\mathbf{k}_2$, which, because the interaction is restricted to the region $|\mathbf{r}_1 - \mathbf{r}_2| < \hbar v_F$, must obey the condition

$$E_F < \frac{\hbar^2 k^2}{2m} < E_F + \hbar \omega_D.$$

In the adjoining figure, interaction is assumed to take place in a shell of thickness $2\hbar\omega_D$ positioned symmetrically about E_F

Diagram to illustrate the simplest interaction between two electrons leading to Cooper pairing. The interaction potential is assumed to be constant ($-V$)

The whiteboard on the right shows the following equations:

$$\Psi(\mathbf{r}) = \frac{1}{L^3} \sum_{\mathbf{k}} g(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}$$

$$\mathbf{r} = \mathbf{r}_1, -\mathbf{r}_2$$

$$-\frac{\hbar^2}{2m} \sum (\nabla_1^2 + \nabla_2^2) \Psi(\mathbf{r}_1, \mathbf{r}_2) + V(\mathbf{r}_1, \mathbf{r}_2) \Psi(\mathbf{r}_1, \mathbf{r}_2) = E \Psi(\mathbf{r}_1, \mathbf{r}_2) = (E + 2E_F^0) \Psi(\mathbf{r}_1, \mathbf{r}_2)$$

$$2E_F^0 + E = \sum_{\mathbf{k}} \frac{\hbar^2 k^2}{2m} g(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}$$

So, let us now use this wave function and calculate the simplest possible case, where we write down the Schrodinger equation for the two under the potential and try to see if there is a bound state. Now, of course, this wave function is too simplistic one can have all possible combinations of such states that will be the most general representation. So, that is what says

most general representation of a two particle state for the case of a non-vanishing interaction is given by this series.

So, one can write down $\psi(r)$ equal to one by L^3 sum over k sum g_k weight factor if your $i k \cdot r$, where r is equal to r_1 minus r_2 ok.

So, the summation of this is confined to the this k equal to k_1 equal to minus k_2 . So, that is that is already there. Now, that restriction means also in terms of energy, that you are confining yourself within a and within an annulus, just as I drew in the other case, in the momentum space. So, this is also in the momentum space, but the corresponding energies should be within E_F plus $\hbar \omega_D$ and E_F minus $\hbar \omega_D$ in that range.

So, that is what as I mentioned the interaction is retarded; retarded means it is separated in time, which means it acts only on a narrow region in the energy space. And, how much, what is that energy space, well it has to be connected to the phonon energies. And, that is exactly that is the lowest, that is the low energy scale, here for because the Fermi level is of course, a very large scale compared to that $\hbar \omega_D$ Debye is much smaller.

And, that is that because phonon is responsible for causing this interaction attractive interaction, it is natural, it can actually be shown easily that d it is attractive, over a region which is in energy space it is above to $\hbar \omega_D$ Debye ok.

(Refer Slide Time: 17:01)

The quantity $|g(\mathbf{k})|^2$ is the probability of finding one electron in state \mathbf{k} and other in $-\mathbf{k}$

$$g(\mathbf{k}) = 0 \quad \text{for} \quad \begin{cases} k < k_F \\ k > \sqrt{2m(E_F^0 + \hbar\omega_0)/\hbar^2} \end{cases}$$

Inserting (2) into (1), multiplying by $\exp(-i\mathbf{k} \cdot \mathbf{r})$ and integrating over the normalization volume gives

$$\frac{\hbar^2 k^2}{m} g(\mathbf{k}) + \frac{1}{V} \sum_{\mathbf{k}'} g(\mathbf{k}') V_{\mathbf{k}\mathbf{k}'} = (e + 2E_F^0) g(\mathbf{k}) \quad (3)$$

The interaction matrix element $V_{\mathbf{k}\mathbf{k}'} = \int V(\mathbf{r}) e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{r}} d\mathbf{r}$ describes scattering of the electron pair from $(\mathbf{k}, -\mathbf{k})$ to $(\mathbf{k}', -\mathbf{k}')$ and vice versa. In the simplest model this matrix element is assumed to be independent of \mathbf{k} and attractive.

$$V_{\mathbf{k}\mathbf{k}'} = \begin{cases} -V_0 (V_0 > 0) & \text{for } E_{\mathbf{k}} < \left(\frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 k'^2}{2m} \right) < E_{\mathbf{k}'} + \hbar\omega_0 \\ 0 & \text{otherwise} \end{cases}$$

Using (3) we get, $\left(\frac{\hbar^2 k^2}{m} - e + 2E_F^0 \right) g(\mathbf{k}) = -A$, where

$$A = \frac{V_0}{V} \sum_{\mathbf{k}'} g(\mathbf{k}') \quad (4)$$

is independent of \mathbf{k} .

