Modern Optics
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Lecture – 05
Maxwell's equations and electromagnetic waves (Contd.)

Today we will discuss the propagation of electromagnetic waves and energy flow this is a very interesting topic and will be useful throughout our discussion Energy Flow and Poynting Theorem.

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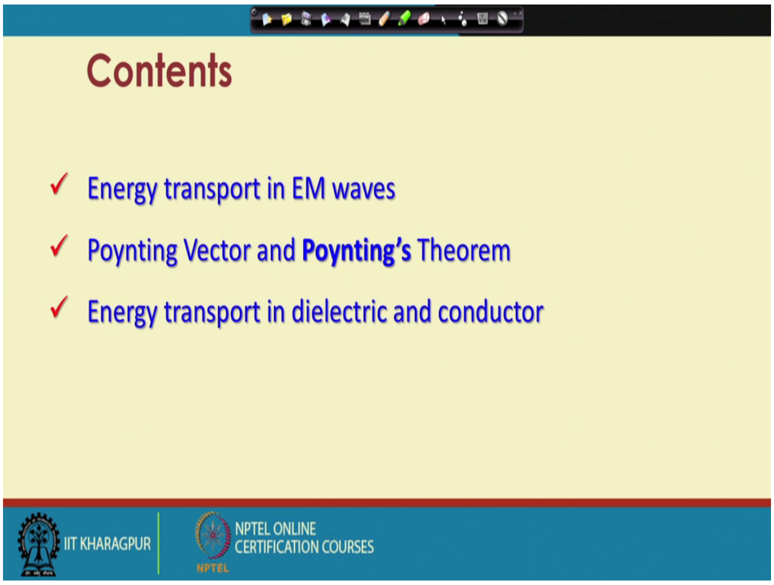

So, we will be discussing under the following heads that energy transport in electromagnetic waves in a certain volume of the space through which the electromagnetic waves are travelling, how the energy is entering and how much of the energy is leaving out of that space that will clearly quantify will try to understand how the loss is taking place within that volume.

Then in the process we will work out the Poynting vector and the Poynting theorem look at each of the terms of the Poynting theorem. Followed by this we will take two example cases, the energy transport in a dielectric medium and we will also considered the transport of energy in a conducting medium.

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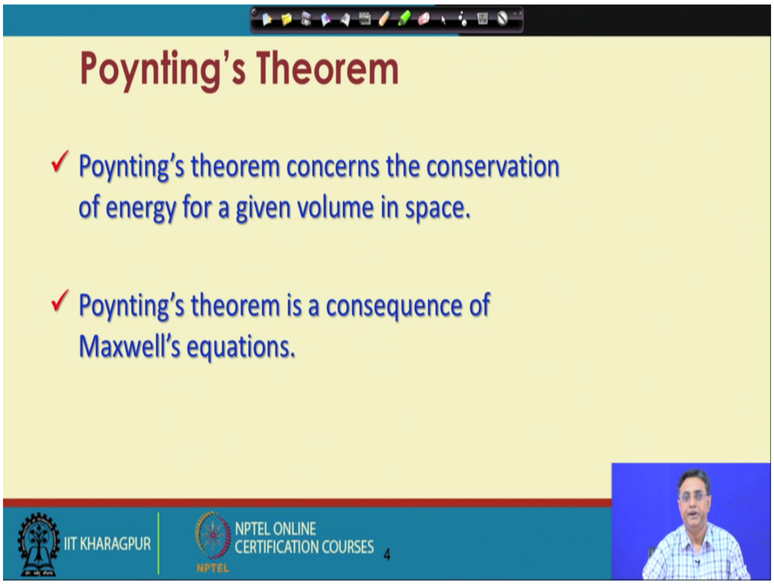

So, Poynting theorem, Poynting theorem basically concerns the conservation of energy for a given volume in space. As I have mentioned that when the electromagnetic waves are travelling through certain region of space within the volume the energy, how much part of the energy is lost and how much part of the energy is flowing out; we will correlate these two things and this will be a will be described through this Poynting theorem which is actually a consequence of the Maxwell's equation and we will start with the Maxwell's equation to arrive at the conclusion.

