Introduction to Non-Linear Optics and it is Applications
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Lecture – 16
Nonlinear Maxwell's Equation

So, welcome back students to the next class of Introduction to Non-Linear Optics and it's Application course. So, in the previous class, we learned about the sum and different frequency generation and also we know how the second harmonics are generated. Today is the lecture number 16 and let us see what we have in today's lecture.

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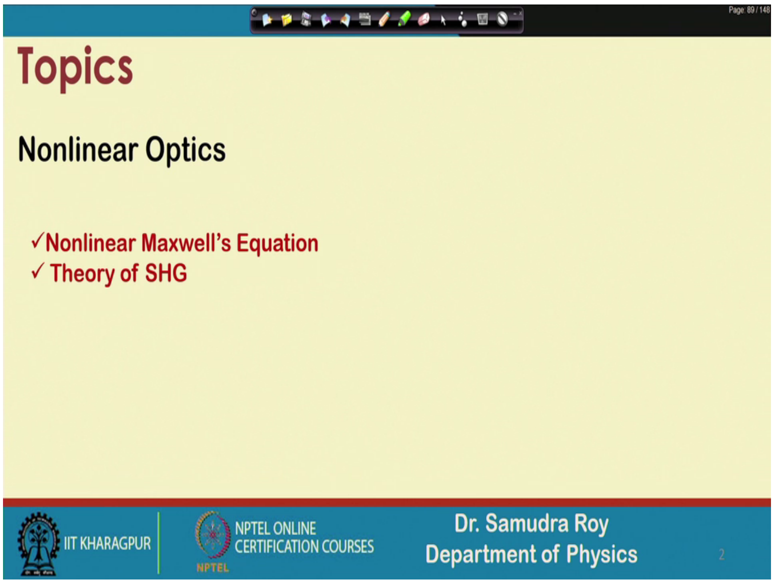

So, today we have 2 very important concept; one is non-linear Maxwell's equation and we will after having after having the knowledge of non-linear Maxwell's equation, we going to learn the theory of second harmonic generation. So far, roughly we understand how the second harmonics are generated. So, when we launch an electric field, because of the frequency mixing, different frequency components are generated. And if we launch one electric field with frequency omega, then we find that we can generate frequency 2 omega which is called the second harmonic generation. But the detail theory is still missing because we do not know exactly what is happening inside the medium and how the second harmonics are evolving, how the second harmonic fields are evolving are still quite hazy.

So, in order to understand the detailing of the second harmonic generation or the basic physics or what is going on inside the material, it is important to know, what is the meaning of non-linear Maxwell's equation. Well, the Maxwell's equation is the governing equation the main governing equation of the electric field. So, when the electric filed is inside the medium, what happened that it will interact with the medium and if the interaction is non-linear in nature, inside the Maxwell's equation, we have some kind of non-linear components. And this non-linear components is nothing, but the non-linear polarization that we have been learning for last few classes.

So, let us try to find out how to how to calculate this non-linear Maxwell's equation or how to derive this non-linear Maxwell's equation.

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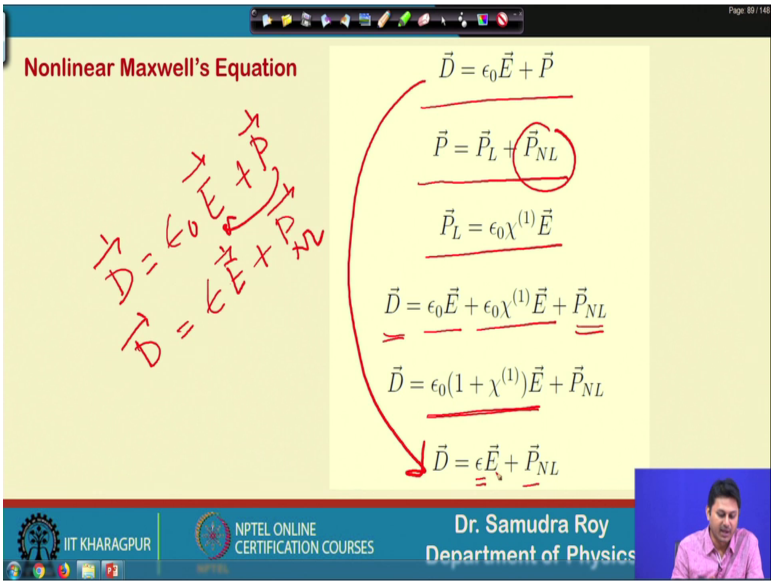

The derivation is quite straightforward and it is something like the way we generate Maxwell's wave equations exactly we will generate in the similar way the non non-linear Maxwell's equation or non-linear Maxwell's wave equation. So, let us start with our fundamental expressions that is D, which is epsilon 0 plus P displacement vector is epsilon 0 E plus P. Now it is not necessary that D and E should be in the same direction because of this polarization can be a function of electric field. But if I launch the electric field that not ensured that the polarization will be in the same direction if the medium specially if the medium is an isotropic in nature, then there is a I mean if I launch an electric field one direction, then the polarization will be another direction.

So, we will come to this point later, but here more importantly we need to introduce the non-linear polarization. So, now, the polarization can be divided into 2 part; one is the linear one and an linear part and another is the non-linear part. The linear part is introduced in this way and it is proportional to the electric field as usual. And if that is the case, then the D can be represented at epsilon 0 E plus epsilon 0 chi 1 E which is a P linear term and then P non-linear term I we just left the P non-linear term as P n l vector, ok. After having the value of D, now it is time to write in compact way, and if I do, then it should be epsilon 0 1 plus chi and E and we know that epsilon 0 1 plus chi is nothing but epsilon 0 and P non-linear will be there.

