Modern Optics
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Lecture – 09
Wave propagation in anisotropic media (Contd.)

So, we have seen the various aspects of the anisotropic medium, particularly the permittivity tensor and the principle axes system.

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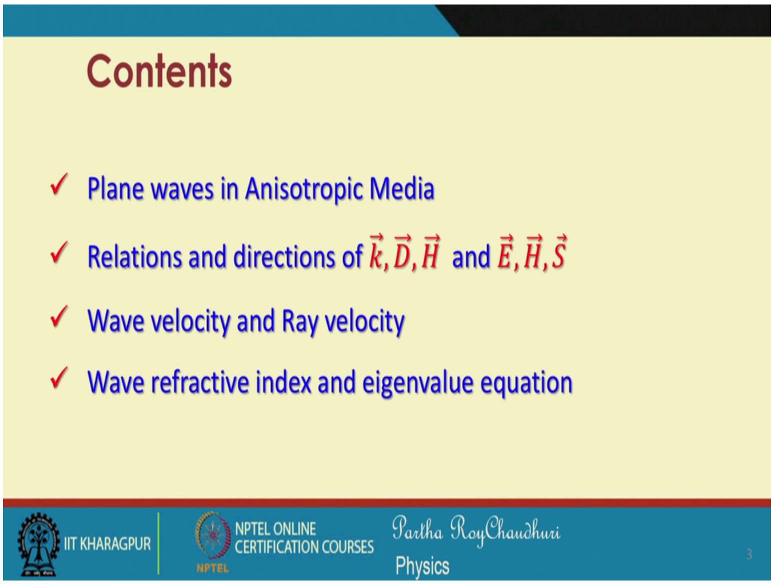

Now, we will discuss the propagation of plane electromagnetic waves in anisotropic media. And in doing so first we will understand the relation between the relation amongst this, this vectors k, vector D, vector H, vector E, H and S; and their directions. And in the process we will also discuss the wave velocity and ray velocity. These are 2 different things that we will try to understand that the wave propagates along k direction whereas, the energy propagates the ray moves along the pointing vector and the direction of the pointing vector. Then we will look at the refractive index and general eigenvalue equation that will represent the anisotropic medium.

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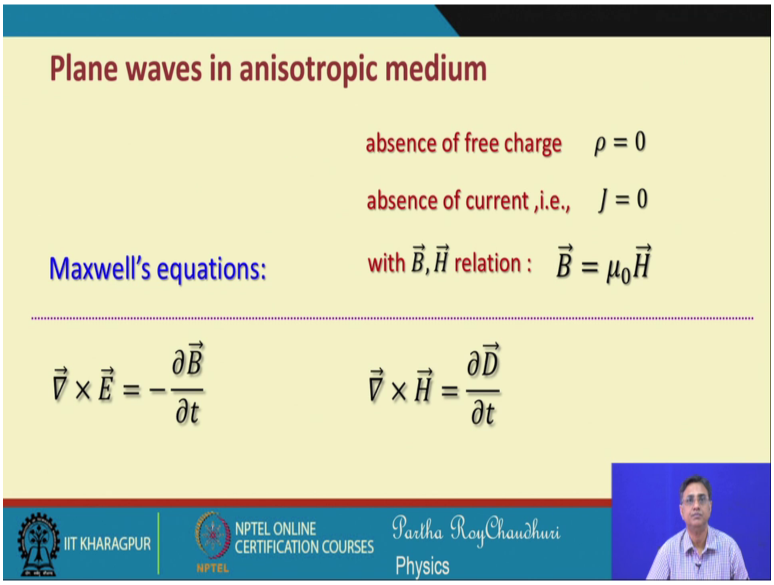

So, let us start again with the medium properties, you again up assume that the medium is free of free charges that is rho equal to 0 and there is no current that is i equal to j equal to 0. And again we considered this constitutive relation B equal to mu naught H. And we also considered the 2 Maxwell's curl equation that is del cross E equal to minus del B del t and del cross H equal to del t del D del t.

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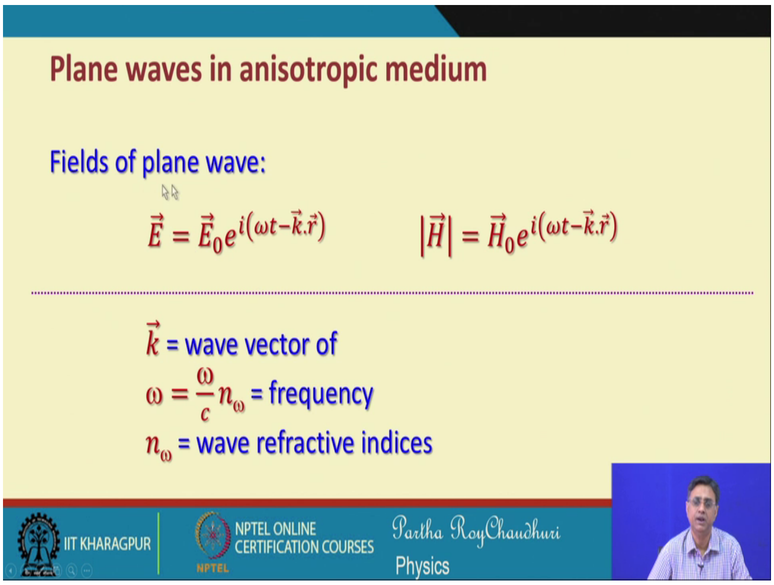

So, we have also seen the solution of the wave equation, the plane wave solution is in the form of E equal to E naught, which is the amplitude of the electromagnetic wave, the electric field vector and the phase in terms of omega t minus k dot r. And similarly for the magnetic field, we can represent this as H naught E to the power of the phase factor omega t minus k dot r, where k is the wave vector omega is the frequency and n omega is the wave refractive indices right.

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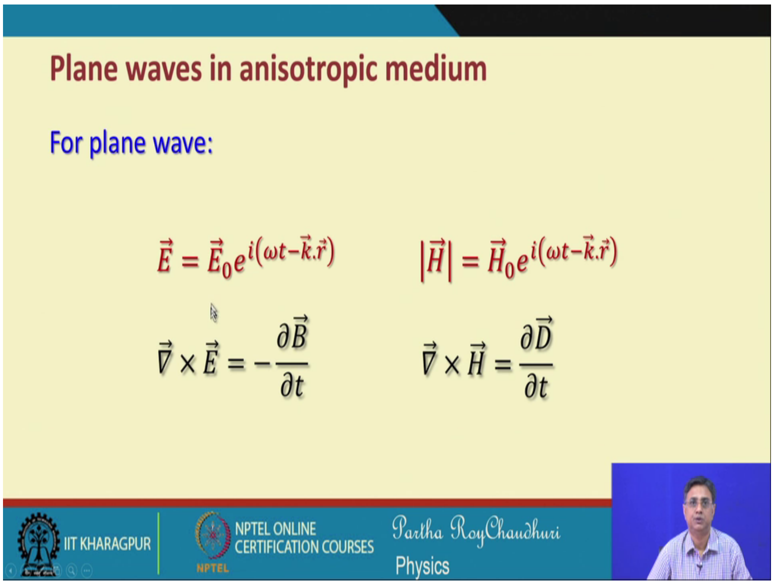

So, if we assume this solution we can plug in the solution into the curl equations for E and H.