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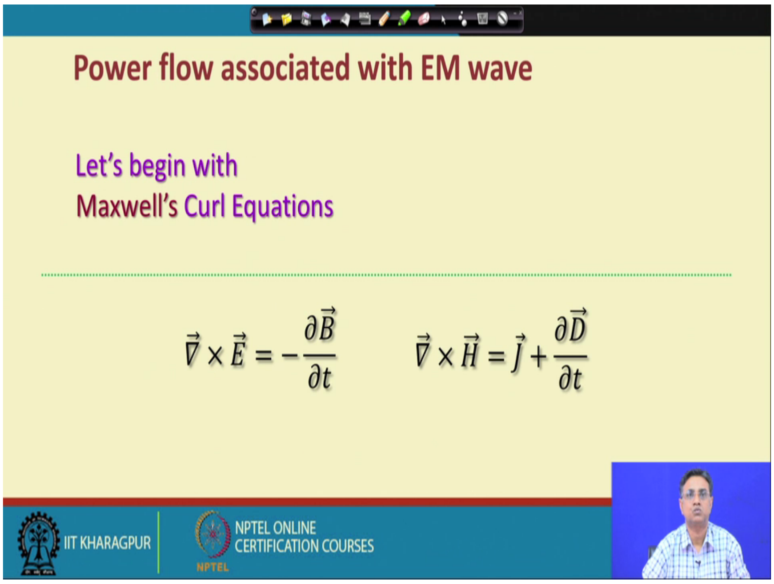

So, let us begin with the Maxwell's curl equations, we have these two famous curls equations that del cross E equal to minus del B del t and del cross H equal to j plus del D del t where B is the magnetic field and D is the displacement current j is the induction current.

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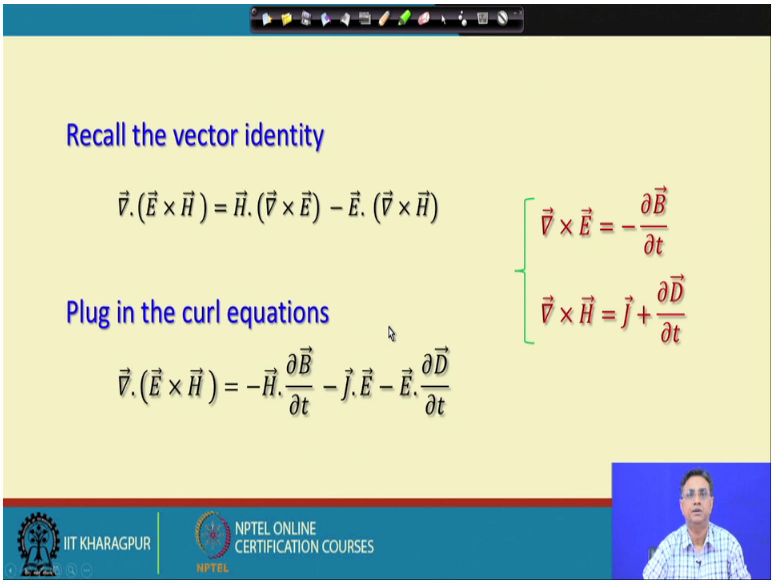

So, starting with these two equations, we will organize the Poynting vector; let us recall that the vector identity that is, del dot E cross H. This left hand side can be written in the form that H dot del cross E minus E dot del cross H.

So, if we plug in the curl equations for this del cross E and for del cross H from here that is del cross equal to we will substitute minus del B del t and for del cross H will substitute J plus del d del t. If you do that this quantity H dot del cross E becomes H dot del B del t, because del cross E is now replaced by minus of del B del t and for del cross H if I substitute this quantity J plus del d del t, then it becomes minus del j dot E minus E dot del d del t. So, this is the equation which involves the E cross H divergence of that.

