Electronic Theory of Solids
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Lecture - 34
Adiabatic Demagnetisation

So what we do now is something which is related to the thermal properties of a magnetic insulator and this is an application which is actually quite relevant and important.

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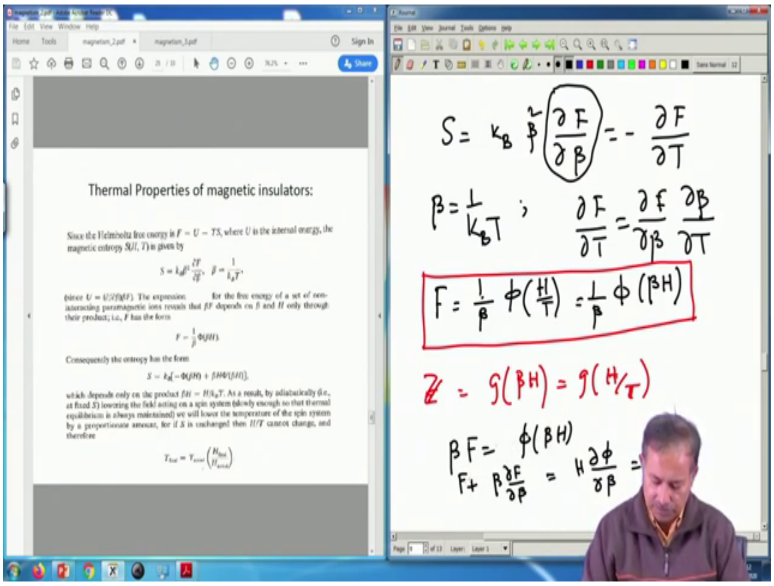

Thermal Properties of magnetic insulators:

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(since $U = \partial(\beta F)/\partial\beta$). The expression for the free energy of a set of non-interacting paramagnetic ions reveals that βF depends on β and H only through their product; i.e., F has the form

$$F = \frac{1}{\beta}\Phi(\beta H).$$

Consequently the entropy has the form

$$S = k_B[-\Phi(\beta H) + \beta H\Phi'(\beta H)],$$

which depends only on the product $\beta H = H/k_B T$. As a result, by adiabatically (i.e., at fixed S) lowering the field acting on a spin system (slowly enough so that thermal equilibrium is always maintained) we will lower the temperature of the spin system by a proportionate amount, for if S is unchanged then H/T cannot change, and therefore

$$T_{final} = T_{initial}\left(\frac{H_{final}}{H_{initial}}\right)$$

$$S = k_B \beta^2 \boxed{\frac{\partial F}{\partial \beta}} = -\frac{\partial F}{\partial T}$$

$$\beta = \frac{1}{k_B T}; \quad \frac{\partial F}{\partial T} = \frac{\partial F}{\partial \beta}\frac{\partial \beta}{\partial T}$$

$$\boxed{F = \frac{1}{\beta}\phi\left(\frac{H}{T}\right) = \frac{1}{\beta}\phi(\beta H)}$$

$$Z = g(\beta H) = g(H/T)$$

$$\beta F = \phi(\beta H)$$

$$F + \beta\frac{\partial F}{\partial \beta} = H\frac{\partial \phi}{\partial \beta} =$$

And this uses a beautiful observation that I have been harping on for quite some time which is that your this partition function if you look at is a function of H by kBT.

Now, this I have mentioned several times in my previous lectures. Now if you then you can calculate the entropy from it. So, the calculations that we did so far for Curie's law for example, they are more or less followed by insulating crystals containing rare earth ions, these rare earths have deep lying 4 F levels which are incomplete unfilled shells and then they have a moment quite large moment which shows up in this susceptibility measurements and magnetic and paramagnetic behavior, and 1 by corresponding 1 by T susceptibility.

What is interesting is that if you look at your S, S is basically kB into beta square del F del beta which is basically minus del F del delta T, this formula of course, you remember that S is equal to negative derivative of free energy with respect to temperature.

Which if you write remember that kB if beta is 1 by kB T and then what you do is that this del F del T you can write as del F del beta del beta del T and you can see this del beta del T gives you this beta square term ok.

So, kB beta square, now the thing that one wants to do is have an interesting application out of the fact that your free energy.