Handwritten on whiteboard:

$$\Psi(\mathbf{r}) = \frac{1}{L^3} \sum_{\mathbf{k}} g(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}$$

$$\bar{\mathbf{r}} = \bar{\mathbf{r}}_1 - \bar{\mathbf{r}}_2$$

$$-\frac{\hbar^2}{2m} \sum (\nabla_1^2 + \nabla_2^2) \Psi(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2) + V(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2) \Psi(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2) = E \Psi(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2) = (E + 2E_F^0) \Psi(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2)$$

$$2E_F^0 + E = -\frac{\hbar^2}{2m} \frac{1}{L^3} \sum_{\mathbf{k}} (-2k^2) g(\mathbf{k}) e^{i\mathbf{k} \cdot \bar{\mathbf{r}}}$$

And, $g(\mathbf{k})$ is the probability of $g(\mathbf{k})$ square of course, is the probability of finding one electron in state \mathbf{k} and other electron in state $-\mathbf{k}$ ok. So, then of course, you also know that $g(\mathbf{k})$ this has to be restricted in this region it is a 0, and in this region again it is 0 because it is within that small region.

(Refer Slide Time: 17:40)

Since the interaction in \mathbf{k} -space is restricted to a shell with an energy thickness of $\hbar\omega_0$ (with $\omega_0 = \text{Debye frequency}$) above E_F^0 the possible \mathbf{k} -states are given by the shaded area in Fig. below. This area and therefore the number of energy-reducing phonon exchange processes - i.e. the strength of the attractive interaction - is maximum for $\mathbf{K} = 0$. It is therefore sufficient in what follows to consider the case $\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k}$, i.e. electron pairs with equal and opposite wave vectors.

The associated two particle wavefunction $\psi(\mathbf{r}_1, \mathbf{r}_2)$ must obey the Schrodinger equation

$$-\frac{\hbar^2}{2m} (\Delta_1 + \Delta_2) \psi(\mathbf{r}_1, \mathbf{r}_2) + V(\mathbf{r}_1, \mathbf{r}_2) \psi(\mathbf{r}_1, \mathbf{r}_2) = E \psi(\mathbf{r}_1, \mathbf{r}_2) = (e + 2E_F^0) \psi(\mathbf{r}_1, \mathbf{r}_2) \quad (1)$$

e is the energy of the electron pair relative to the interaction-free state ($V = 0$), in which each of the two electrons at the Fermi level would possess an energy $E_F^0 = \hbar^2 k_F^2 / 2m$. The two-particle function in this case consists of two plane waves

$$\left(\frac{1}{\sqrt{V}} e^{i\mathbf{k}_1 \cdot \mathbf{r}_1} \right) \left(\frac{1}{\sqrt{V}} e^{i\mathbf{k}_2 \cdot \mathbf{r}_2} \right) = \frac{1}{V} e^{i(\mathbf{k}_1 \cdot \mathbf{r}_1 + \mathbf{k}_2 \cdot \mathbf{r}_2)}$$

Handwritten on whiteboard:

$$\Psi(\mathbf{r}) = \frac{1}{L^3} \sum_{\mathbf{k}} g(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}$$

$$\bar{\mathbf{r}} = \bar{\mathbf{r}}_1 - \bar{\mathbf{r}}_2$$

$$-\frac{\hbar^2}{2m} \sum (\nabla_1^2 + \nabla_2^2) \Psi(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2) + V(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2) \Psi(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2) = E \Psi(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2) = (E + 2E_F^0) \Psi(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2)$$

$$2E_F^0 + E = -\frac{\hbar^2}{2m} \frac{1}{L^3} \sum_{\mathbf{k}} 2k^2 g(\mathbf{k}) e^{i\mathbf{k} \cdot \bar{\mathbf{r}}}$$

Now, just use the Schrodinger equation which is what is written here? So, it is minus \hbar^2 cross square by so, it is we are doing a 2 particle problem, \hbar^2 cross square by twice m , $\Delta^2 \psi(r_1, r_2)$ or I can write as they right, you can also write I mean it is of course, a function of r_1 minus r_2 as we have seen. So, you can also write it as r_1, r_2 which is the way it is written here, but anyway.