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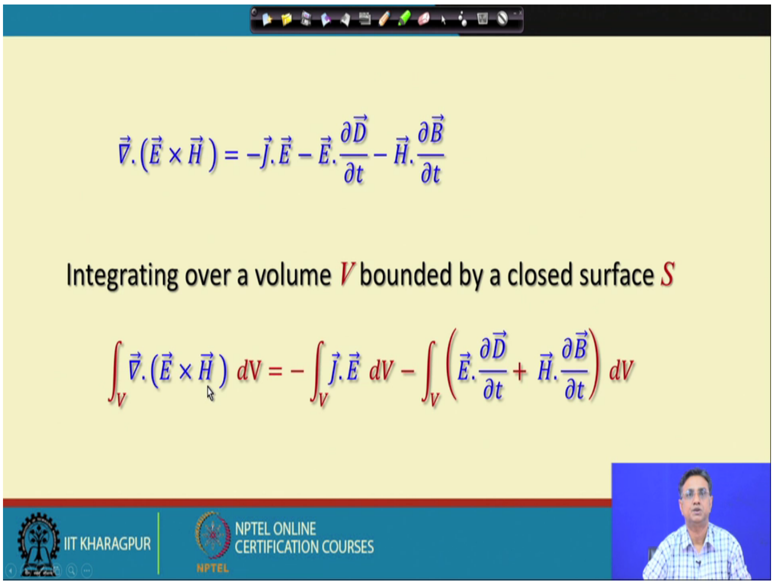

So, let us look at this; del dot E cross H minus B equal to minus j E minus E dot del d del t minus H dot del t. We have just rearranged this equation in a fashion that, we have taken this quantity and this quantity together which is a like and this quantity we have placed separately. The mission the reason is very clear that we want to look at these terms these two terms put together and also this term J dot E d v.

Now, let us consider a certain volume of space through which this electromagnetic wave is travelling and let us considered that the area bound surface area bound by this volume is S is a close surface. So, we can we can write this equation, if you take if integrate over this volume, we can write this equation del dot E cross H d V is equal to minus J E d V minus E dot del D del t plus H dot del B del t.

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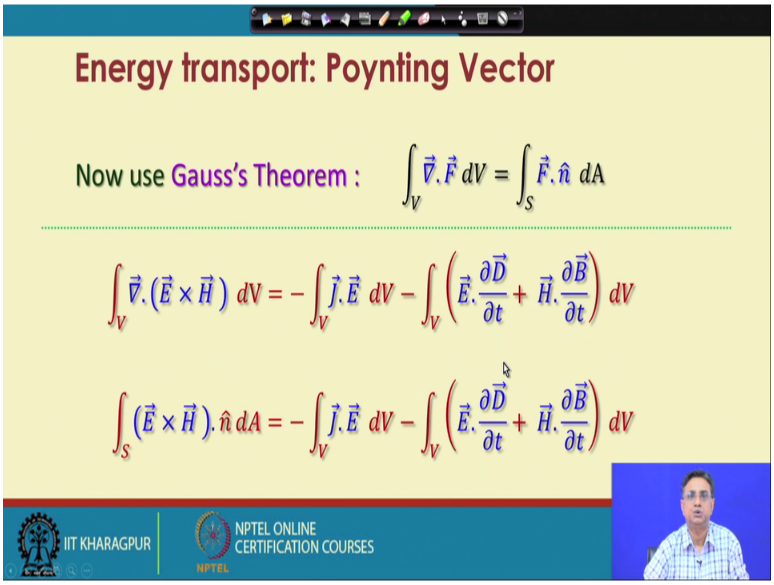

Now, look at these two terms we will use Gauss's theorem to see what is meant by this quantity del dot E cross E E cross H. So, Gauss's theorem states that it relates the volume integral to a surface integral for a vector field H the divergence of this vector field over is a over a certain volume is equal to the closed surface integral of the vector field.

So, by replacing this F by E cross H that is if we F if we substitute for F this E cross H, then we can re write this equation del dot E cross H d V equal to the right hand side which you have seen as it is. Now if I if I apply this Gauss's theorem, I can write this equation as surface integral of E cross H that is E cross H dot n unit vector d A integral of that will be equal to the right hand side.