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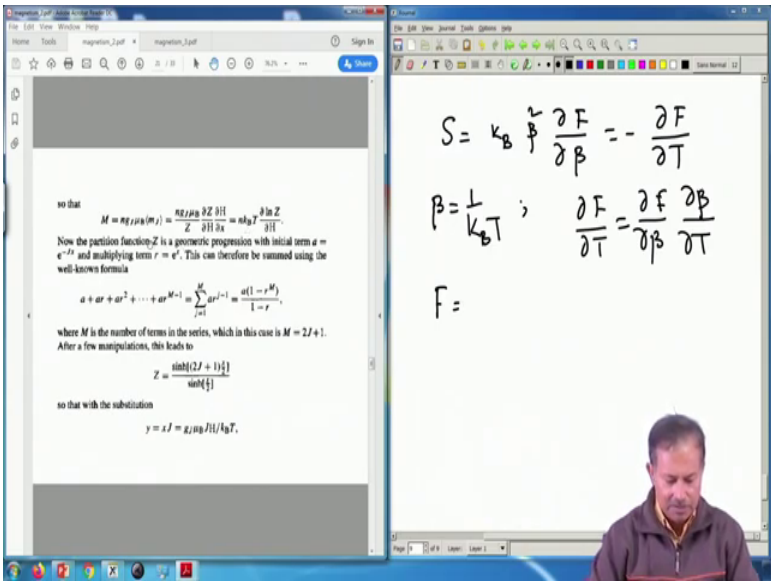

so that

$$M = n g_J \mu_B \langle m_J \rangle = \frac{n g_J \mu_B}{Z} \frac{\partial Z}{\partial H} \frac{\partial H}{\partial x} = n k_B T \frac{\partial \ln Z}{\partial H}.$$

Now the partition function Z is a geometric progression with initial term $a = e^{-Jx}$ and multiplying term $r = e^{x}$. This can therefore be summed using the well-known formula

$$a + ar + ar^2 + \cdots + ar^{M-1} = \sum_{j=1}^{M} ar^{j-1} = \frac{a(1-r^M)}{1-r},$$

where M is the number of terms in the series, which in this case is $M = 2J+1$. After a few manipulations, this leads to

$$Z = \frac{\sinh[(2J+1)\frac{x}{2}]}{\sinh(\frac{x}{2})}$$

so that with the substitution

$$y = xJ = g_J \mu_B J H / k_B T,$$

$$S = k_B \beta^2 \frac{\partial F}{\partial \beta} = - \frac{\partial F}{\partial T}$$

$$\beta = \frac{1}{k_B T} ; \quad \frac{\partial F}{\partial T} = \frac{\partial F}{\partial \beta} \frac{\partial \beta}{\partial T}$$

$$F =$$

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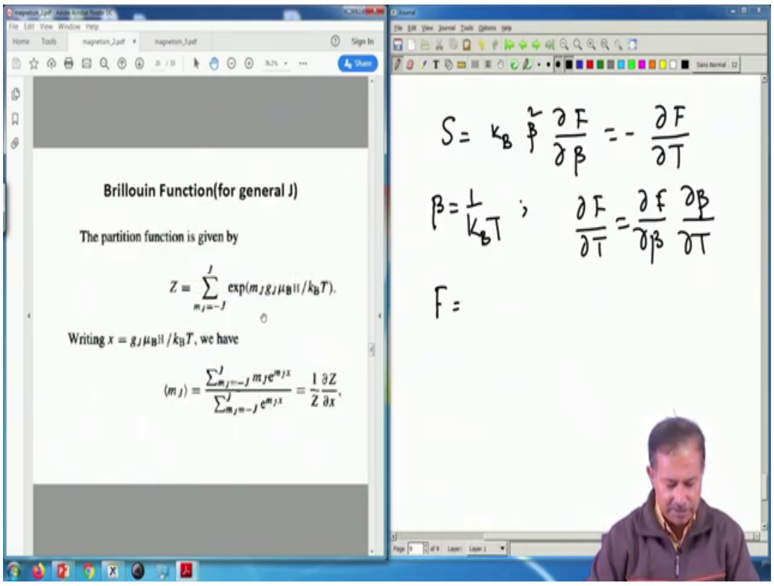

Brillouin Function(for general J)