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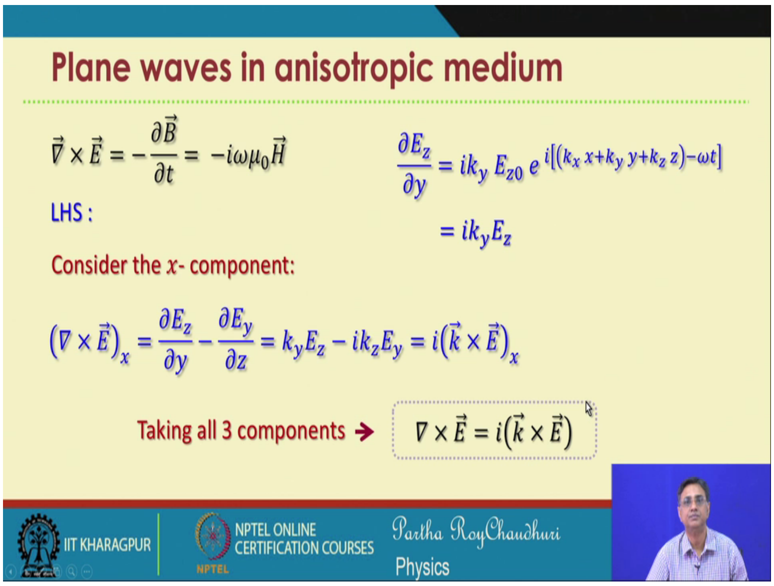

And let us see that; what happens if we consider only the x component of the curl. We have seen in the similar derivation in the case of isotropic medium. So, in this case del cross E, if you take the x component this will give you del E z del y minus del E y del z. Which is equal to k y E z minus i k z E y, there should be 1 i missing here. So, i into k cross E the x component, because this quantity can also be represented as k cross E. So, del E z del y this gives you i k y E z therefore, for x component if this is true for if you consider all other components like, del cross E y components del cross E z component all of them will contribute this quantity. And if you put together then del cross E can be written as i into k cross E. So, this is the general relation of the electric field, but this del cross E is equal to minus del B del t. So, we will bring in this quantity on the right hand side.

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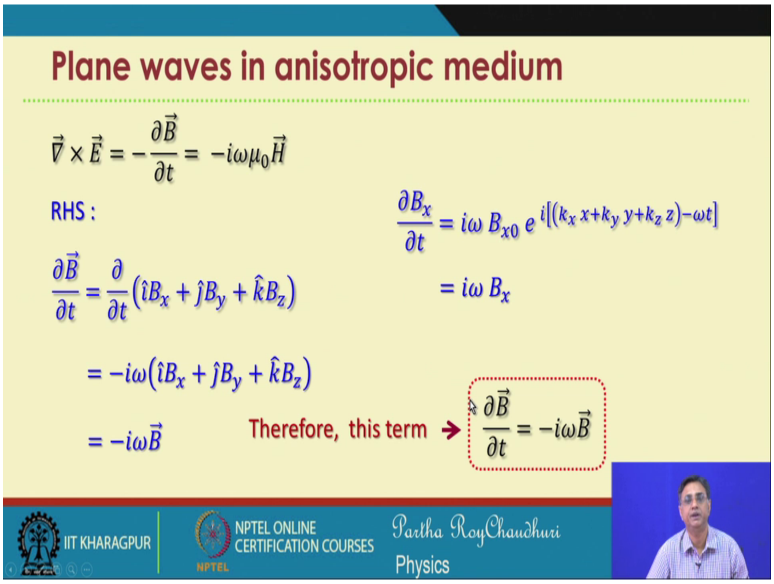

So, del cross E the right hand side gives you del B del t this vector. So, i omega i B x plus j cap B y plus k cap B z put together will give you; minus i omega B. Therefore, so this is for x component for y component del B y del t will give you i omega B y and del B z del t will give you i omega B z. So, therefore, this term that is the right hand side del B del t will give you simply minus i omega B.

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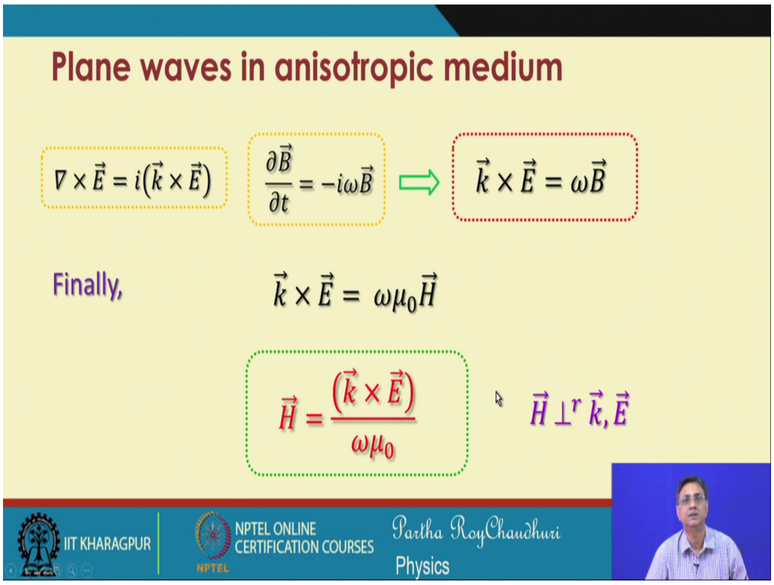

So, we have the left hand side as well as the right hand side, if you bring them together then we can write k cross E equal to omega mu naught H. And this we have seen in a similar way in the case of isotropic medium also. So, H can be represented in this form; that k cross E by omega, this is very interesting to note that. And this property will be used everywhere when you know the electric field how to bring out the magnetic field using this relationship, but from here we can understand that this the direction of H is in a plane is perpendicular to the plane which contains this k and E vector.

That means H is perpendicular to both k and E that is, the propagation direction and the electric field direction, but that does not mean that k and E are mutually perpendicular we have to see how they follow this relationship.