So, if I look from this equation to this equation, what essentially is changing? D was epsilon 0 E plus P this P is a P total. Now I combine this P linear term here and in this P linear term, we have the information of the refractive index that is the material information, that is in we basically inserting that material information into the system through refractive index and that is why, this epsilon 0 become epsilon and since a non-linear part is associated with this.

So, this non-linear part will be left alone and it will be simply that. So, from here to here, we make such certain modification and the modification is essentially we just separate out the non-linear polarization term and the rest part is same. If non-linear polarization is 0, then D equal to epsilon 0 E is the standard form that we usually find when we calculate the Maxwell's equations and the corresponding derivations ok.

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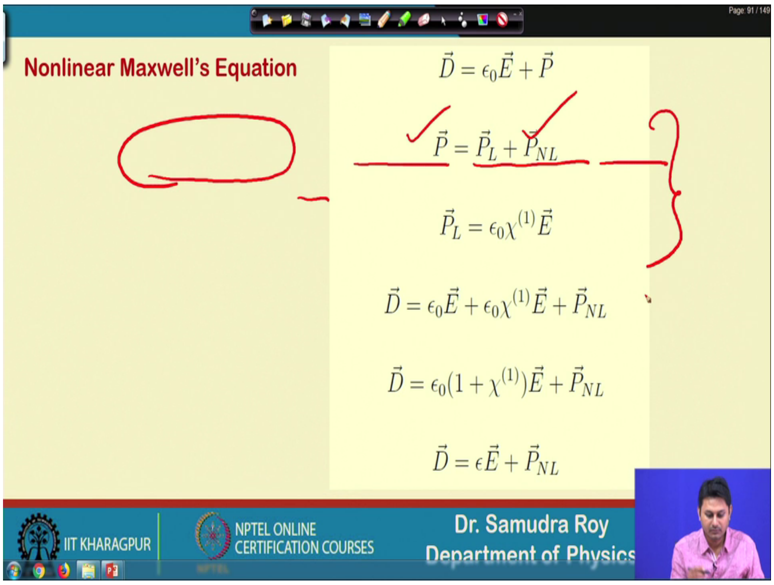

Ok.

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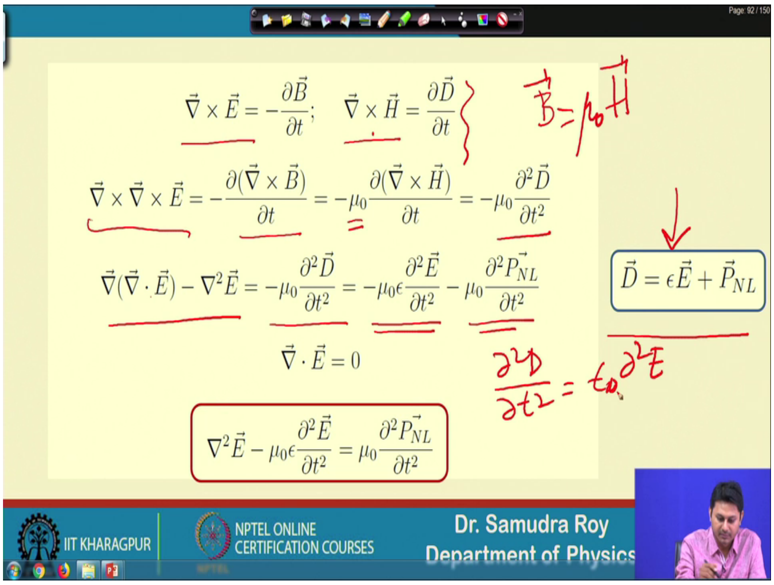

After having the knowledge of D with non-linear polarization as shown here, in the this is the expression that we figure out. Now we start deriving the Maxwell's equation with this condition with this condition. So, now, again we will going to use 2 Karl equation. So, Karl of E is minus del B del t. Karl of H is minus del D del t. So, now, if I take Karl both the side from this equation, so, Karl of B will be this and if I take Karl of H both the Karl of H both the cases. So, I use of Karl of ah if I from this first equation, if I do Karl of B, it will Karl of B and this Karl of B I use this one and then it will be Karl of H with mu 0.

Because, B is mu 0 H and then Karl of H I will going to use this term. So, it will be the second derivative with respect to second order derivative with respect to time and D. So, this Karl of Karl of E can be represented in a standard form we know it is a grad of it is a divergence grad of divergence of E minus of grad square E and then the left hand side right hand side is minus epsilon 0 D 2 D d t square. This term is now represented into 2 part; one is this and another is this, because my D is epsilon 0 E plus P non-linear.

So, when I write double derivative of D essentially in the right hand side I need to write this is not this is epsilon, ok. So, let me clear it and do it once again.