The partition function is given by

$$Z = \sum_{m_J=-J}^{J} \exp(m_J g_J \mu_B H / k_B T).$$

Writing $x = g_J \mu_B H / k_B T$, we have

$$\langle m_J \rangle = \frac{\sum_{m_J=-J}^{J} m_J e^{m_J x}}{\sum_{m_J=-J}^{J} e^{m_J x}} = \frac{1}{Z}\frac{\partial Z}{\partial x},$$

$$S = k_B \beta^2 \frac{\partial F}{\partial \beta} = - \frac{\partial F}{\partial T}$$

$$\beta = \frac{1}{k_B T} \; ; \quad \frac{\partial F}{\partial T} = \frac{\partial F}{\partial \beta} \frac{\partial \beta}{\partial T}$$

$$F =$$

If you remember your previous viewgraphs that, remember this Z see there is this Z was always a function of H by kBT. So, let us see the simpler one.

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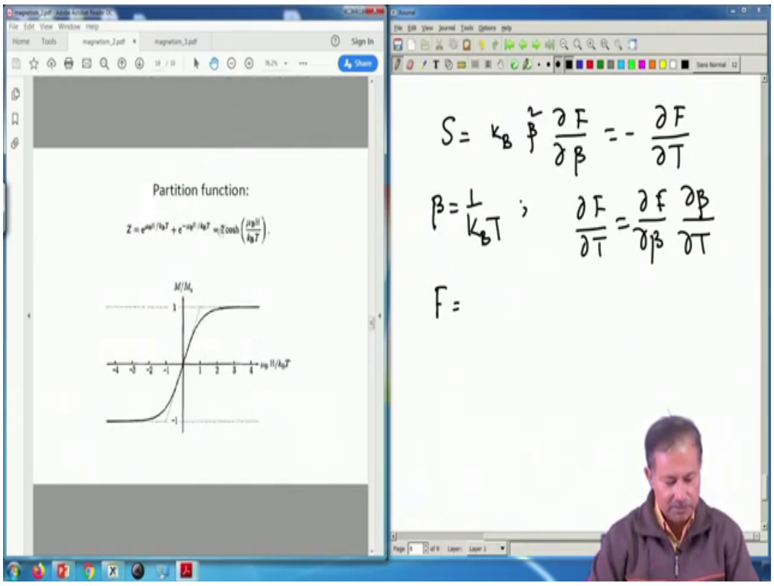

Partition function:

$$Z = e^{\mu_B H / k_B T} + e^{-\mu_B H / k_B T} = 2\cosh\left(\frac{\mu_B H}{k_B T}\right).$$

$$S = k_B \beta^2 \frac{\partial F}{\partial \beta} = - \frac{\partial F}{\partial T}$$

$$\beta = \frac{1}{k_B T} \; ; \quad \frac{\partial F}{\partial T} = \frac{\partial F}{\partial \beta} \frac{\partial \beta}{\partial T}$$

$$F =$$

So, if you calculate the partition function it is just 2 cos hyperbolic mu BH by kBT for j equal to half, for j equal to large values of large j any j there are these other factors, but this formula is still more or less obeyed except for some numerical some numbers additional numbers here.

But the thing is that F is minus kBT log Z; S is basically log of Z times 1 by beta. So, that will do what I mean that will be a just as I said here, but Z is a function of H by T. So, F will be a function of just 1 by beta phi of beta H some function of beta H. So, F is a function of beta 1 by beta some function phi of H by T or you can write it as 1 by beta as a function since a functional relation is beta H ok; kB being a constant you can always do that.

So, this is interesting, that fact that H is only a function of 1 by beta times, a function of a quantity which is a function of H by T this is going to be used. Now I can immediately do the derivative that I want to do to calculate the entropy ok..

So, before I just do the calculations let me just try to tell you what I am trying to do. This is interesting observation that we have meant so far that your partition function is a function of, is just a function of some function g of beta H right; that means, g of H by kT H by T, some function of I can write it as g of H by T.

Now this thing says that if I reduce my H, I have to necessarily reduce my T also. So, H by T is the factor that goes in. So, this is used, as a suppose I keep I want to keep a quantity, suppose I want to keep Z fixed and then I am reducing my H then you have to reduce your T, because it is a function of H by T.