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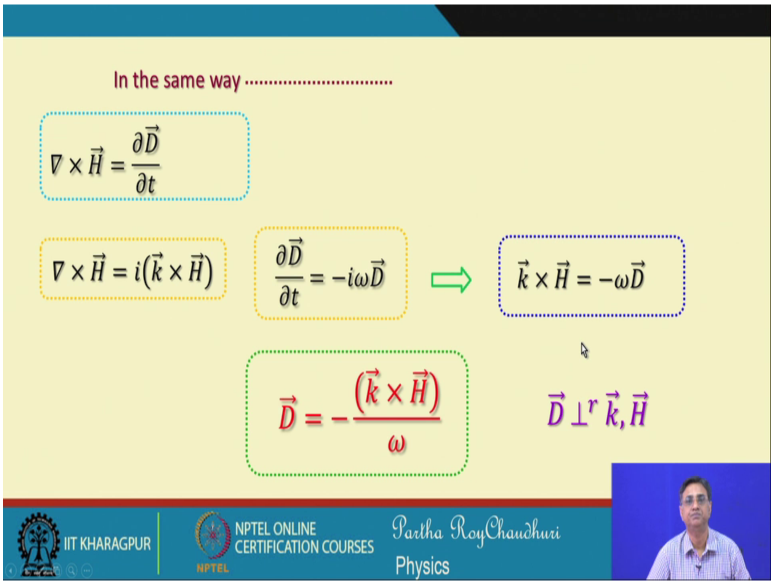

And to do that we will follow in the same way the del cross H equal to del D del t, where this del cross H will give you i k cross H in the same way and for the right hand side that is del D del t will give you minus i omega D. So, if you arrange this put together these 2 things then you can write k cross H is equal to minus omega D. So, this is again a relation relationship which tells you about the direction of the k, H and D. And you can see that D can be written in this from; which is again a very useful property that, when you know the displacement vector you can explore the direction of the of the magnetic field with respect to the direction of propagation.

So, from this relation you can see that D is perpendicular to a plane which contents k and H. So, D is perpendicular to the propagation vector and is also perpendicular to the magnetic field direction. So, therefore, we have these 2 things that H is perpendicular to k H is also perpendicular to k E and D is perpendicular to k and D is also perpendicular to H.

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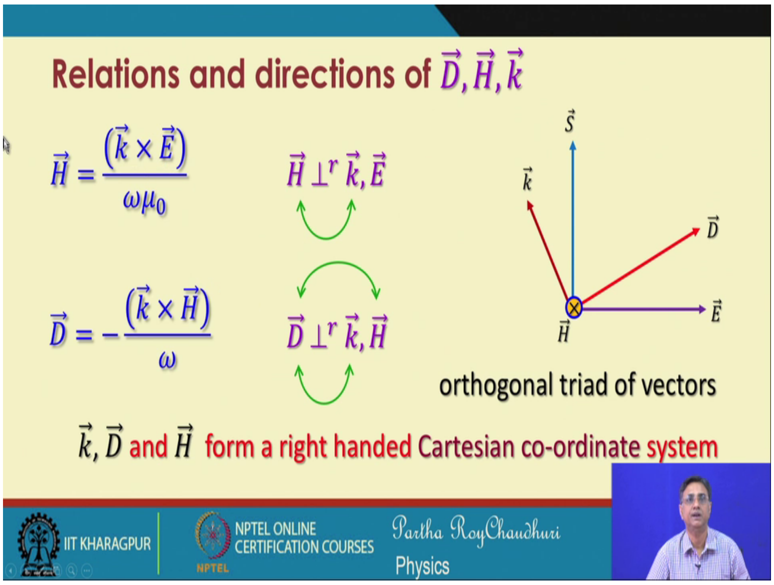

So, let us put all of them together from this relation we find H is perpendicular to k, H is perpendicular to E and likewise D is also perpendicular to k. So, if you take this property H perpendicular to k and D perpendicular to k, then k D and H they form a right handed Cartesian coordinate system. That is this k, D and H they are forming a coordinate, right handed coordinate system.

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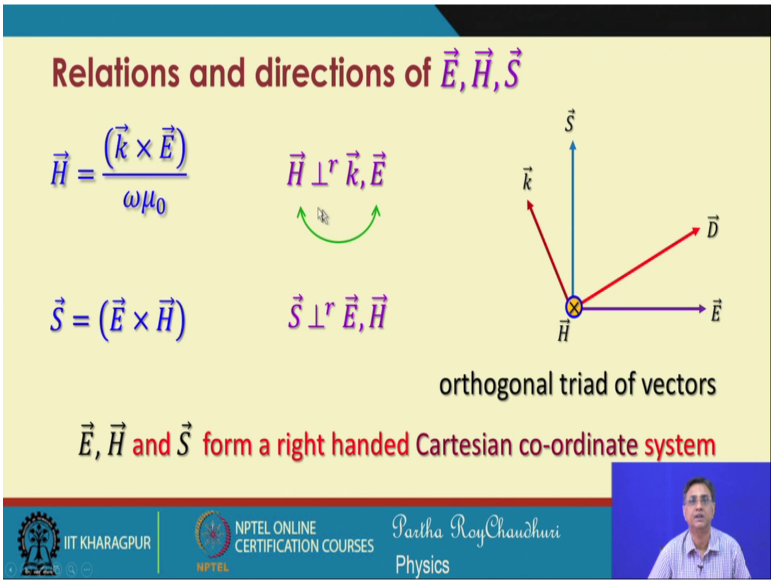

And, for the k E and H, we can simply locate that H is perpendicular to E H is perpendicular to E and k we have seen individually they are perpendicular to H and perpendicular to E. So, S is perpendicular to E and H. So, you have this direction of propagation of the pointing vector that is; perpendicular to the H vector and E vector. So, this E, H and S again form a right handed Cartesian coordinate system.

So, you have seen that k D H, k D H forms a Cartesian coordinate system and this E H S they form Cartesian right handed Cartesian coordinate system, but in general they are they are different in terms of their angle.

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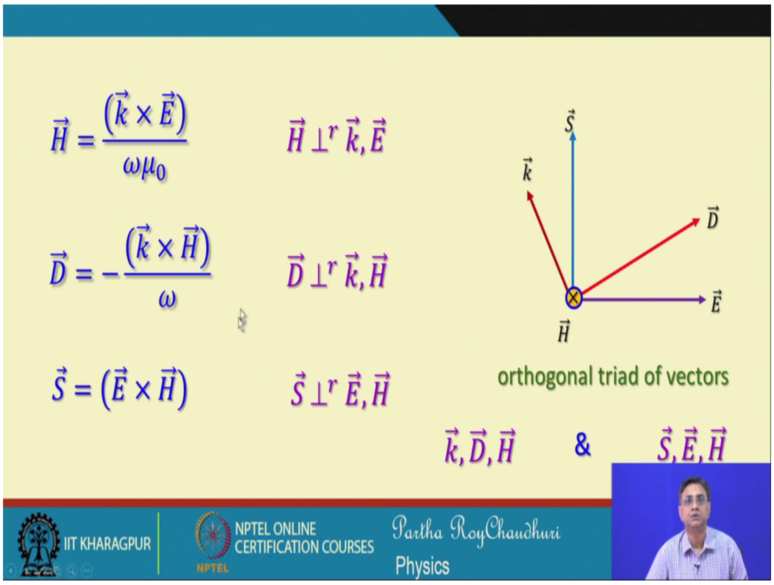

So, to summarize this H is perpendicular to k and E, D is perpendicular to k and H, S is perpendicular to E and D, E and H. So, you have k D H one set you have S E H another set.