So, let me just put that $\Delta^2 \psi(r_1, r_2) + V(r_1, r_2) \psi(r_1, r_2) = (E - E_F) \psi(r_1, r_2)$, which let us take as $\epsilon + E_F \psi(r_1, r_2)$, why does one do that? Ok. So, let us see why one does this, the reason is that suppose the two electrons were exactly at the Fermi level then the total energy would be E_F . And, now here we have taken them slightly above the Fermi level so; that means, there is an energy very small amount of energy ϵ .

Now, at the end of the day if it turns out that this ϵ is becoming negative; that means, there is a solution which actually stabilizes it this kind of it two particle state. So, these two particles form a bound state. And, that is why I am taking this and the endgame is to show that this ϵ has a negative value. And, that is what we will do and that is what Leon Cooper also did. He showed that under certain conditions the conditions allow outline again this ϵ is indeed possible to be negative ok.

So, let us now do one thing, that we multiply this equation which is this equation by. So, let us just write down for example, first this $e^{i(k \cdot r)}$ in this equation. So, this will be minus \hbar^2 cross square by twice m sum over K $\Delta^2 \psi(r_1, r_2)$ we will operate $e^{i(k \cdot r)}$. So, it will give you $2 \times \hbar^2 k^2$ here times $g(k)$ ok. And, then of course, this E to the power $i(k \cdot r)$ by L^3 and all that ok. Then there is this V term so, that V term is here this V term $V(r_1, r_2)$.

(Refer Slide Time: 21:25)

Since the interaction in k -space is restricted to a shell with an energy thickness of $\hbar\omega_D$ (with ω_D = Debye frequency) above E_F the possible k -states are given by the shaded area in Fig. below. This area and therefore the number of energy-reducing phonon exchange processes - i.e. the strength of the attractive interaction - is maximum for $\mathbf{K} = 0$. It is therefore sufficient in what follows to consider the case $\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k}$, i.e. electron pairs with equal and opposite wave vectors.

The associated two-particle wavefunction $\psi(\mathbf{r}_1, \mathbf{r}_2)$ must obey the Schrödinger equation

$$-\frac{\hbar^2}{2m} (\nabla_1^2 + \nabla_2^2) \psi(\mathbf{r}_1, \mathbf{r}_2) + V(\mathbf{r}_1, \mathbf{r}_2) \psi(\mathbf{r}_1, \mathbf{r}_2) = E \psi(\mathbf{r}_1, \mathbf{r}_2) \quad (1)$$

ϵ is the energy of the electron pair relative to the interaction-free state ($V = 0$), in which each of the two electrons at the Fermi level would possess an energy $E_F^2 = \hbar^2 k_F^2 / 2m$. The two-particle function in this case consists of two plane waves

$$\left(\frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\mathbf{r}_1} \right) \left(\frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\mathbf{r}_2} \right) = \frac{1}{V} e^{i\mathbf{k}\cdot(\mathbf{r}_1 + \mathbf{r}_2)}$$

$\frac{1}{L^3} \sum_{\mathbf{k}} V(\mathbf{r}_1, \mathbf{r}_2) g(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}$
 $\rightarrow E \sum_{\mathbf{k}} g(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}$

So, let us just put it in $V \int \int d\mathbf{r}_1 d\mathbf{r}_2 \sum_{\mathbf{k}} g(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}$ and of course, the \mathbf{k} sums are always restricted within that window, but ok. So, that is there and there is 1 by L^3 .

So, that is the V term giving and the e term gives me E times sum over $\mathbf{k} g(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}$. So, these are the two terms in addition to the other one that I have ok.

(Refer Slide Time: 22:24)

The most general representation of a two-particle state for the case of a non-vanishing interaction is given by the series,

$$\psi(\mathbf{r}_1 - \mathbf{r}_2) = \frac{1}{V} \sum_{\mathbf{k}} g(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} \quad (2)$$

The summation is confined to pairs with $\mathbf{k}_1 = -\mathbf{k}_2$, which, because the interaction is restricted to the region $\hbar\omega_D$, must obey the condition

$$E_F < \frac{\hbar^2 k^2}{2m} < E_F + \hbar\omega_D$$

In the adjoining figure, interaction is assumed to take place in a shell of thickness $2\hbar\omega_D$ positioned symmetrically about E_F

Diagram to illustrate the simplest interaction between two electrons leading to Cooper pairing. The interaction potential is assumed to be constant (V_0)

$\frac{1}{L^3} \sum_{\mathbf{k}} V(\mathbf{r}_1, \mathbf{r}_2) g(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}$
 $\rightarrow E \sum_{\mathbf{k}} g(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}$

(Refer Slide Time: 22:32)