So, entropy as I will show is a quantity which can be kept constant in an adiabatic transformation and so, therefore, one can use the fact that this quantities of being a function of only H by T in the ratio not individually H not individually T its ratio that appears always. That can therefore, be used in an adiabatic process if you go through an adiabatic process by keeping S fixed then actually you do it as I will show, you do it alternately adiabatic and isothermal..

And therefore, you can reduce the temperature of a system whose entropy is dominantly coming from magnetic entropies. So, that is the thing that we are going to do that is a practical application that is actually being done and that is used and. So, that is why it is beautiful application from a simple observation of a mathematical form that the partition function is a function of H by T not single H not single T.

So, that is the pretext the prelude to what I am going to do ok. So, let us now start doing the calculation. So, beta times F is phi of beta H ok. So see I need this to calculate entropy, I need del F del beta, then I will multiply by kB into beta square. So, this will be I will take derivative on both sides; so, this will be 1 plus F plus beta del F del beta equal to del phi and del beta, which I will write as phi prime.

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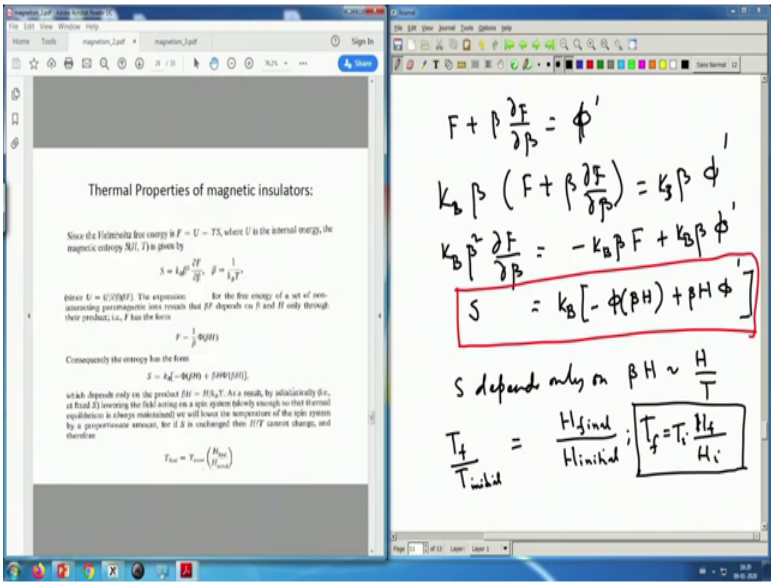

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$$T_{final} = T_{initial}\left(\frac{H_{final}}{H_{initial}}\right)$$

$$F + \beta\frac{\partial F}{\partial \beta} = \phi'$$

$$k_B\beta\left(F + \beta\frac{\partial F}{\partial \beta}\right) = k_B\beta\phi'$$

$$k_B\beta^2\frac{\partial F}{\partial \beta} = -k_B\beta F + k_B\beta\phi'$$

$$\boxed{S = k_B[-\phi(\beta H) + \beta H\phi']}$$

S depends only on $\beta H \sim \frac{H}{T}$

$$\frac{T_f}{T_{initial}} = \frac{H_{final}}{H_{initial}}; \quad \boxed{T_f = T_i\frac{H_f}{H_i}}$$

So, F plus beta del F del beta equal to phi prime, I have to multiply it by kB into beta square remember. So, it is here on the left hand side you can see I have to multiply it by kB into beta square to get my entropy; kB into beta square into F plus beta del F del beta equal to kB beta square phi prime..

So, kB beta square del F del beta, so from this I find out kB beta square del F del beta ok. So, I do not multiply by I multiply only by kB beta, therefore, I have this so, this will give me immediately KB beta square del F del beta and that is then equal to minus kB beta F plus kB beta phi prime.

Now F is 1 by beta. So, that will cancel my beta here and so the result that I will get is kB into minus phi beta H. So, I am using the fact that F is equal to 1 by beta phi of beta H. So, then this gives me plus beta H phi prime. Remember this phi prime sorry there was a H sitting there because I took a derivative with respect to beta sorry. So, this is correct, but there is a H sitting here. See, this is beta H so, I took a derivative with respect to beta; so H will come out and that is this beta H phi prime.