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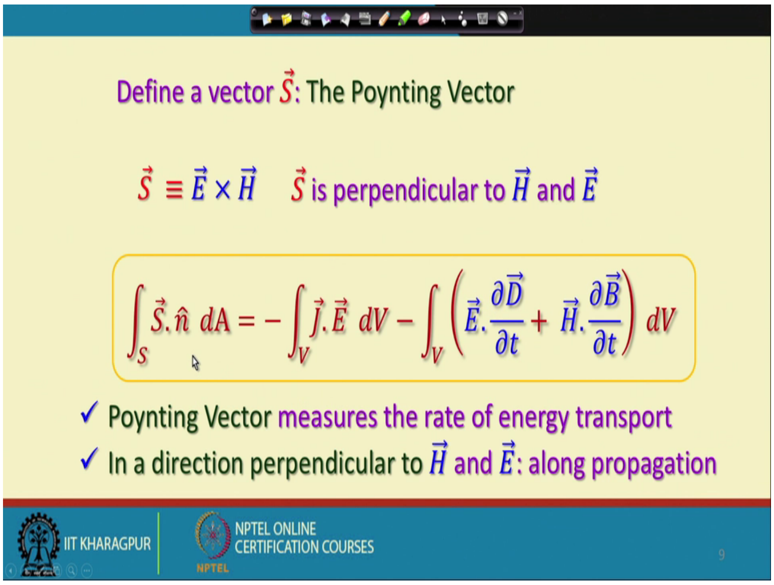

Now, we will define the Poynting vector the Poynting vector is defined as the cross product of E field and the magnetic field. So, s is equal to E cross H, you can see from this definition that S is perpendicular to E also S is perpendicular to H; that means, this Poynting vector is a vector which is perpendicular to the plane containing the electric field and the magnetic field; that means, if this Poynting vector has to represent certain quantity, then the direction of that quantity will be perpendicular to the to the plane that contains the electric field and magnetic field and we know that electric and magnetic field they are also perpendicular to each other, as a result this S E and H they form a right handed tired of vectors.

Now, if we replace this E cross H by this quantity Poynting vector S, then S dot n d A surface integral of that will be equal to the quantity which we have described before that is J dot E d V minus E dot del D del t H dot del B del t volume integral of that. So, it tells you that the Poynting vector measures the rate of energy transport and the energy transport it happens in a direction which is perpendicular to both the magnetic field and electric field and; that means, it is along the direction of propagation direction. So, the propagation direction is the direction along which the energy transport takes place.

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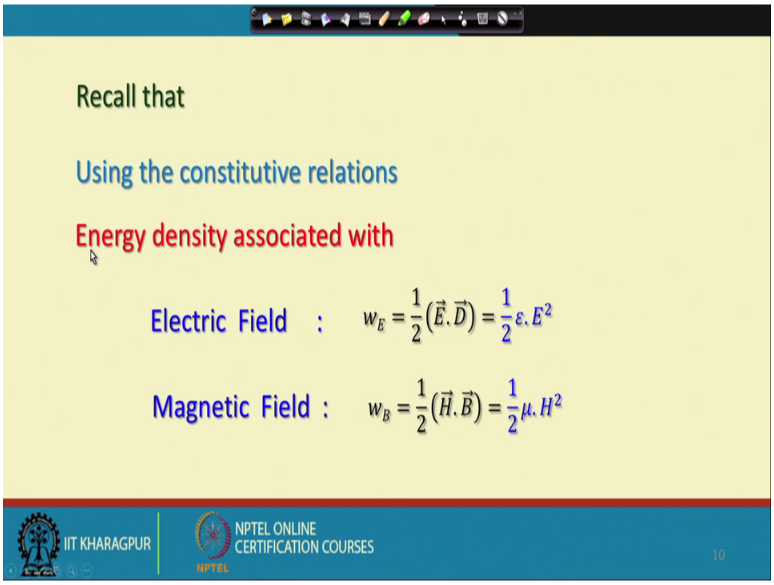

Now, let us recall that using the constitutive relations, the energy density that is associated with the electric field is half E dot D that is equal to half epsilon E square and the magnetic field energy density associated with the magnetic field is in the same way half H dot B which is equal to half mu dot H mu into a H square.