The quantity $|g(\vec{k})|^2$ is the probability of finding one electron in state \vec{k} and other in $-\vec{k}$

$$g(\vec{k}) = 0 \text{ for } \begin{cases} k < k_F \\ k > \sqrt{2m(E_F^0 + \hbar\omega_0)} \end{cases}$$

Inserting (2) into (1), multiplying by $\exp(i\vec{k}' \cdot \vec{r})$ and integrating over the normalization volume gives

$$\frac{\hbar^2 k^2}{m} g(\vec{k}) + \frac{1}{L^3} \sum_{\vec{k}'} g(\vec{k}') V_{\vec{k}\vec{k}'} = (\varepsilon + 2E_F^0) g(\vec{k}) \quad (3)$$

The interaction matrix element $V_{\vec{k}\vec{k}'} = \int V(\vec{r}) e^{i(\vec{k}-\vec{k}') \cdot \vec{r}} d^3r$ describes scattering of the electron pair from $(\vec{k}, -\vec{k})$ to $(\vec{k}', -\vec{k}')$ and vice versa. In the simplest model this matrix element is assumed to be independent of \vec{k} and attractive.

$$V_{\vec{k}\vec{k}'} = \begin{cases} -V_0 (V_0 > 0) & \text{for } E_F^0 < \left(\frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 k'^2}{2m} \right) < E_F^0 + \hbar\omega_0 \\ 0 & \text{otherwise} \end{cases}$$

Using (3) we get $\left(\frac{\hbar^2 k^2}{m} - \varepsilon + 2E_F^0 \right) g(\vec{k}) = -A$, where

$$A = \frac{\hbar^2}{L^3} \sum_{\vec{k}'} g(\vec{k}') \quad (4)$$

is independent of \vec{k} .

Handwritten note:

$$\frac{1}{L^3} \sum_{\vec{k}} V(\vec{r}_1, \vec{r}_2) g(\vec{k}) e^{i\vec{k} \cdot (\vec{r}_1 - \vec{r}_2)}$$

$$\rightarrow \varepsilon \sum_{\vec{k}} g(\vec{k}) e^{i\vec{k} \cdot (\vec{r}_1 - \vec{r}_2)}$$

$$\int e^{-i\vec{k}' \cdot \vec{r}} d^3r$$

Now, what one does is that one uses a trick, so that you can get rid of this sum over \vec{k} and so that is what is written here, if you multiply both sides by $e^{-i\vec{k}' \cdot \vec{r}}$ and then integrate over all \vec{r} ok.

So, let me show you what happens to one of these terms. For example, this one, this one had $g(\vec{k})$ to the power i so, this had minus $\hbar^2 k^2$.

(Refer Slide Time: 23:10)

The quantity $|g(\vec{k})|^2$ is the probability of finding one electron in state \vec{k} and other in $-\vec{k}$

$$g(\vec{k}) = 0 \text{ for } \begin{cases} k < k_F \\ k > \sqrt{2m(E_F^0 + \hbar\omega_0)} \end{cases}$$

Inserting (2) into (1), multiplying by $\exp(i\vec{k}' \cdot \vec{r})$ and integrating over the normalization volume gives

$$\frac{\hbar^2 k^2}{m} g(\vec{k}) + \frac{1}{L^3} \sum_{\vec{k}'} g(\vec{k}') V_{\vec{k}\vec{k}'} = (\varepsilon + 2E_F^0) g(\vec{k}) \quad (3)$$

The interaction matrix element $V_{\vec{k}\vec{k}'} = \int V(\vec{r}) e^{i(\vec{k}-\vec{k}') \cdot \vec{r}} d^3r$ describes scattering of the electron pair from $(\vec{k}, -\vec{k})$ to $(\vec{k}', -\vec{k}')$ and vice versa. In the simplest model this matrix element is assumed to be independent of \vec{k} and attractive.

$$V_{\vec{k}\vec{k}'} = \begin{cases} -V_0 (V_0 > 0) & \text{for } E_F^0 < \left(\frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 k'^2}{2m} \right) < E_F^0 + \hbar\omega_0 \\ 0 & \text{otherwise} \end{cases}$$

Using (3) we get $\left(\frac{\hbar^2 k^2}{m} - \varepsilon + 2E_F^0 \right) g(\vec{k}) = -A$, where

$$A = \frac{\hbar^2}{L^3} \sum_{\vec{k}'} g(\vec{k}') \quad (4)$$

is independent of \vec{k} .