So, this is my S. So, that is the expression that written on the left hand side, entropy has this formula that kB into minus phi beta H plus beta H phi prime. Again the same statement that S depends only on this quantity, S depends only on beta H which is H by T right. So, suppose you want to keep S fixed which is basically what we do in adiabatic changes then you can lower your field, if you lower your field then you have to lower your temperature because your S is fixed and this is the ratio by which phi depends on both H and T.

So, that means, if you can adiabatically decrease your field then you can decrease your temperature. So, these are of course, the technical things that you have to keep fix. As a result by adiabatically that is at fixed S lowering the field acting on a spin system by adiabatically one means slowly enough so that thermal equilibrium is always maintained one can lower the temperature of a spin system by the proportionate amount..

I mean that is exactly what I have been saying. So, you can right then that suppose I have a T, I have going through suppose I started with an H initial and I landed with H final from initial to final I went.

So, then this will just be equal to the same as T initial right, this ratio has to be maintained because S is kept fixed, so that means, the T f is Ti into Hf by Hi, f and i are final and initial. So, this is. So, if you reduce your Hf your T final has to reduce and this is how it is actually done.

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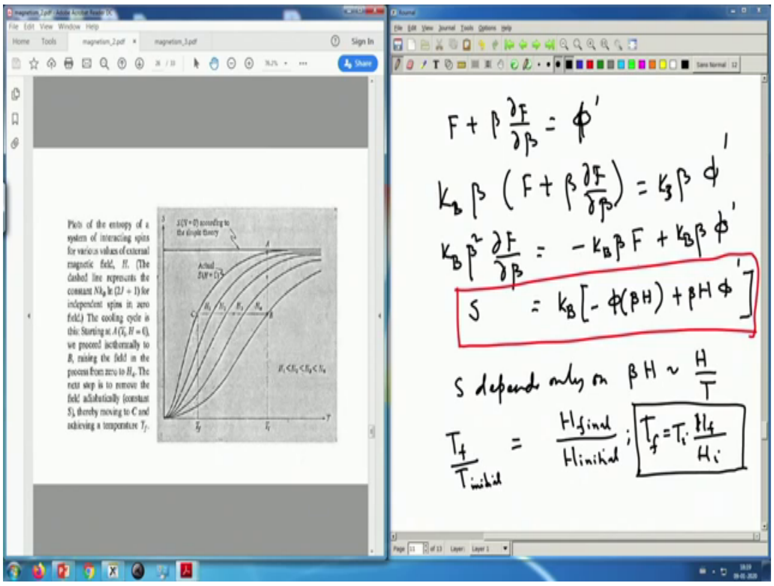

Plots of the entropy of a system of interacting spins for various values of external magnetic field, H. (The dashed line represents the constant $Nk_B \ln(2J+1)$ for independent spins in zero field.) The cooling cycle is this: Starting at $A(T_i, H = 0)$, we proceed isothermally to B, raising the field in the process from zero to H_4. The next step is to remove the field adiabatically (constant S), thereby moving to C and achieving a temperature T_f.

$$F + \beta \frac{\partial F}{\partial \beta} = \phi'$$

$$k_B \beta \left(F + \beta \frac{\partial F}{\partial \beta} \right) = k_B \beta \phi'$$

$$k_B \beta^2 \frac{\partial F}{\partial \beta} = -k_B \beta F + k_B \beta \phi'$$

$$\boxed{S = k_B \left[-\phi(\beta H) + \beta H \phi' \right]}$$

S depends only on $\beta H \sim \frac{H}{T}$

$$\frac{T_f}{T_{initial}} = \frac{H_{final}}{H_{initial}} ; \boxed{T_f = T_i \frac{H_f}{H_i}}$$

So, this picture is already quite nice, so, let me just follow this picture. What one does here is that one this is the entropy versus temperature plot.

You have several curves for S for example, remember the simple theory that we have been doing tells us that if you keep your H equal to 0 then of course, there is no temporary independence at all. So, that is dangerous thing because that will violate the third law of thermodynamics, but anyway..

So, let us that means, that theory that simple minded calculation that we are doing that it depends only on beta H has to give in give has to, give away at some point and at very low fields very low temperatures.

So, it is actually a very low temperatures, this is not correct, but let us just remind you that there is this pitfall that the simple this theory predicts that at H equal to 0 you have a completely flat curve no dependence on temperature we will come back to it discuss it. Now, so the actual H is like this at H equal to 0 actual S at H equal to 0 and then of course, this for higher values of H this side it is C to B the value of H increases..