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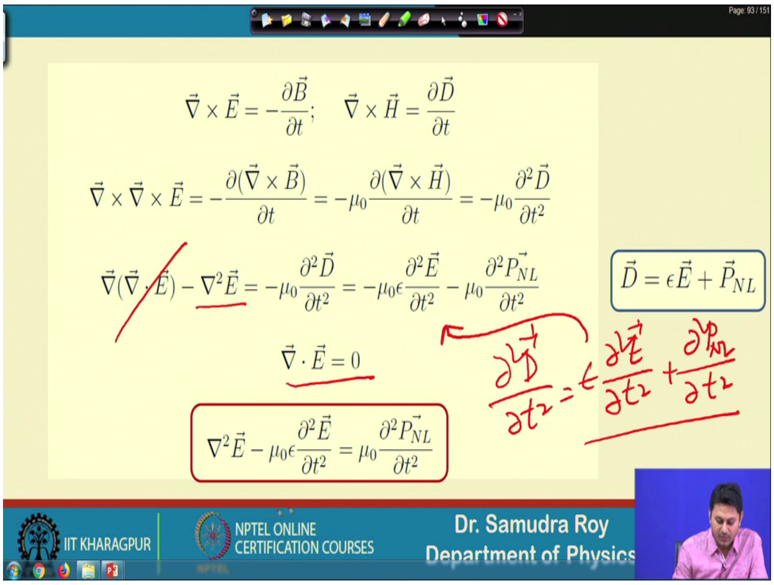

So, I am making the double derivative of D d is this. So, it will right hand side it is simply the double derivative of E plus the double derivative of P n l. So, I making if I use this it will be simply this quantity. So, now, grad dot E is 0. So, this term will cancel out. So, eventually we have these is equal to the right hand side. So, these is the equation that is given in inside the border. We have one very important aspect we find in this equation that in normally this term is not there.

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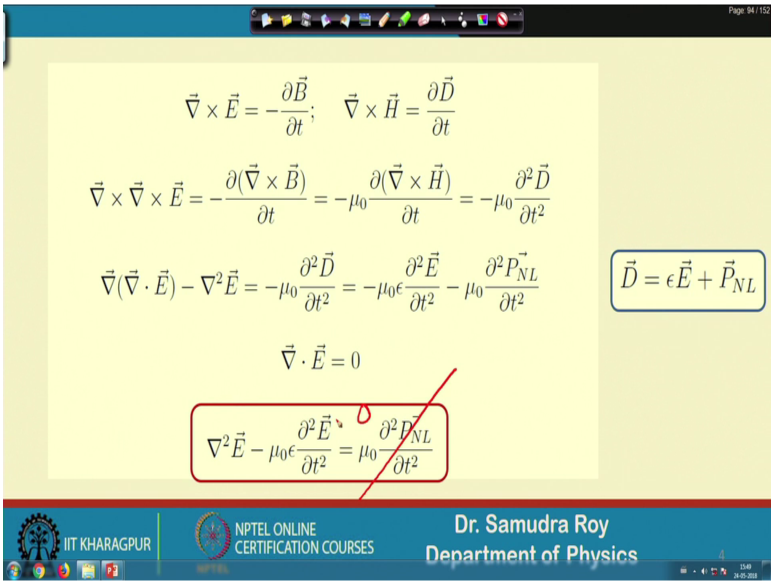

When we consider everything is linear or the polarization is a linear polarization, we have this is equal to 0 and then we have some kind of solution because this is a plane wave the plane wave solution is there. So, we have already done in our basic ah basic linear optics course. So, now, in state of having 0 term what we have is a source term.

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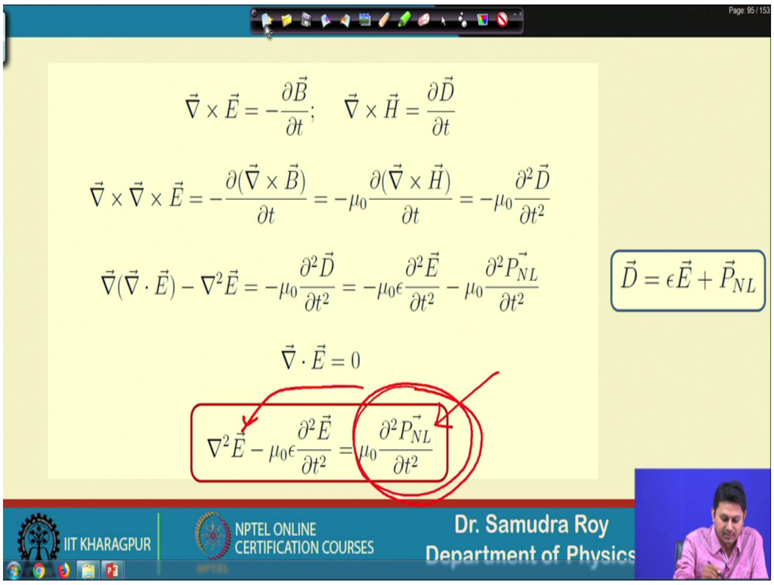

This is basically is source term. This P non-linear term basically is source term of the electric field that mean, whatever the electric field will be there inside the medium in this differential equation we have a source term. And this source term is essentially coming with a fact that mine P non-linear is there. So, P non-linear is basically feeding some kind of energy to generate electric field and that is the same this picture we have already shown with a schematic diagram, that when the dipoles are vibrating in a non-linear fashion, you will start generating some kind of electric field.

And this electric field will have a different frequency component exactly the similar concept we have through this equation, where we find after deriving the Maxwell's equation considering the non-linear polarization term. We have something like this a source term.