Handwritten note:

$$\sum_{\vec{k}} \frac{\hbar^2 k^2}{m} g(\vec{k}) \int e^{i(\vec{k}-\vec{k}') \cdot \vec{r}} d^3r$$

$$\frac{\hbar^2 k^2}{m} g(\vec{k}) + \frac{1}{L^3} \sum_{\vec{k}'} g(\vec{k}') V_{\vec{k}\vec{k}'} = (\varepsilon + 2E_F^0) g(\vec{k})$$

$$V_{\vec{k}\vec{k}'} = \int V(\vec{r}) e^{i(\vec{k}-\vec{k}') \cdot \vec{r}} d^3r$$

$$(\vec{r}, -\vec{k}) \xrightarrow{\text{scattering}} (\vec{r}', -\vec{k}')$$

So, this becomes plus because this is a there is a minus that comes in sorry there is minus 2 K square, because e to the power i k del square of i k will give you e to the power i k dot r del square operator will give you minus k square twice, that will give you minus 2 K square.

So, the result is h cross square k square by m sum over k g k e to the power i k minus k prime dot r integral over r ok. So, that is what you will get from the first term. For example, now this integral gives me a delta function and that delta function just reduces one of these integrals this, this whole thing becomes just 1 k the k summation vanishes. So, that is this term h cross square, k square by m of g of k.

So, this is k equal to k prime. So, I can replace it write as k prime here, but since k i mean I can choose that k prime as my k. So, it does not really matter this equation instead of writing for k prime I can write the equation for k. So, that is what I am doing.

But, the summation vanishes because k is now fixed to k prime. And, that k prime I am calling as k here plus 1 by L cube g of k prime V k prime sum over k prime equal to epsilon plus 2 E F naught g of k ok. So, this V k k prime is basically just there the remember that integral, that we had so, let me just show it here yeah.

(Refer Slide Time: 25:39)

The slide on the left contains the following text:

Since the interaction in k -space is restricted to a shell with an energy thickness of $\hbar\omega_D$ (with ω_D = Debye frequency) above E_F^0 , the possible k -states are given by the shaded area in Fig. below. This area and therefore the number of energy-reducing phonon exchange processes - i.e. the strength of the attractive interaction - is maximum for $\mathbf{K} = \mathbf{0}$. It is therefore sufficient in what follows to consider the case $k_1 = -k_2 = k$, i.e. electron pairs with equal and opposite wave vectors.

The associated two particle wavefunction $\psi(r_1, r_2)$ must obey the Schrödinger equation

$$-\frac{\hbar^2}{2m} \nabla^2 (\psi_1 + \psi_2) \psi(r_1, r_2) + V(r_1, r_2) \psi(r_1, r_2) = E \psi(r_1, r_2) = (e + 2E_F^0) \psi(r_1, r_2) \quad (1)$$

e is the energy of the electron pair relative to the interaction-free state ($V = 0$), in which each of the two electrons at the Fermi level would possess an energy $E_F^0 = \hbar^2 k_F^2 / 2m$. The two-particle function in this case consists of two plane waves

$$\left(\frac{1}{\sqrt{2}} e^{i\mathbf{k}\cdot\mathbf{r}_1} \right) \left(\frac{1}{\sqrt{2}} e^{i\mathbf{k}\cdot\mathbf{r}_2} \right) = \frac{1}{2} e^{i\mathbf{k}\cdot(\mathbf{r}_1 + \mathbf{r}_2)}$$

The diagram shows two overlapping circles in k -space, representing the Fermi sea and the interaction shell. The shaded region is the intersection of the two circles.

The whiteboard on the right shows the following derivations:

$$\sum_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}^2}{m} g(\mathbf{k}) \int e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}} d^3r$$

$$\frac{\hbar^2 \mathbf{k}^2}{m} g(\mathbf{k}) + \frac{1}{L^3} \sum_{\mathbf{k}'} g(\mathbf{k}') V_{\mathbf{k}, \mathbf{k}'}$$

$$= (e + 2E_F^0) g(\mathbf{k})$$

This is the one where we put this sum over k this one.

(Refer Slide Time: 25:46)

The most general representation of a two-particle state for the case of a non-vanishing interaction is given by the series,

$$\psi(\mathbf{r}_1 - \mathbf{r}_2) = \frac{1}{L^3} \sum_{\mathbf{k}} g(\mathbf{k}) e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}}$$

The summation is confined to pairs with $\mathbf{k} = \mathbf{k}_1 = \mathbf{k}_2$ which, because the interaction is restricted to the region $|\mathbf{r}| \leq a$, must obey the condition

$$E_{\mathbf{k}}^0 < \frac{\hbar^2 k^2}{2m} < E_{\mathbf{k}}^0 + \hbar\omega_D$$

In the adjoining figure, interaction is assumed to take place in a shell of thickness $2\hbar\omega_D$ positioned symmetrically about $E_{\mathbf{k}}^0$