So, what you do is that, see the you remember the dashed line represents the constant Nkb log 2 j plus 1 that is what I was saying that you have, no dependence on temperature, and 2 j plus 1 is the degeneracy of jth level. So, that is the that was the entropy at very high temperature that is correct. That is what you see all these curves at very high temperature go to this NkB 2 j plus 1 log 2 j plus 1 value.

Anyway, so the process here is that you start from point A cooling cycle is at A with T i and H equal to 0, then proceed isothermally to a point B. So, this is the point B you come down isothermally, so, entropy is reducing here by ok. So, this is from A point which is H equal to 0 to B point remember is H has increased..

Next step is to remove the field adiabatically. So, we come from A to B by raising the field isothermally, then adiabatically we now need to start reduce reducing the field and go from say B to C which is again H equal to 0 the actual curve, but you may you can stop at H 2 or somewhere, but basically you reduce your field. So, you come along this way, but adiabatically which means your entropy is constant.

And then you see what is happened, your temperature here was Ti the temperature here is Tf; so, you actually managed to reduce your temperature of the system by doing by this process. So, you start from a very low field, come to a higher field isothermally and then go to a lower field adiabatically..

And this process can then be then this point C becomes your initial point and then you keep on doing the same process isothermally go down then adiabatically go to the left and by this process you can repeatedly applying this process you can reduce the temperature of the system.

Now of course, this is a beautiful scheme, but as I said that this kind of simple theory really does not work and it breaks down at some point. So, where does it and why does it break down? It breaks down simply because the fact is that it cannot be that the spins at spins only depend the entropy only depends on the moments right, in a real system that is not going to happen. Of course, at very low temperatures these contribution from spin moment to the entropy is probably large in an insulator and the phonon contribution there is small.

But then they remember that this also says that our H equal to 0, if you have H equal to 0 then you have no T dependence that will violate third law of thermodynamics. So, at some point definitely at a lower temperature at some temperature somewhere this theory breaks down and this you have to consider these other effects that coming particularly the effects at low temperature which are very important at the these moments talk start talking to each other..

And because there is no other effect thermal effect is small and other energy scales have already been very small this interaction energy between spins of different between moments of different ions or different moments basically are not negligible anymore and that will restore the physical law that S has to go to 0 at T going to 0.

And, but then of course, this is used, so, this is used in certain materials which some insulating materials containing the rare arts and paramagnetic salts, and there of course, it works to take your temperature you start at a temperature which is around few degrees and then it can take you down to about 100th of a degree Kelvin and so on..

Below that of course, it will not work, but this reduction by a factor of 100 can be achieved using adiabatic demagnetization, this is called the adiabatic demagnetization procedure by which you can reduce the temperature..

And it is a cooling basically is a procedure for cooling refrigeration and its interestingly its more or less pollution free because nowadays that is a very important issue because you are using not burning any fuel or not spending too much of electricity and so on. We were using a magnetic cycle to do it of course, to do the magnetic cycle you need some energy, but turns out it is a very safe and nice method to go down in temperature, but as I said it has its limits and it stops working below a certain temperature.

So, if you want to use it be careful and choose your material very carefully whose magnetic cycle you will be using and then of course, your its of course, you use a magnetic insulator, but even then you will not be able to go down to extremely low temperatures, for that there are other methods which we will not discuss here..

So, basically this is an application of thermodynamics of a magnetic system and what have we used? That is a very simple mathematical observation and which is actually guided by physics as I have been discussing, that you know if you want to raise H and also T then these two actually oppose each other and therefore, your partition function for example, will reflect that fact that your magnetization will reflect that fact that your magnetic.