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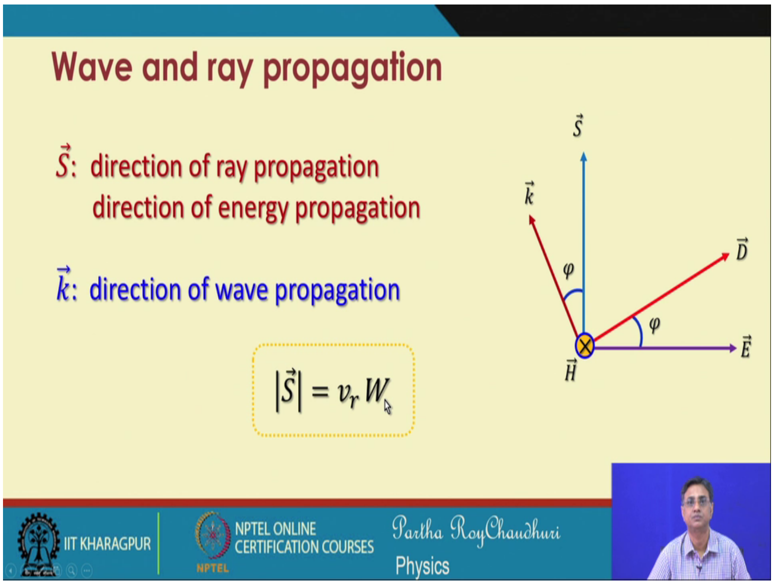

This S represents the direction of the ray propagation and we have seen in the earlier discussion that while talking about the pointing vector, that it is the vector along which the energy propagates. So, the ray propagation and energy propagation the take place along the pointing vector direction whereas, the k vector it tells you about the direction of propagation of the electromagnetic wave. And this pointing vector is related to the energy density, by this equation that S mod of that is equal to the ray velocity multiplied by the energy density.

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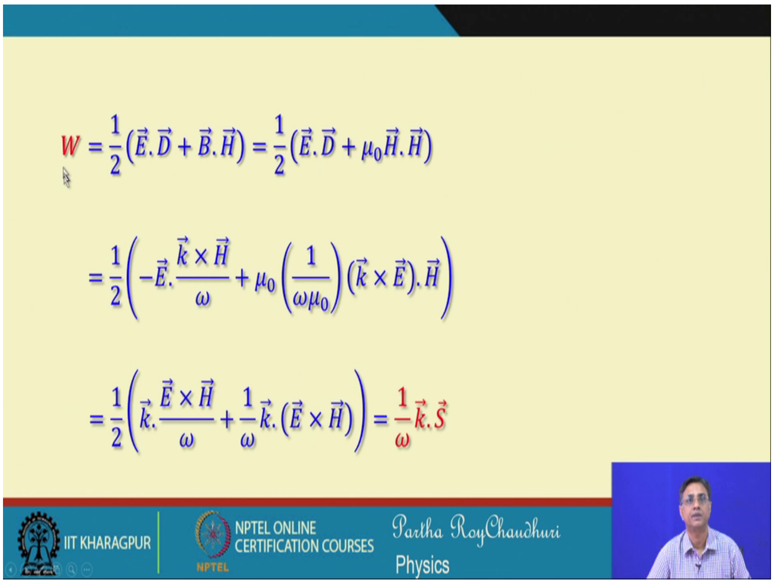

So, omega the electrical and magnetic energy density can be represented as half E dot D plus B dot H. If I use this constitutive relation for B then half E dot D plus mu naught H dot H, we can write for the energy density of the electromagnetic waves. So, if we simplify this we can write this equal to half minus E k cross H, because for D we can write k cross H by omega with a minus sign. And for B we can write that k cross E by mu omega and 1 mu is already here. So, k cross E dot H, we can see that k dot E cross H by omega and k dot E cross H by omega. So, 2 of them are identical terms and there is a half outside. So, if I take put together then we can see that this will represent simply the pointing vector E cross H.

So, I can write this energy density associated with the electromagnetic waves is equal to 1 upon omega times k dot S that propagation direction and the direction of the pointing vector. This is a very interesting finding, it tells you that the energy density associated, with the propagation of the electromagnetic waves is a component of the k vector along the direction of the along the direction of the pointing vector that is along which the energy flows.

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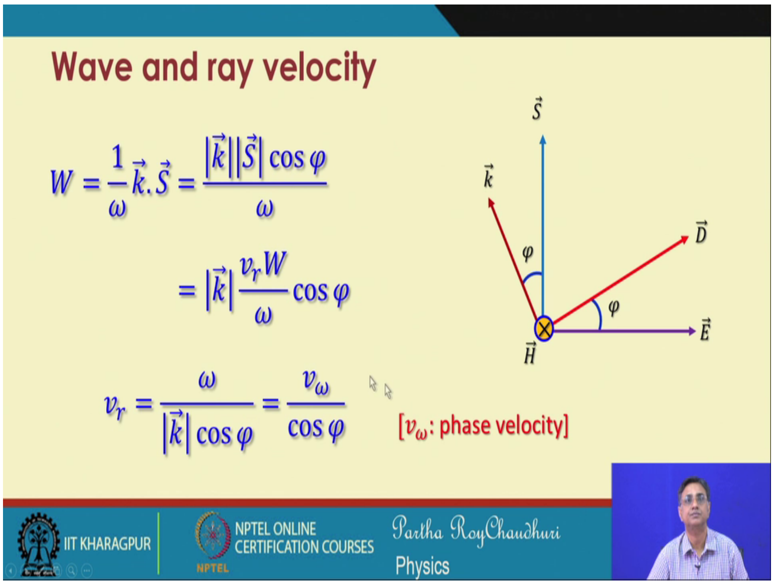

Now, let us look at the wave velocity and ray velocity. You see that the energy density just now we have seen that the energy density can be represented as 1 upon omega k dot S. So, if I use this relation W equal to 1 upon omega k dot S, but k dot S we can write explicitly like mod of k mod of S by omega into cosine phi, where phi is the angle between the direction of propagation of the electromagnetic wave and the direction of propagation of the energy that is the pointing vector. So, this angle is phi. So, the wave in general travels along this direction in an anisotropic medium whereas, the energy flow takes place along this direction. So, we can rewrite this equation S cos omega t S cos first phi will give you that v r into omega into W by omega cosine phi.

But, this you can write in this form because v r omega will give you this 1 upon cosine phi. So, we are you can write in this form omega by k mod cosine phi. So, we get a beautiful relation that v r equal to v omega by cosine phi, because omega by k gives you the wave velocity. So, the ray velocity and the magnitude of the ray velocity and that of the wave velocity they are related by a factor of cosine phi, which is the angle between the direction of propagation of the energy. Whereas, k the 1 which represents the direction of propagation of the wave they make an angle phi between them.