Diagram to illustrate the simplest interaction between two electrons leading to Cooper pairing. The interaction potential is assumed to be constant (V_0)

Handwritten equations on the whiteboard:

$$\sum_{\mathbf{k}} \frac{\hbar^2 k^2}{m} g(\mathbf{k}) \int e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}} d^3r$$

$$\frac{\hbar^2 k^2}{m} g(\mathbf{k}) + \frac{1}{L^3} \sum_{\mathbf{k}'} g(\mathbf{k}') V_{\mathbf{k}\mathbf{k}'}$$

$$= (\epsilon + 2E_{\mathbf{k}}^0) g(\mathbf{k})$$

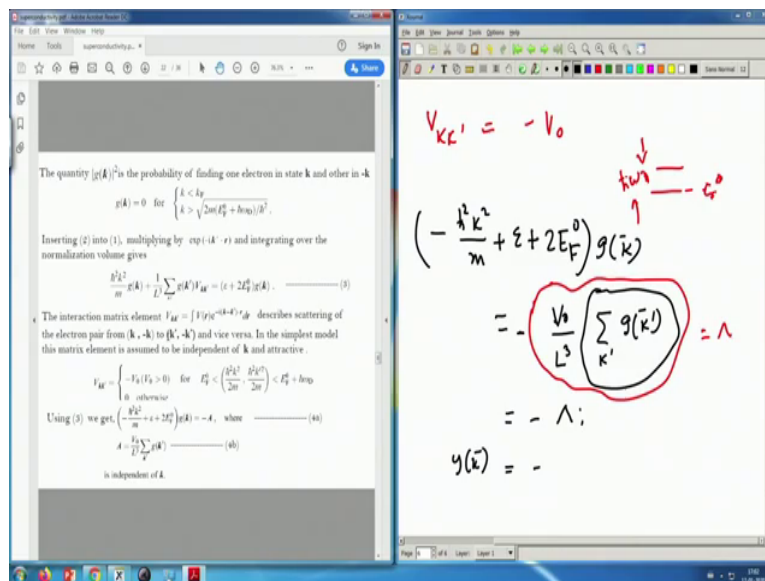
$$V_{\mathbf{k}\mathbf{k}'} = \int V(\mathbf{r}) e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}} d^3r$$

So, then you this is multiplied by V r_1 r_2 and then I multiply again by e to the power minus i k prime r_1 minus r_2 and then sum over integral over r . So, that whole thing is now called g of just put it in and you will see that, this is, this is what g of V of k k prime is V of k k prime is just this integral V of r is r_1 minus r_2 e to the power i k minus k prime.

So, let me just check the signs i k r_1 minus r_2 . So, I have plus i k minus k prime dot r d^3r . So, this is my V of k k prime. So, this is r independent, r is integrated over it only depends on k and k prime. So, this V k k prime basically describes scattering of the electron pair from k minus k . So, from k minus k state to k prime minus k prime.

So, this scattering is described by the potential ok fine. So, now, now comes this approximation of course, this is a complicated equation there is a sum over k k prime here. So, you cannot solve it so easily. So, this is my equation this I have to solve and then there is a k prime sum. So, each k is connected to all other k each k prime is connected to all other k s. So, I cannot solve it analytically here ok . So, what one does is that one now motivates physically.

(Refer Slide Time: 28:09)



So, that physical motivation is already there we already know that the $V_{\mathbf{k}\mathbf{k}'}$ is negative for a region where V_0 is positive. For where these \mathbf{k} and \mathbf{k}' are such that E_F^0 and $E_F^0 + \hbar\omega_D$ within this window the corresponding energy is $\hbar\omega_D$ within that region.

So, they these two energies of the two electrons are just above the Fermi level, but within the range of $\hbar\omega_D$. So, just above Fermi level E_F^0 in a region, which is just $\hbar\omega_D$ above it. So, the both the energies of these original electrons the electrons that are participating that are scattering are here.

So, with this scattering is basically the two electrons at $\mathbf{k} - \mathbf{k}$ going to $\mathbf{k}' - \mathbf{k}'$. So, and in all these for all these \mathbf{k} and \mathbf{k}' the energies must be within this region and that is what this condition here tells you ok. So, if you do that then of course, your life becomes much simpler because now $V_{\mathbf{k}\mathbf{k}'}$ is no longer dependent on \mathbf{k} or \mathbf{k}' .