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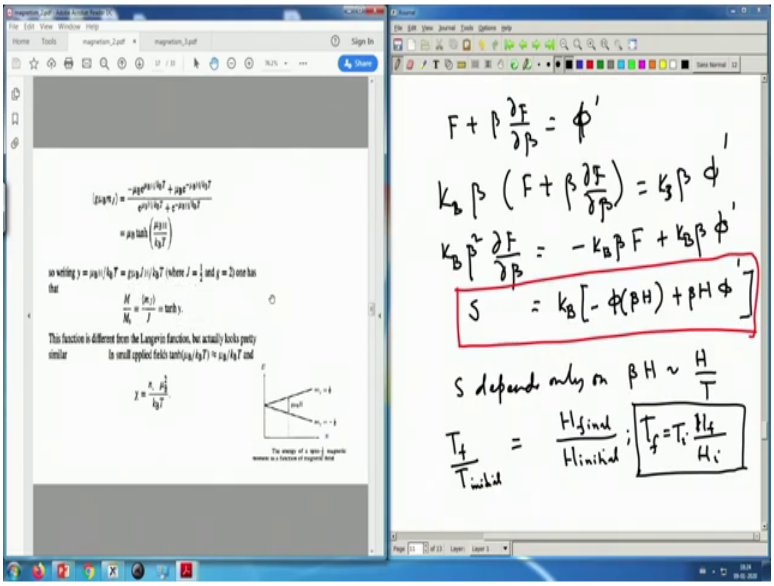

$$\langle \mu_B m_J \rangle = \frac{-\mu_B e^{-\mu_B B/k_B T} + \mu_B e^{\mu_B B/k_B T}}{e^{\mu_B B/k_B T} + e^{-\mu_B B/k_B T}}$$

$$= \mu_B \tanh\left(\frac{\mu_B B}{k_B T}\right)$$

so writing $y = \mu_B B/k_B T = g\mu_B J B/k_B T$ (where $J = \frac{1}{2}$ and $g = 2$) one has that

$$\frac{M}{M_s} = \frac{\langle m_J \rangle}{J} = \tanh y.$$

This function is different from the Langevin function, but actually looks pretty similar. In small applied fields $\tanh(\mu_B B/k_B T) \approx \mu_B B/k_B T$ and

$$\chi = \frac{n\mu_0 \mu_B^2}{k_B T}.$$

The energy of a spin-$\frac{1}{2}$ magnetic moment as a function of magnetic field

$$F + \beta \frac{\partial F}{\partial \beta} = \phi'$$

$$k_B \beta \left(F + \beta \frac{\partial F}{\partial \beta} \right) = k_B \beta \phi'$$

$$k_B \beta^2 \frac{\partial F}{\partial \beta} = -k_B \beta F + k_B \beta \phi'$$

$$\boxed{S = k_B \left[-\phi(\beta H) + \beta H \phi' \right]}$$

S depends only on $\beta H \sim \frac{H}{T}$

$$\frac{T_f}{T_{initial}} = \frac{H_{final}}{H_{initial}} ; \boxed{T_f = T_i \cdot \frac{H_f}{H_i}}$$

For example these kind of a magnetization for example, is a function only y and because; obviously, the magnetization has to decrease as we increase temperature, magnetization has to order become larger magnetic moments have to order magnetization becomes larger if you put more field.

So, this H divided by T is the common ratio that dictates the play of field and temperature, and if that is true then of course, if you keep the quantity like entropy which depends only on this H by T fixed then by changing H you can change T.

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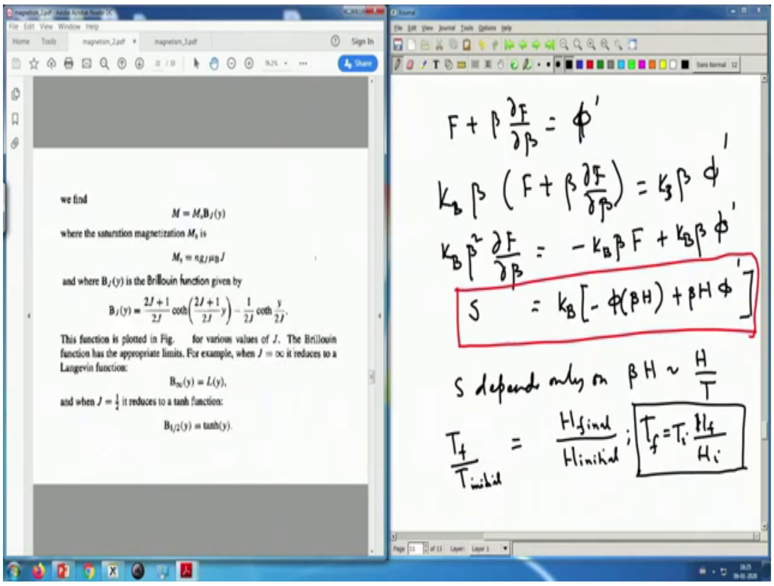

we find

$$M = M_s B_J(y)$$

where the saturation magnetization M_s is

$$M_s = n g_J \mu_B J$$

and where $B_J(y)$ is the Brillouin function given by

$$B_J(y) = \frac{2J+1}{2J}\coth\left(\frac{2J+1}{2J}y\right) - \frac{1}{2J}\coth\frac{y}{2J}.$$