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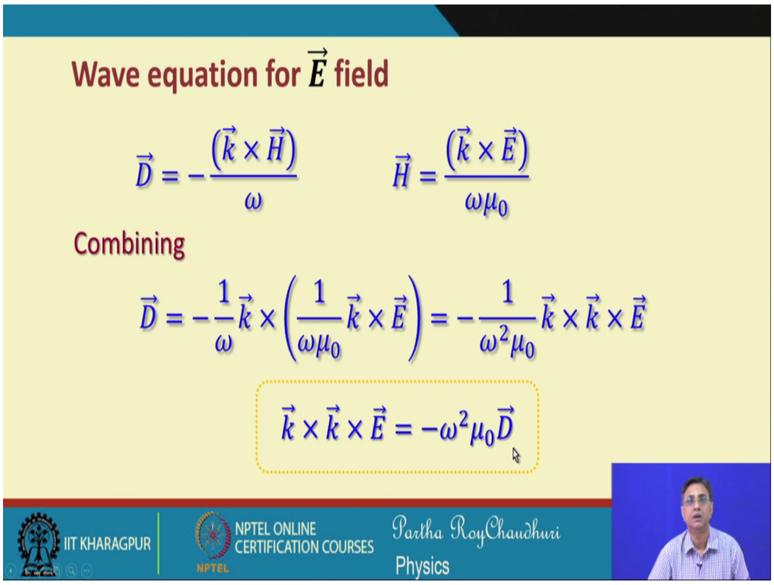

So, the wave equation for E field can be arrange from these 2 equations we have seen D equal to minus k cross H by omega, and H equal to k cross E by mu omega mu naught omega. So, if I combine these 2 equations, that is if I write in place of H this quantity in place of H if I substitute then we can write D equal to minus 1 by omega k cross. And for H we have brought in this 1 by omega mu naught k cross H k cross E. So, you can write this as omega square mu naught minus of that k cross k cross E. So, this is a vector triple product. So, with end up with this because the right hand side is omega square mu naught D. So, we can write this equation in this form k cross k cross E is equal to minus omega square mu naught D.

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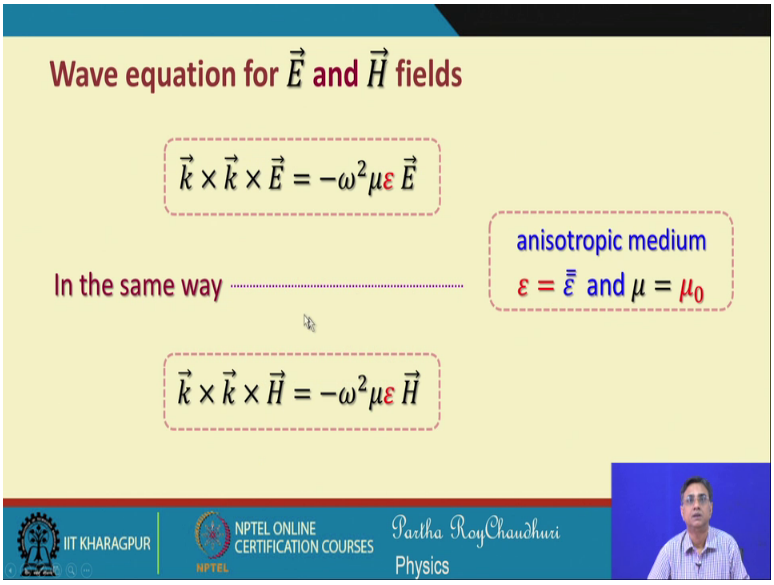

So, this represents the wave equation because D you can now write as epsilon E, where epsilon in case of anisotropic medium is a permittivity tensor. And similarly for the magnetic field we can also arrive at this equation k cross k cross H equal to minus omega square mu epsilon H, by following the same recipe. So, we have a set of equations representing the wave equation for the electric field and the magnetic field in an anisotropic medium.

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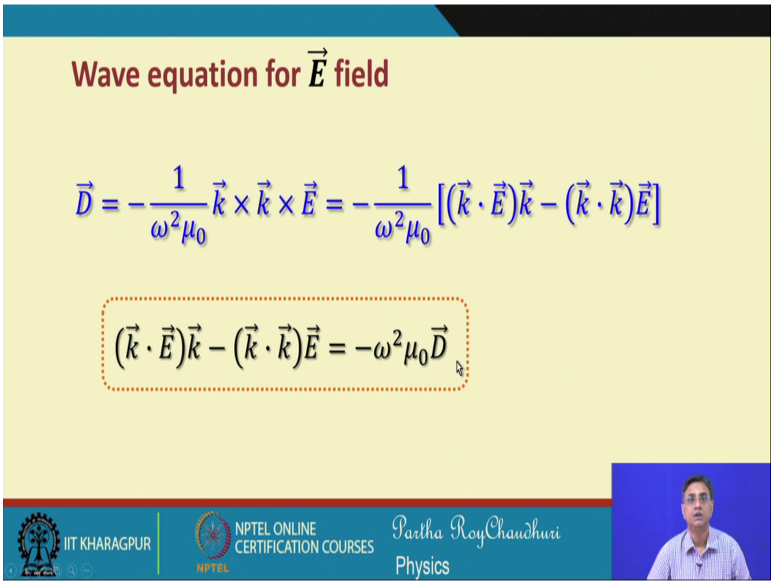

So, the wave equation for E field can be written in this form, if you just reorganize this equation that is E equal to k cross k cross E by minus omega square mu epsilon naught.

Then 1 by omega square mu can be written in this form k cross k cross E so, but k cross k cross E which is equal to k dot E k minus k dot k E is equal to this, but you know that k dot E is equal to 0, we have seen that the direction of propagation and the electric field they are mutually orthogonal to each other. So, therefore, this quantity does not arise, so we can write that equation minus k square E is equal to minus omega square mu naught D.

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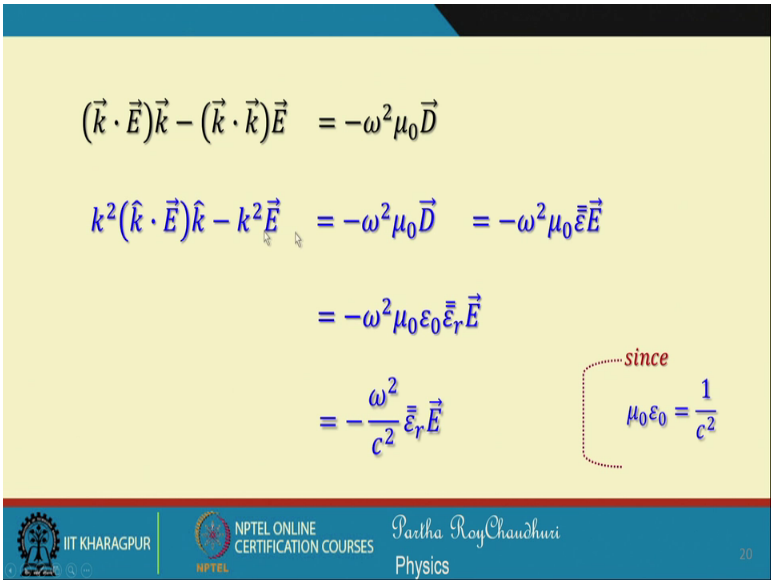

So, we write this equation because as are mentioned k dot E equal to 0. So, this minus k square E is equal to is equal to minus omega square mu naught D, but D we can write explicitly in the form of in terms of the permittivity of the free space the relative permittivity, which is a tensor multiplied by this electric field. So, we can write that minus k square E is equal to minus omega square by c square epsilon r into k, because omega square mu naught epsilon naught is equal to 1 by c square. So, this term put together will give you omega square by c square.