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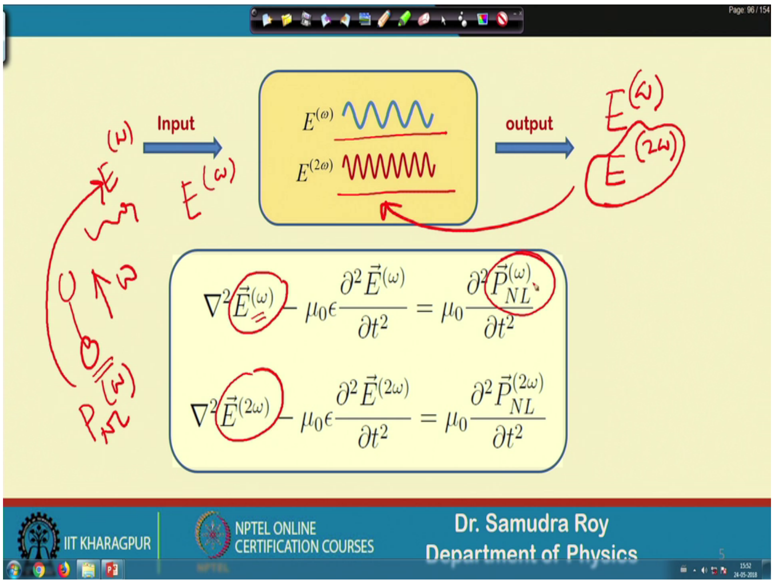

So, the next thing is what happened when we have the electric field do the system having frequency components. So, we have already derived the non-linear Maxwell's wave equations. So, this is the picture. We will launch an electric field and we are we will get we put some kind of input and we will get output outside. But in the system inside the system what happened that we have 2 frequency components.

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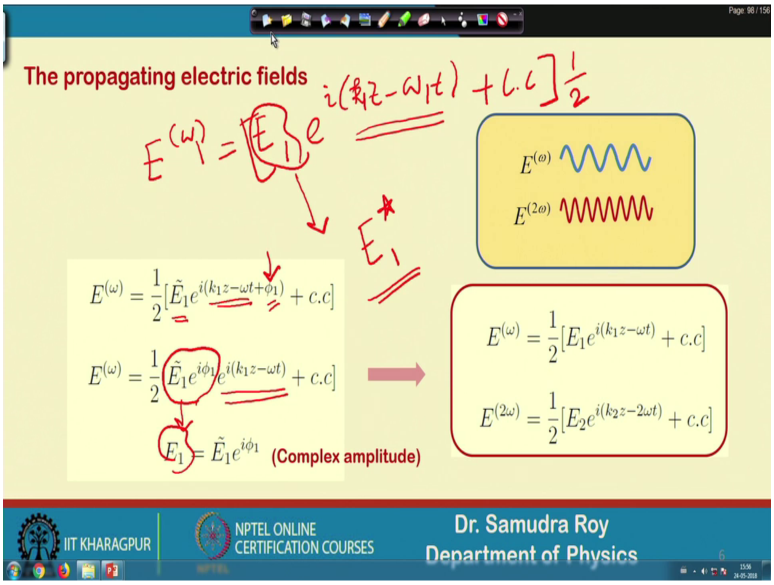

So if I launch, if I launch electric field E of omega, if I consider only the second harmonic generation, then inside this system is second harmonic field will generate. So, output; obviously, we will have E omega and E of 2 omega so; that means, E of 2 omega will going to be here somewhere. So, in this figure, we find that inside the material which is non-linear in nature. We have the fundamental wave and also the second harmonic both are present.

So, for both the cases the non-linear Maxwell's equation of the non-linear wave equation is valid. So, for 2 waves we have 2 equations. Basically, the equation is same; the electric field component E omega and electric field component 2 omega are different. But the equation is same for electric field component E omega what happened the source term is P non-linear omega as I mentioned. So, when we launch the electric field, what happened the dipole? If this is the dipole, it will going to vibrate at frequency omega and start generating E field of omega.

So, that is why, the source term is nothing, but the P non-linear which is vibrating at the frequency omega. So, that is the source of the electric field which is generating and the frequency of the electric field as same as the frequency at which the polarization is vibrating. Here, the polarization is vibrating with the frequency omega non-linear polarization and as a result, we are getting some kind of electric field which is also vibrating with a frequency omega exactly in the similar way.

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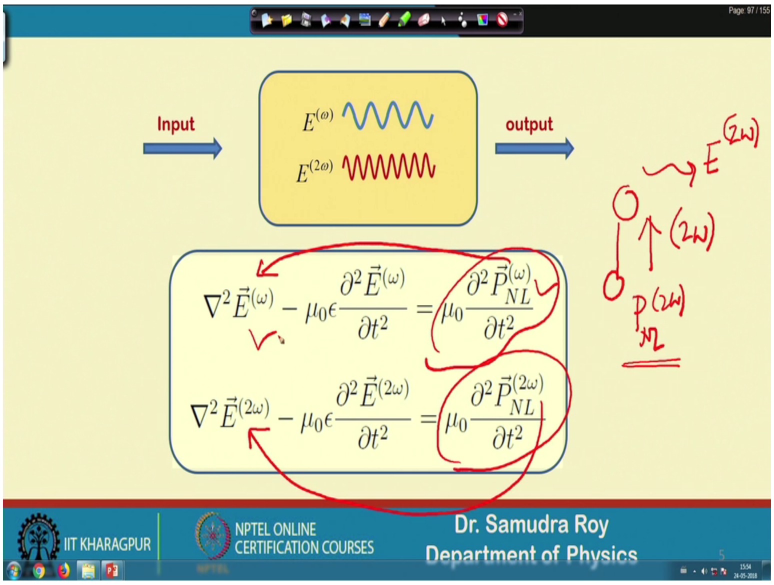

We can say that, if this is my dipole, suppose and it is now vibrating with a frequency 2 omega and when it is vibrating with a 2 omega, my non-linear polarization can be represented 2 omega. This is the way it is vibrating and as a result, it is generating a frequent generating electric field with the same frequency 2 omega. So, that means, the generation of 2 omega is essentially supported by the non-linear polarization which is having which is having a component of 2 omega.