This function is plotted in Fig. for various values of J. The Brillouin function has the appropriate limits. For example, when $J = \infty$ it reduces to a Langevin function:

$$B_\infty(y) = L(y),$$

and when $J = \frac{1}{2}$ it reduces to a tanh function:

$$B_{1/2}(y) = \tanh(y).$$

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$$k_B\beta^2\frac{\partial F}{\partial\beta} = -k_B\beta F + k_B\beta\phi'$$

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$$\frac{T_f}{T_{initial}} = \frac{H_{final}}{H_{initial}}\ ; \boxed{T_f = T_i\cdot\frac{H_f}{H_i}}$$

And this is a beautiful way of very nice way cute way of using a mathematical fact or a physical fact to lower the temperature of course in every method you use a physical fact but this is a magnetic system and by using you do not associate magnets with for refrigeration right, but here for example, a magnetic interactive magnetic phenomenon that simple thermalization and ordering these two opposing processes they appear in the ratio and therefore, that can be used to a clever way in a clever way to give you this cooling refrigeration.

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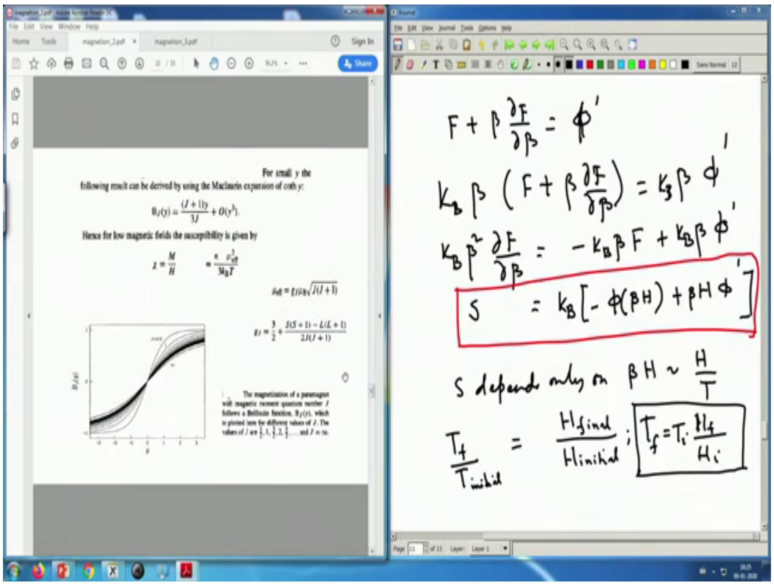

For small y the following result can be derived by using the Maclaurin expansion of coth y:

$$B_J(y) = \frac{(J+1)y}{3J} + O(y^3).$$

Hence for low magnetic fields the susceptibility is given by

$$\chi = \frac{M}{H} \approx \frac{n\mu_0\mu_{\text{eff}}^2}{3k_B T}$$

$$\mu_{\text{eff}} = g_J\mu_B\sqrt{J(J+1)}$$

$$g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}$$

The magnetization of a paramagnet with magnetic moment quantum number J follows a Brillouin function, $B_J(y)$, which is plotted here for different values of J. The values of J are $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$ and $J = \infty$.

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S depends only on $\beta H \sim \frac{H}{T}$

$$\frac{T_f}{T_{\text{initial}}} = \frac{H_{\text{final}}}{H_{\text{initial}}}\ ; \boxed{T_f = T_i\frac{H_f}{H_i}}$$

It is a so beautiful way to do apply a physical phenomenon which is unrelated normally to another field of science, which is technology here where you want to go down in temperature cooling and so, this is a complete subject of cryogenics that people study to do these kind of things, and this is one of the methods of cooling down from few degrees down to about 100 or so of a degree by using certain paramagnetic salts insulating paramagnetic salts.

So, next what we will do is that we will go to another interesting phenomenon which in the next class I will discuss.