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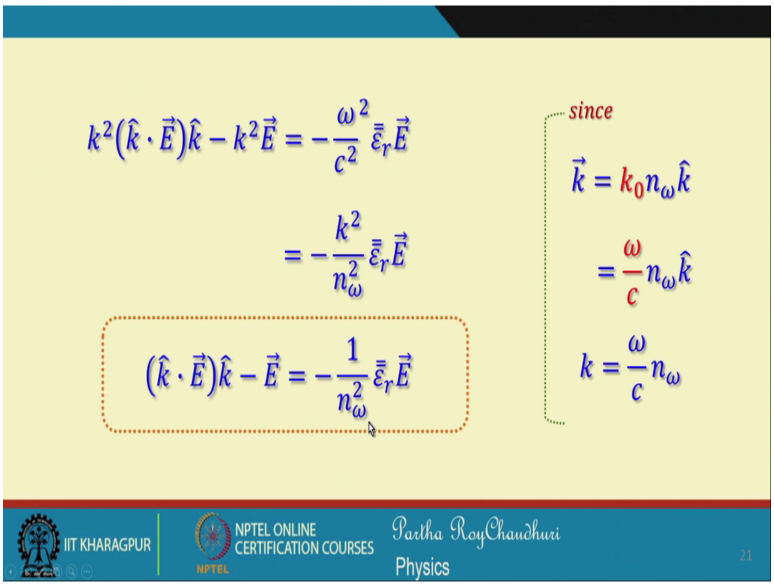

So, we can write this equation that this k square k dot E anyway this quantities 0. So, minus k square E equal to, minus k square E equal to 0. We will work with this equation k equal to k naught n omega k cap. So, that is a direction of the k with the magnitude of k add n omega. So, we can write k equal to omega by c. So, you can rewrite this equation again in this form.

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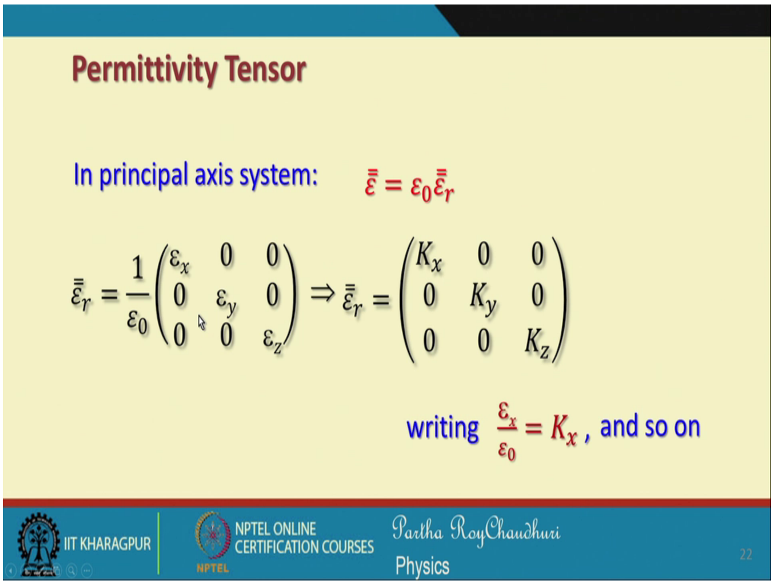

So now, in the principal axis system epsilon r is equal to 1 by epsilon naught epsilon x epsilon y epsilon z. So, this relative permittivity takes the form of this, if I replace this epsilon x by epsilon naught by a quantity k the dielectric constant k x k y and k z then this can be written the epsilon r permittivity tensor for the principle axes system can be written in this form.

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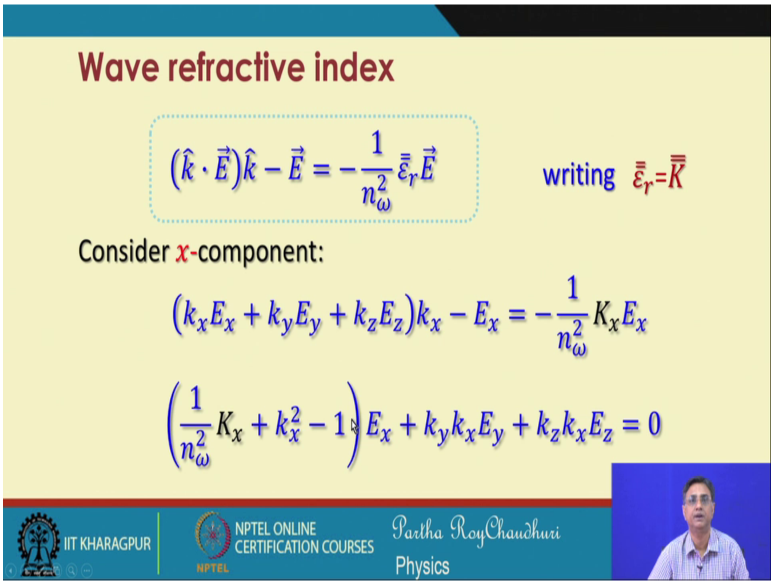

And, then we can rewrite this equation k dot E, k cap minus E equal to 1 by n omega square epsilon r into E. And then writing this equation in this form we can write; the x component of the of this equation as k x E x plus k y E y plus k z E z into k x minus E for E, E x for E and the x component of this quantity 1 by n omega square k x E x. If I rearrange this equation then you can write in this way 1 by, because I put all k xs together, k y E y together and E z together. So, you can rewrite this equation in this form. And this is for x component for the y and z component we can write similar equations.

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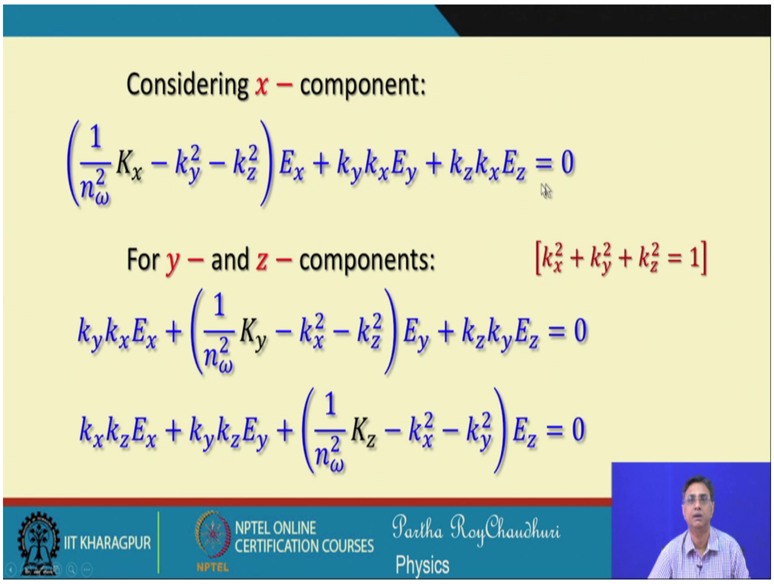

So, considering x component we have obtained this equation, if you considered y and z component we can write 2 more equations representing the y and z components in this from. You can see that now this term the similar term has gone into E y and in the z component this similar term has gone into E z. So, you end up with a set of 3 simultaneous equations, which can be put together into the in the form of a matrix equation and the equation will look like this.