Now, if I look to this differential equation, in this differential equation in the source term, both the cases we have P non-linear omega and P non-linear 2 omega, 2 source term are there. This source term basically responsible for generating electric field with omega. This source term is important for generating the electric field 2 omega, but the equation is same for both the cases; only the components are different. So, in the next step, that is a very important to find out what should be this value and what should be this value so that, we can solve at the end of the day we can solve our Maxwell's equation ok.

The electric field, we know very clearly what should be the electric field because; we have been using the form of this electric field for long. So, electric field with frequency component omega we just use as E 1 E to the power of i k 1 z minus omega 1 t plus complex conjugate of that with a half term, so that, it this is real this is the normal way to represent the electric field and that is the frequency component of E omega, but here one thing you should remember that this E 1 and this is a phase, but some additional phase should also be there.

So, here we write E 1 tilde which is a real quantity and then E to the power i k z minus omega t that is the component related to the plane wave and on top of that we have some additional phase which is phi one. So, this phi one can be represented as E to the power I phi one multiplied by E one t and write only the k one z minus omega t in a separate way. So, if I do then I can write E one is E one tilde E to the power of I of phi one.

So, now, this is complex quantity. So, this is amplitude, but normally the concept is amplitude is a always real quantity, but here we absorb the phase additional phase term to make it more general and find that the amplitude is now complex in nature. So, that in the future, we may have the term like E 1 star this is the term which basically gives you the idea that E 1 which is essentially the amplitude may have some phase associated with that. So, that is why it is a complex.

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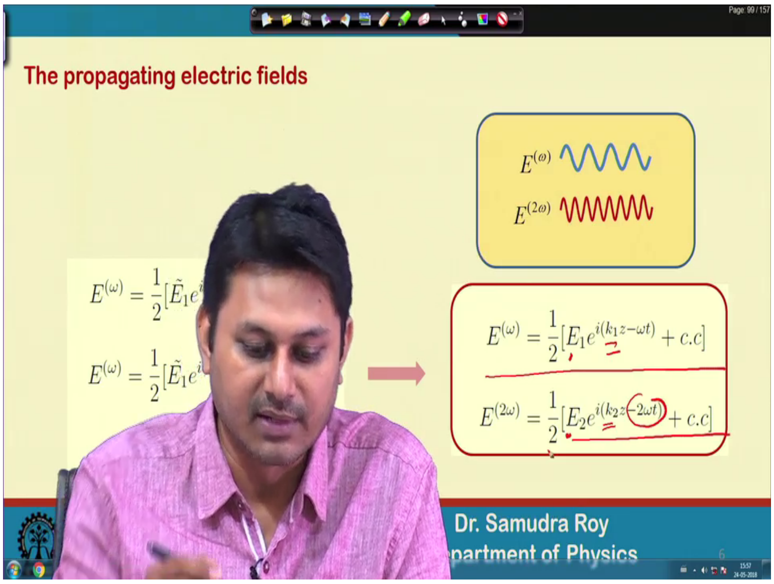

So, simple thing here, how to find out what is my E omega and E 2 omega. So, in the slide we can find that E omega is nothing but E 1 E to the power of k 1 z minus omega t E 2 E to the power of k 1 z minus omega t E 2 E to the power k 2 minus 2 omega t because this is 2 omega. So, that is why, 2 omega frequency is there k 1 is a propagation constant for the frequency component omega or the fundamental wave and k 2 is the frequency component of second harmonic. So, these E omega and E 2 omega are the electric field and this electric field has this kind of forms remember that E 1 and E 2 are now complex amplitude in general ok.

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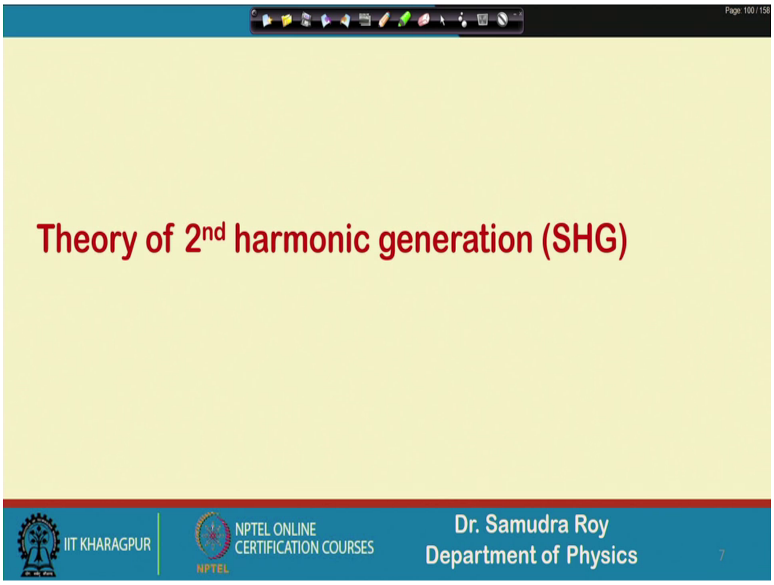

Now, we going to learn as I mentioned the theory of second harmonic generation.