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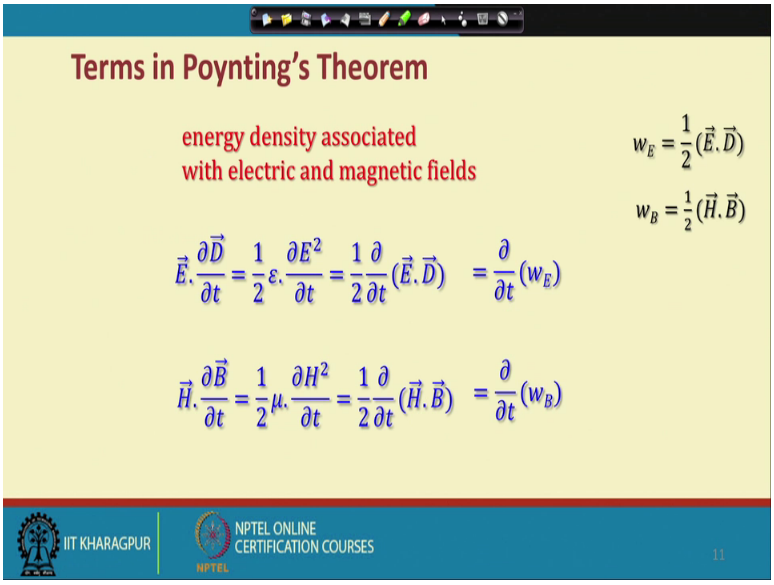

So, if we use definition of the of the electric and magnetic field energy density, then we can write that E dot del D del t which is equal to half of epsilon is replacing this D that is D equal to E into epsilon. So, half epsilon into del E square del t which can be rewritten in this form half del d del del t of E dot D which is equal to del del t of the energy density w E electrical energy density.

So, we have used this electrical energy density here in place of this you can take half inside. So, it gives you this quantity and in the same way if we proceed, then for H dot del B del t, we can write half mu into del t del del t of H square which is equal to half of del del t of H dot B and in turn this gives you del del t of w B which is the energy density associated with the magnetic field. Therefore, these two terms which are appearing in the equation can be replaced by the sum of these two terms; that means, we can represent this sum of these two terms is equal to del del t of the sum of the energy density is associated with the electric and magnetic fields.

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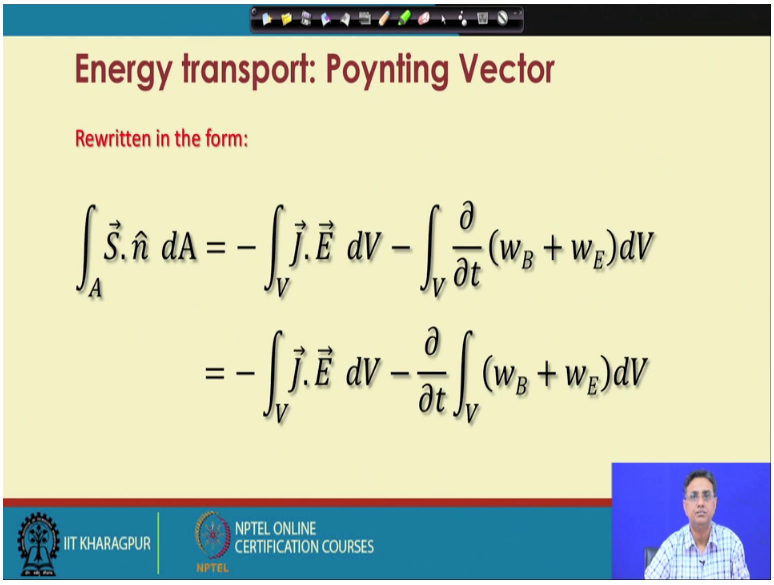

So, if we rewrite this equation that S dot n this left hand side the Poynting vector surface integral of that will be equal to J dot E d V and this quantity. Just now we have seen that del del t of del B w B and w E d V is the translated form of the individual variation of the electric and magnetic fields. So, we can write this equation if you take del del t outside the integral, then we can write that it represents the total energy, energy density associated with the magnetic field and electric field over this volume and del del t of that that is the rate of change.