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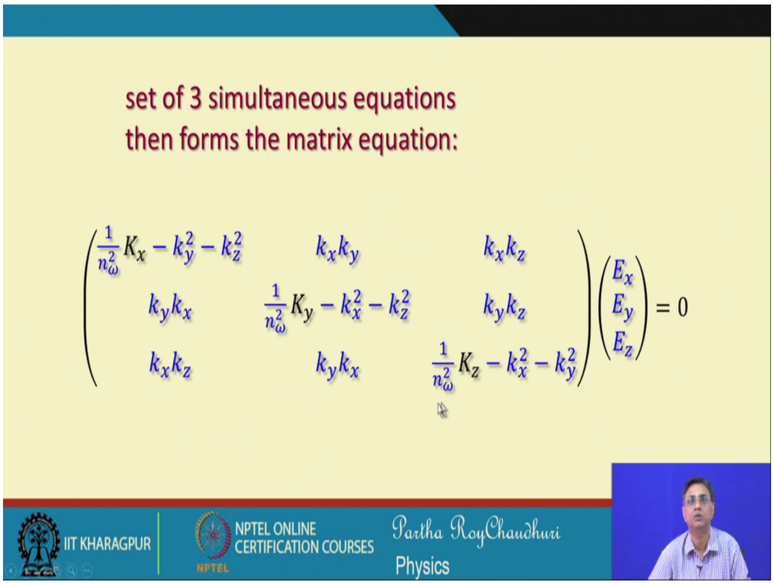

So, 1 by n omega square K x minus k y square minus k z square and dissimilar term here in all diagonal positions and k x k y k y k x k z k k x k z k z k x. So, they will be occupying the up diagonal position. So, this equation this matrix equation will be the characteristic equation which will represent the an behavior of anisotropic medium.

Now, this matrix equation to solve this we have for a non-trivial solution, the determinant of this equation will be equal to 0.

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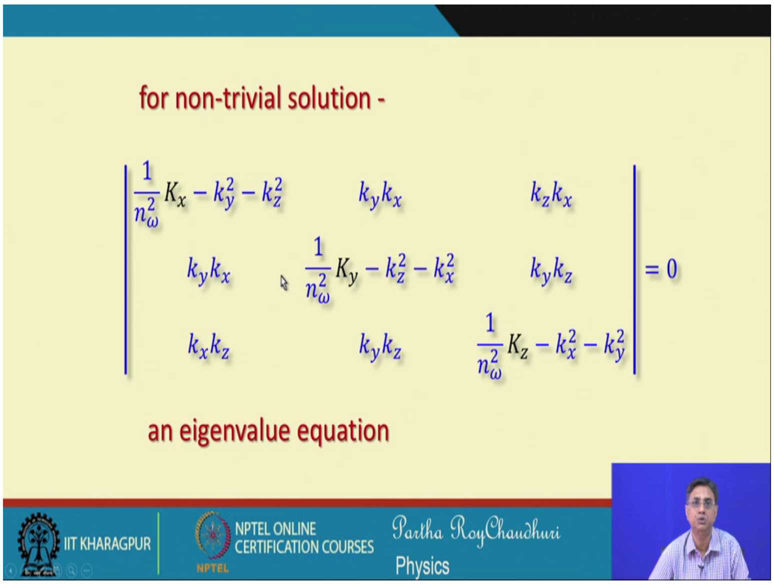

So, this is an eigenvalue equation which will give you the values of n omega.

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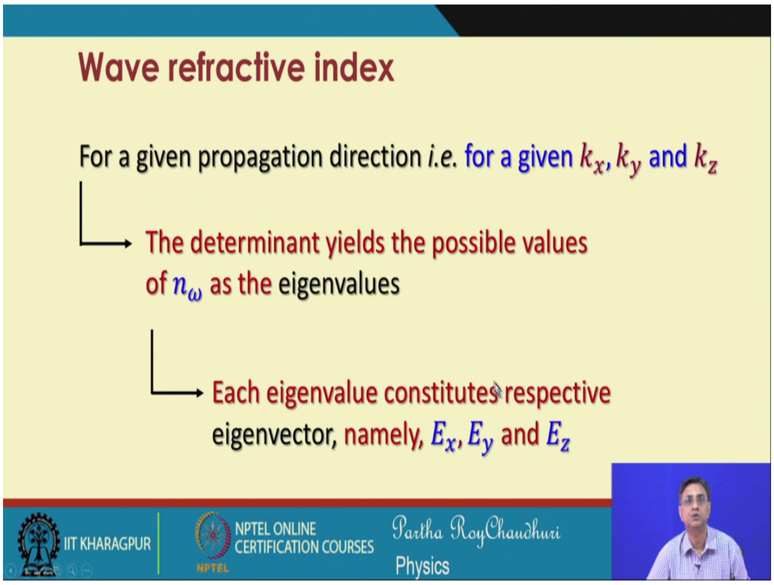

And, to solve this you will see that; we have to consider a direction of propagation of the electromagnetic waves that is, if you specify the direction of propagation then we know the k x, k y and k z. The determinant equation will yield the possible values of n omega as the Eigenvalue, because it involves only k x, k y and k z these are the properties capital K x capital K y and capital K z these are the properties of the medium, this is the property these are the properties of the direction of the electromagnetic wave. So, knowing this direction and knowing the property of the medium we can we can find out the values of n omega with respect to that direction of propagation. Each eigenvalue constitutes the respective eigenvectors the E x, E y and E z electric field components.