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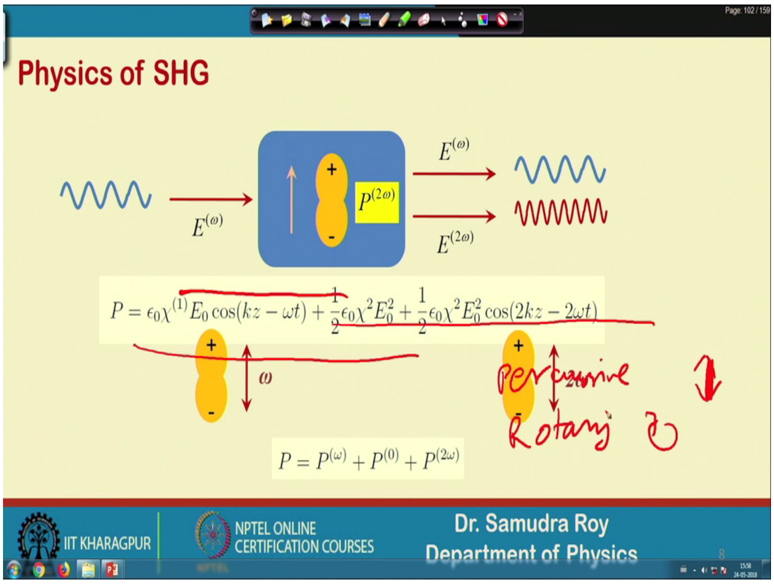

So, if I go back to the picture, as I mention that this picture, we have already shown in our previous classes that if I launch an electric field of E omega, the polarization will now going to vary by these are the polarization terms.

Several time I mention that for this polarization term, we have a frequency component P omega and P 2 omega that is important here I am not now I am not considering about the optical rectification term. So, this is now vibrating with the frequency omega. This is now vibrating with the frequency 2 omega. This is a source term of 2 omega electric field and this is a source term of omega electric field ok.

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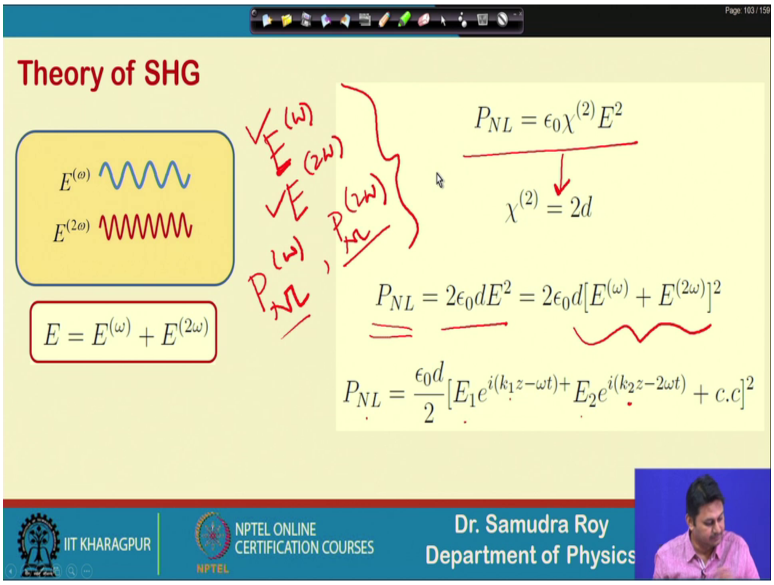

So, I know what is my electric field, right. So, electric field I have already derived. Now, the next quantity that we need to derive it is P non-linear of 2. So, I need to. So, E omega I know E 2 omega, I know next thing is to find out P non-linear omega. What will be the form of that or P non-linear of 2 omega? Because, in order to use the Maxwell's equation or solve Maxwell's equation, this four quantity is important;, this term and this term we know what is the what is the functional form of that, but this and this we need to figure out.

So, if I want to find out the second harmonic. So, P non-linear is nothing, but epsilon 0 chi 2 E square. Now, here we you going to use this chi 2 as 2 D this is some historical reason people are using this D mac matrix rather using chi 2. So, we will also follow the same thing and going to use in different books you find that this D they represent the chi 2 in terms of D.

But there is a related relation between chi 2 and D and that relation is chi 2 is equal to 2 multiplied by D. So, now, if I use this chi 2, then my P non-linear is 2 epsilon D and E square. Now, this E square term is containing 2 frequency component omega and 2 omega. So, omega 1 plus omega 2 and square of that so, P non-linear is now having E 1 E to the power of i k z minus omega t E 2 E to the power of i k z minus omega t and plus complex conjugate of that and square of that and half of that. Because, if I go back to our previous slide, let us go back to our previous slide.

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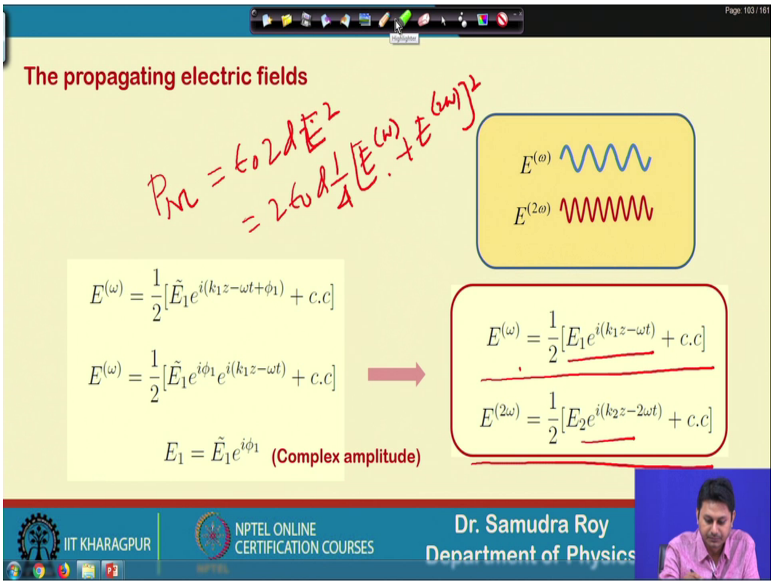

So, in our previous slide, here you will find that E 1 and E 2 is this. Once we have E 1 and E 2 in this form, now we need to find out what is my P non-linear. So, P non-linear has different frequency component, but we are using this notation and then E square. So, when I make E square. So, E is now 2 of epsilon 0 D 1 by 4 and square means E omega plus E of 2 omega square and E omega is this.