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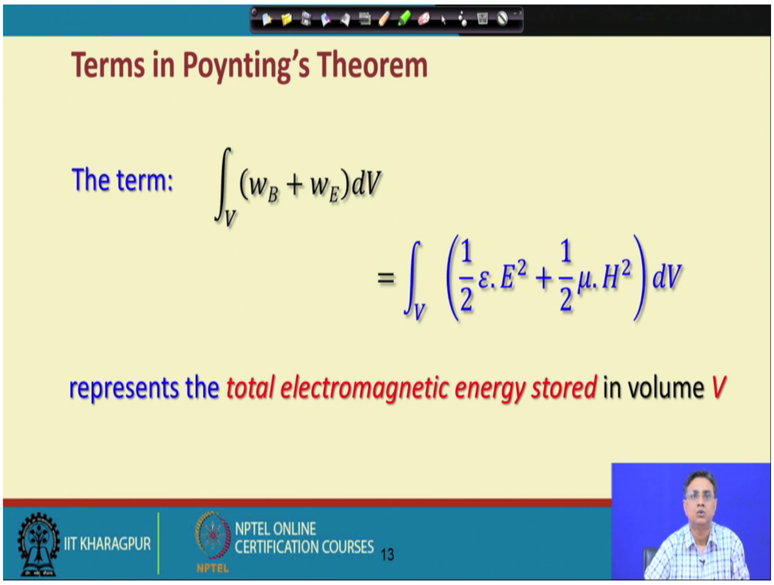

So, let us let us look at this term that that the energy density some of the energy density is for the magnetic field and for the electric field is equal to this just now we have seen. But because this is the total energy density with the electric field and magnetic field, so this quantity actually represents the total electromagnetic energy stored in the volume V.

So, total we have to magnetic energy in a volume V and if you take del del t of that that will represent the change in the total electromagnetic energy that is stored within that volume. If we place a minus sign before that it should definitely represent that this amount of energy will be the loss.

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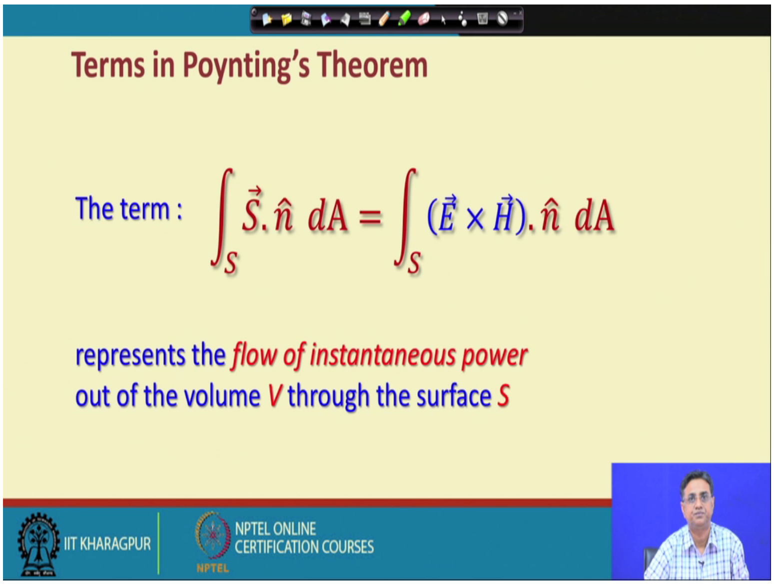

And if you look at this term that is S dot n d A which is equal to this, this actually represents the instantaneous power flow of instant instantaneous power instantaneous power out of the volume V through the surface S.