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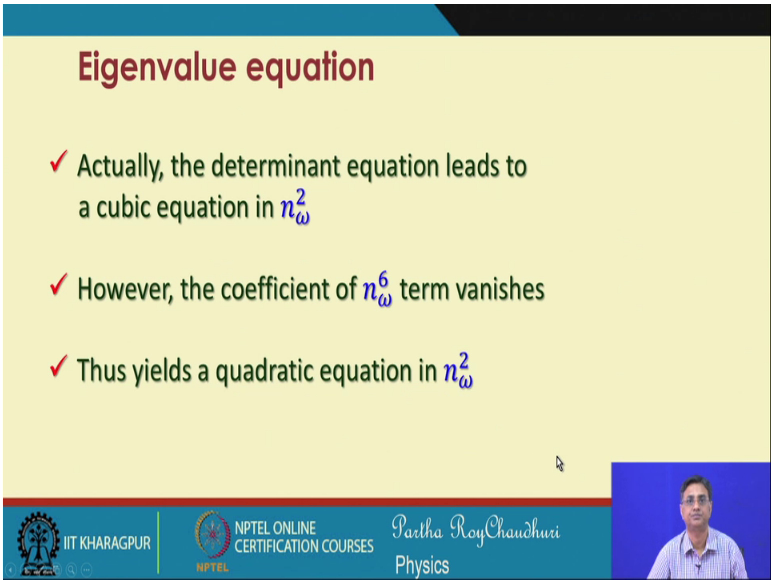

So, actually this determine that this determinant equation because it is it involves n square n square and n square. So, this will give you a cubic equation of n square that is, n to the power of 6. So, with this determinant equation gives you a cubic equation in ns n square. So, that the coefficient of n square of 6 will be there, but interestingly all the coefficients of n to the power n omega to the power of 6 will vanish, resulting in a quadratic equation in n omega square.

So; that means, we will have a quadratic equation which will involve only n omega square that is, there will be 2 values of n omega for this equation.

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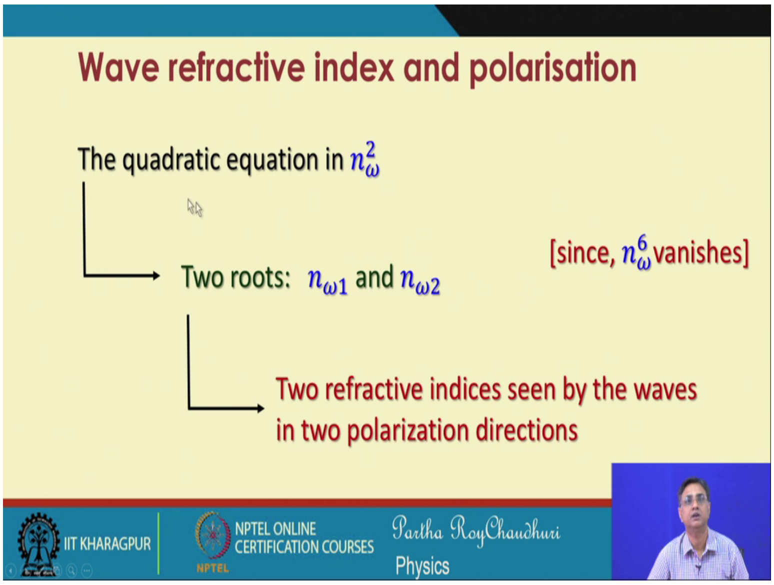

So, there will be 2 roots each of the roots will be called n omega 1 and n omega 2. These are the 2 refractive indices waves refractive indices as seen by the waves along the directions when given the particular direction of propagation of the electromagnetic waves. So, these 2 refractive indices seen by the waves along the direction will be the direction of polarization. We will learn more about this polarization direction at the connection with the refractive indices.

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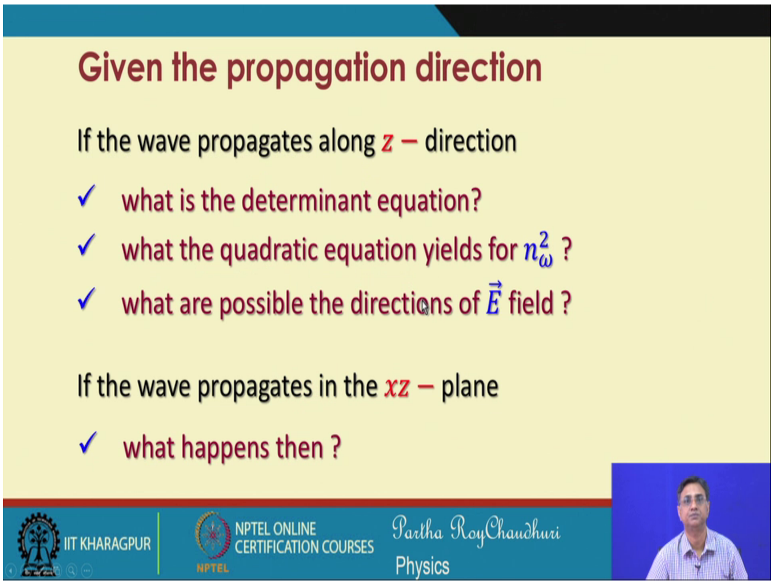

So, the wave propagates along z direction, if you consider this situation then with this determinant equation we should be able to obtain the n omega values because this characteristic equation the eigenvalue equation should tell me that there will be 2 n omega values 2, wave refractive indices associated with any direction of propagation of the electromagnetic waves in the medium. So, if I specify the direction of propagation of the wave in the medium for example, a case when the electromagnetic wave, which is the simple most case that it propagates along the z direction. Then there will be to a n omega values which will correspond to correspond to the 2 mutually orthogonal access that is x and y.

So, in that case what will be the mathematically, what will be the quadratic equation that gives you for n omega square. If I can find out the n omega values for this particular direction of propagation, then we should be able to find out the direction of electric fields. If the wave propagates in the x z plane then also we can do the same operation, because this is a situation which is different from the simple case of the wave propagating along z direction. So, when we analyze this situation that the wave is propagating along x z plane or y z plane or z x plane, then what happens? So, we will try to study this properties in the next section.

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So, in this case we have seen that how to write the wave equation for the plane waves travelling in anisotropic medium, and in the in doing so we actually established the relations in between the k, D and H vector fields, and also the E H. And S we have seen that this 3 k, D, H and E, S, H they individually form a an orthogonal tired of vectors. We have also seen the wave velocity and the ray velocity their relationship, we have seen that the wave propagates along the k direction with a velocity, which is in an isotropic anisotropic medium in general different from the ray velocity which travels along the direction of the pointing vector that it is S. And they are related this wave velocity and the ray velocity, they are related by factor of cosine phi, which is the angle between the direction of the wave propagation and that of the ray propagation.

After that we tried to work out the eigenvalue equation for which is also the characteristic equation for the anisotropic medium, and from there we have learnt how to solve this equation to extract the 2 different refractive indices the those are associated with a given direction of propagation of the electromagnetic waves. If we consider various cases of the propagation of electromagnetic waves in different directions, knowing the property of the of the medium we can identify we can find out the roots of the eigenvalue equation, which will give you the 2 refractive indices associate with the propagation of the wave along the direction. And the direction of electric field vector will be the direction of polarization.

Because there are 2 refractive indices, there will be 2 different directions of electric field vectors and therefore, we can electromagnetic wave there would be, there will be only 2 direction of polarizations in general. And these 2 direction of polarizations will be the direction of the D vectors that is, the displacement vectors. So, in the next section we will also discuss the individual cases of the wave propagating along a simple x axis or z axis, and also the case where the wave is propagating along a y z plane or x z plane or so.

Thank you.