So, that is why, this quantity plus this quantity plus complex conjugate and square of that exactly the same thing we have done in this that slide. So, I should go back to that slide where making the square term here. So, E 1 E to the power of k z minus omega t E 2 E to the power of k z minus omega 2 plus this square of that. So, P non-linear is this.

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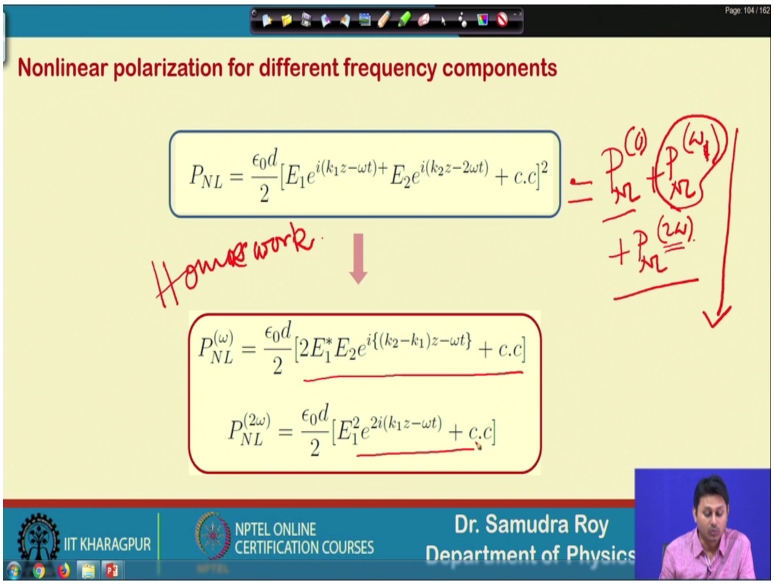

So, now, what we will try to find out what is P non-linear omega and P non-linear 2 omega because, when we do this entire calculation it is for sure.

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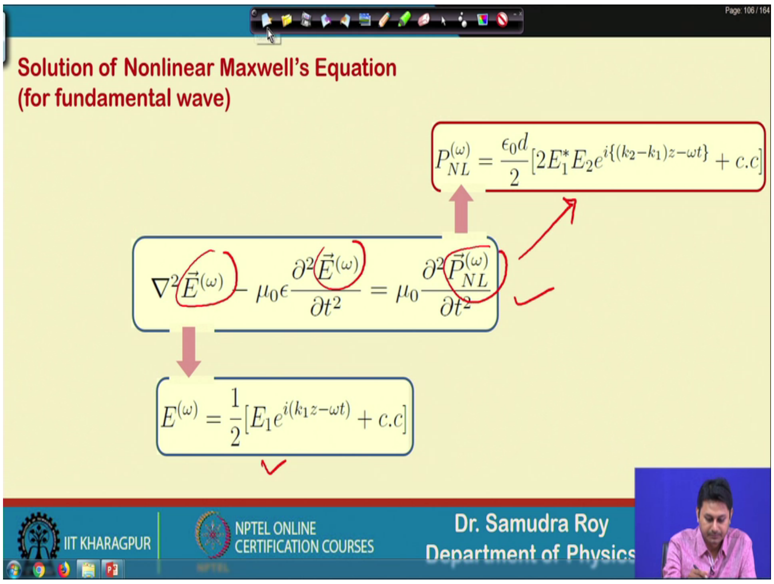

It is for sure that I will have I will have some component like P 0 P non-linear 0 plus P non-linear omega 1 sorry P non-linear omega because, there is no omega one plus P non-linear 2 omega this 3 components should be there. So, P non-linear omega is a frequency component that is generated with a frequency omega and P 2 omega is a frequency component of and P non-linear with frequency component 2 omega. So, now, if we do that. So, I can also put this is some sort of homework that how one can generate P non-linear omega from this P non-linear omega with this term. But I will going to explain that, but I want you the student to make it by your own hand do that by your by your own hand.

So, let me let me try to understand that how this P omega components 2 omega components are there P omega and 2 omega. So, omega components will have. So, when I make a square of this term. So, what will happen? We have one term which is square of this; that means, 2 omega will be there one term we have a square of this things.

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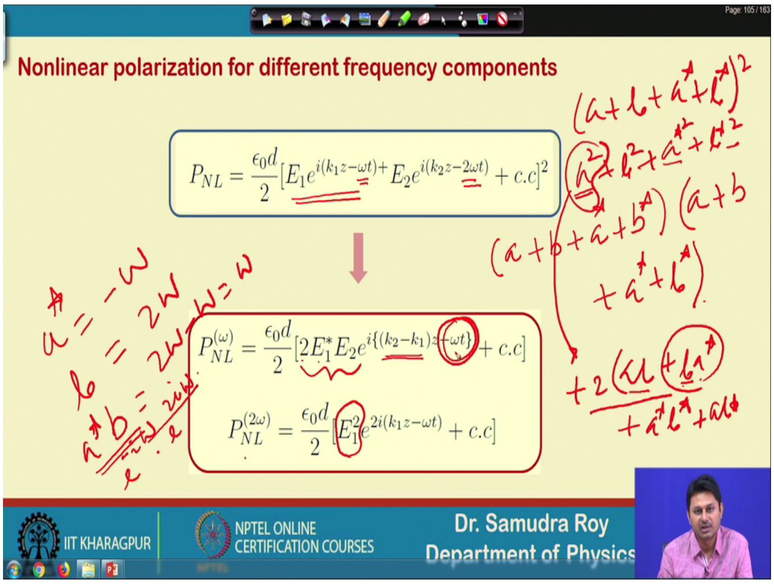