So, immediately a huge simplification occurs what you get is $\left(-\frac{\hbar^2 k^2}{m} + \epsilon + 2E_F^0 \right) g(\mathbf{k}) = -\frac{V_0}{L^3} \sum_{\mathbf{k}'} g(\mathbf{k}')$ ok. So, this really is an enormous simplification, because look at this right hand side this object here is now \mathbf{k} independent, because all \mathbf{k}' 's are summed here ok. So, this is just a

constant, I can just write it as sum minus lambda it is just a number ok. So, this is just a k independent object is just a number sitting here.

On the left hand side of course, I still have g of k. So, the lambda is equal lambda is this entire object, this entire object is called lambda ok. See, there is a minus V 0. So, I have to be careful. So, this was g k k prime.

(Refer Slide Time: 31:36)

The most general representation of a two-particle state for the case of a non-vanishing interaction is given by the series,

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{D}} \sum_{\mathbf{k}} g(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)}$$

The summation is confined to pairs with $\mathbf{k} = \mathbf{k}_1 = -\mathbf{k}_2$ which, because the interaction is restricted to the region $|\hbar\omega| < \epsilon_0$, must obey the condition

$$E_F^0 < \frac{\hbar^2 k^2}{2m} < E_F^0 + \hbar\omega_0.$$

In the adjoining figure, interaction is assumed to take place in a shell of thickness $2\hbar\omega_0$ positioned symmetrically about E_F

Diagram to illustrate the simplest interaction between two electrons leading to Cooper pairing. The interaction potential is assumed to be constant ($-V_0$)

Handwritten notes on the whiteboard:

$$V_{\mathbf{k}\mathbf{k}'} = -V_0$$

$$k\omega = \epsilon - \epsilon_0$$

$$\left(-\frac{\hbar^2 k^2}{m} + \epsilon + 2E_F^0\right) g(\mathbf{k})$$

$$= \frac{V_0}{L^3} \sum_{\mathbf{k}'} g(\mathbf{k}') = \Lambda$$

$$= -\Lambda;$$

(Refer Slide Time: 31:37)

Since the interaction in k -space is restricted to a shell with an energy thickness of $\hbar\omega_0$ (with ω_0 = Debye frequency) above E_F^0 the possible k -states are given by the shaded area in Fig. below. This area and therefore the number of energy-reducing phonon exchange processes - i.e. the strength of the attractive interaction - is maximum for $\mathbf{k} = \mathbf{0}$. It is therefore sufficient in what follows to consider the case $\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k}$, i.e. electron pairs with equal and opposite wave vectors.

The associated two particle wavefunction $\psi(\mathbf{r}_1, \mathbf{r}_2)$ must obey the Schrodinger equation

$$-\frac{\hbar^2}{2m} (\Delta_1 + \Delta_2) \psi(\mathbf{r}_1, \mathbf{r}_2) + V(\mathbf{r}_1, \mathbf{r}_2) \psi(\mathbf{r}_1, \mathbf{r}_2) = E \psi(\mathbf{r}_1, \mathbf{r}_2) = (\epsilon + 2E_F^0) \psi(\mathbf{r}_1, \mathbf{r}_2)$$

ϵ is the energy of the electron pair relative to the interaction-free state ($V = 0$), in which each of the two electrons at the Fermi level would possess an energy $E_F^0 = \hbar^2 k^2 / 2m$. The two-particle function in this case consists of two plane waves

$$\left(\frac{1}{\sqrt{D}} e^{i\mathbf{k} \cdot \mathbf{r}_1}\right) \left(\frac{1}{\sqrt{D}} e^{i\mathbf{k} \cdot \mathbf{r}_2}\right) = \frac{1}{D} e^{i\mathbf{k} \cdot (\mathbf{r}_1 + \mathbf{r}_2)}$$

Handwritten notes on the whiteboard:

$$V_{\mathbf{k}\mathbf{k}'} = -V_0$$

$$k\omega = \epsilon - \epsilon_0$$

$$\left(-\frac{\hbar^2 k^2}{m} + \epsilon + 2E_F^0\right) g(\mathbf{k})$$

$$= \frac{V_0}{L^3} \sum_{\mathbf{k}'} g(\mathbf{k}') = \Lambda$$

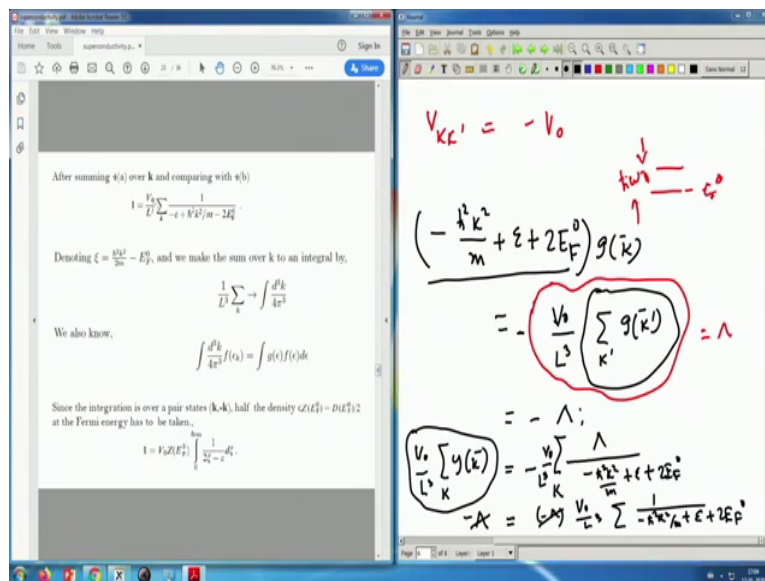
$$= -\Lambda;$$

Let me just see how it, whether there is a minus sign epsilon is $V k k$ prime ok. So, this is h cross square k square by m this thing equal to $V k k$ prime that is correct.

Student: (Refer Time: 31:57).

So, this was also minus yeah sign here ok. So, my remember h cross square k square by m has been brought to the right. And, so this is ok. So, that is that is perfectly all right. So, in this condition the this whole thing in the left hand side is just now a lambda.

(Refer Slide Time: 32:26)



So, let us just go one more step that is independent of k . Now, what one can do is that one just writes g of k equal to ok . So, let me just check the once more, how they defined it? Ok. Ok. Fine g of k is then what I do is that I multiply it by V_0 by lambda cube and sum over k . And, this is equal to lambda by sum over k lambda by h cross square k square by m plus epsilon plus $2 E F 0$ ok.

So, all I have done is that I have first written $g k$ equal to minus lambda divided by this thing and then I summed over this both sides by multiplying by V_0 by lambda L cube ok. Now, you can see that this thing is again minus lambda. So, this is becoming lambda minus lambda and that cancels with this minus lambda here.

And, so, this left hand side is my minus lambda something the right hand side is V_0 by L^3 sum over all this 1 by h^3 minus h^3 cross square k square by m plus c plus this ϵ_0 plus $2 E_F$ and then there was this minus lambda. So, these two cancel I get the equation that is written here. So, that is that is this equation ok. And, this is the equation we have to solve.

(Refer Slide Time: 34:50)

The slide on the left contains the following text and equations:

After summing $\psi(a)$ over k and comparing with $\psi(b)$

$$1 = \frac{V_0}{L^3} \sum_{\mathbf{k}} \frac{1}{\epsilon_0 + \hbar^2 \mathbf{k}^2 / m - 2E_F}$$

Denoting $\xi = \frac{\hbar^2 \mathbf{k}^2}{2m} - E_F$, and we make the sum over k to an integral by,

$$\frac{1}{L^3} \sum_{\mathbf{k}} \rightarrow \int \frac{d^3k}{4\pi^3}$$

We also know,

$$\int \frac{d^3k}{4\pi^3} f(\epsilon_{\mathbf{k}}) = \int g(\epsilon) f(\epsilon) d\epsilon$$

Since the integration is over a pair states $(\mathbf{k}, -\mathbf{k})$, half the density $g(\epsilon) = D(\epsilon)/2$ at the Fermi energy has to be taken,

$$1 = V_0 Z(E_F) \int \frac{1}{\xi^2 - \epsilon^2} d\epsilon$$

The whiteboard on the right shows the following equations:

$$\xi = \frac{\hbar^2 k^2}{2m} - E_F$$

$$\int_{\mathbf{k}} \frac{d^3k}{4\pi^3} f(\epsilon_{\mathbf{k}}) = \int_{\epsilon} g(\epsilon) f(\epsilon)$$

Now, this we redefine just a definition one uses a definition as ξ equal to $\hbar^2 k^2$ square by m minus E_F ok. So, that is one is defining with the 2 here ok. So, that is that 2 is so this is like a single particle energy minus the Fermi level. So, original electronic energy measured from the Fermi level. So, that is what this is?

So, do you define this and then your equation becomes then you use the standard the standard relations that sum over k can be converted to an integral over k , because there is a large number of k 's as we did in the free electron theories. And, then of course, there is this other relation that we use that, any integral like this of a function, which is a function of ϵ only can be converted to a density of states integral over energy. This is an integral over k whereas; this integral is our energy. So, this is doable.

So, with this we will now start solving this Cooper problem and see what happens.