So, we have 4 omega term will be there and complex conjugate terms are there so, that means, we have a term say this is a plus b plus a star plus b star complex conjugate means the star of that multiplication of a plus b plus a star plus b star it is something like this

So, when we do that it is nothing, but a plus b plus a star plus b star square of that when you do the square of all this things we will have a square term plus b square term plus a star square term plus b star square term plus 2 of a b this couple term plus b a star plus a star b star plus plus a b star this kind of terms you will get. So, now, every time you need to find out what is a frequency component of that. So, a square term has a frequency of 2 omega because, this is omega b square term will having a frequency component of 4 omega. Because, it is square of that a star square is a frequency component of minus of 2 omega this is minus of four omega.

But when the cross terms are there then, there is a possibility that we will have some terms which has a frequency component omega and how you get the frequency component of omega very easily you can find that if I make a star b which is which is this term then what is a frequency component a star has a frequency component of minus omega b as a frequency component of 2 omega if I add it a star b will be the frequency component 2 omega minus omega which is omega why add this multiplication the frequency component is inside the exponential terms.

So, it will be E to the power i omega and then it should be multiplied by E to the power of one is E to the power minus omega and multiplied by E to the power of 2 i of omega. So, when you multiplied that this frequency component will go into add up. So, when we add up then we will have a k term like k 2 minus k 1 and then the frequency of omega. So, you will find that we will have a frequency component of omega, but most importantly this term E 1 star and E 2.

In the similar way, if I try to find out what is the frequency component of 2 omega for P non-linear it is nothing, but the first one that is a square term because only when you make the square term only the first square; that means, a square will give you 2 omega. So, that is why the amplitude is epsilon E one square here in this case the amplitude is 2 a b. So, this 2 multiplication by a star b. So, that is why 2 multiplication of a star and b with the corresponding exponential terms.

So, this exponential term basically gives you the frequency components and when we have all the frequency components, I can determine what is my P non-linear 2 omega and what is my P non-linear omega. So, now, we have all the information in our hands.

So, we are in the position to solve the Maxwell's equation because the P non-linear term and the electric field term both are there the solution of the Maxwell's equation for fundamental waves how to find out how the E omega will going to evolve if I try to find out so, I will going to use.

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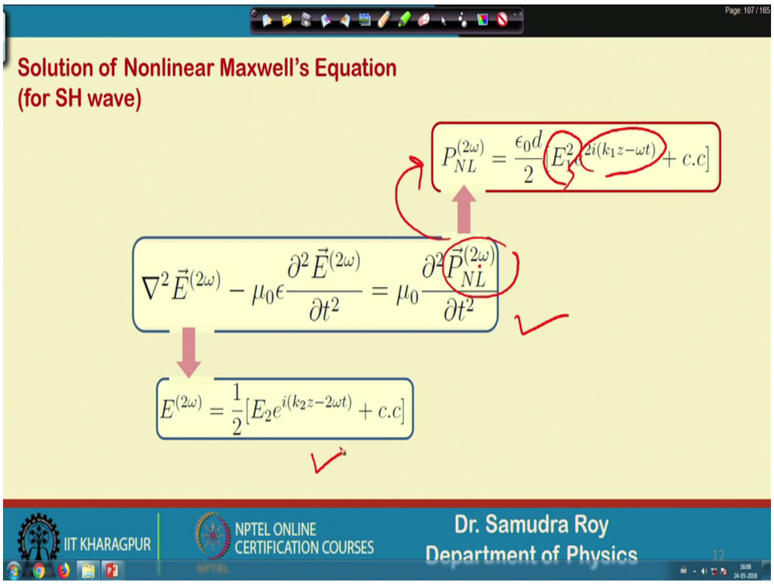

We will going to use sorry we will going to this equation this another equation where E omega E omega is this and P non-linear omega is this that we just derived in the previous in the previous page also for 2 omega also for 2 omega we will have. This is how the second harmonics will going to evolve. We will have this equation this is a non-linear Maxwell's equation and P non-linear we just extract that P non-linear is nothing but, this where the frequency component P non-linear 2 omega is the frequency component. And here, we find the frequency component is really 2 omega the important thing is the amplitude here.

So, we will have the expression of P non-linear 2 omega we have the already we have the expression of E. So, both the information is now in our hand. So, the next step is to solve the non-linear Maxwell's equations. So, now in the next class today we will going to I mean we understand how the non-linear Maxwell's equation can be useful to understand the second harmonic.

So, in the second harmonic, we find that it is important to find out the source term that is P non-linear 2 omega and P non-linear omega and once we have the functional form of both P and E with different frequency component we can use that in the Maxwell's equation. And readily, we can find the evaluation equation of the second harmonic wave and the fundamental waves. So, in the future classes, we just solve these non-linear Maxwell's equation with certain approximation and try to find out how second harmonic wave will going to evolve. So, with this note let me conclude here thank you very much for your attention. So, see you in the next class.

Thank you.