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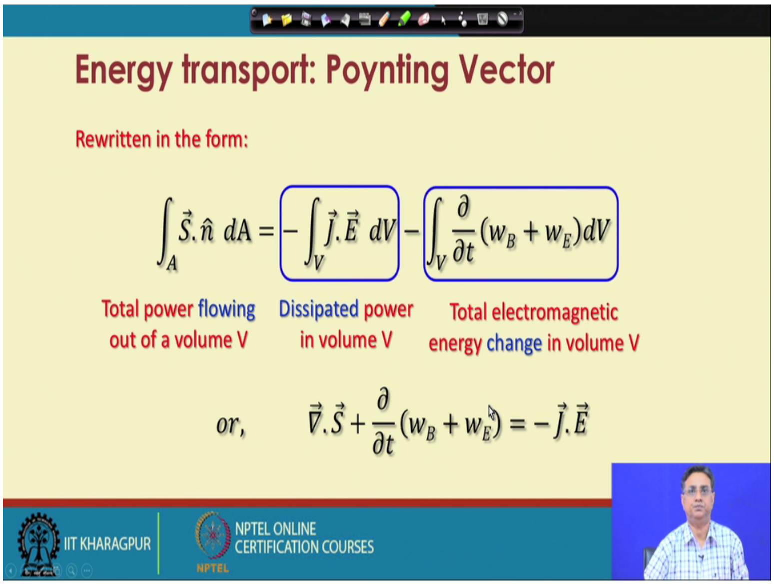

Now, again rewriting this equation in this form, this left hand side that is the term which contains the Poynting vector represents the total power flowing out of the volume V which is equal to the power in the volume dissipated power in the volume and also the total electromagnetic energy change in the volume. This is just now I have mentioned that this quantity del del this quantity w B plus w E that is the energy densities of the electric field and the magnetic field integrated over the volume d V represents the total volume within that space, total energy stored within that space and if you take the time derivative of that it represents the change in that energy and since there is a minus sign, so it represents that the loss of the energy from that space.

So, if you look at this equation, the energy flow through that volume will be equal to the loss of the energy from that volume and the dissipated power within that volume. So, that is very consistent and it really represents the conservation of energy; del dot S because this quantity we have called that power dissipation because del dot S del dot S and this del B del t the continuity equation is equal to minus J dot E.

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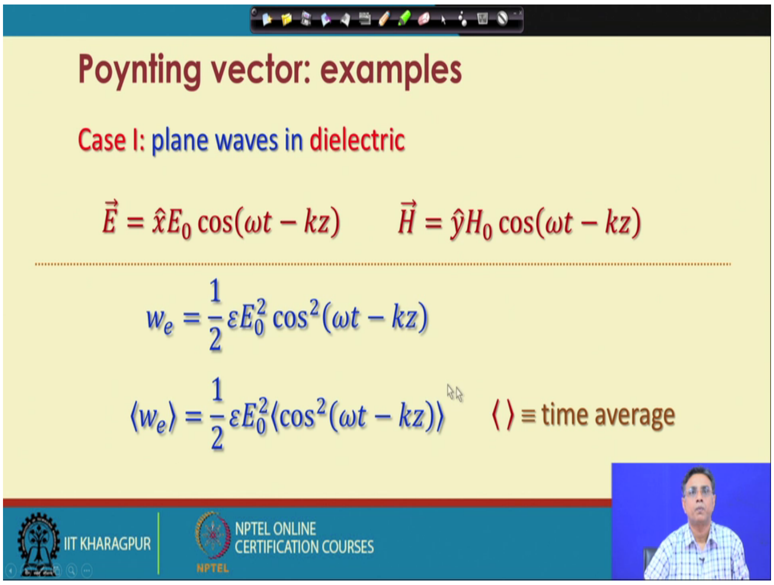

So, we will consider two different cases one is the dielectric and through which a plane wave propagating and how this Poyting vector measures the flow of energy and followed by that we will consider another situation when the electromagnetic wave is propagating in a conducting medium. So, in this case we can represent the electric field E as x cap E naught cosine omega t minus k z as if the wave is propagating along the z direction, then the magnetic field can be written as this because electric field is x polarized and the magnetic field is y polarized.

So, the energy density associated with the electric field can be calculated as the integral of this, then half of E 0 square cosine square omega t k z. So, this quantity if I take the time average time average of this, that is integral over a complete cycle; then we can write that the value of this will be because for 0 to twice pi by omega time that is t will give you the value half and you already have half. So, omega E the time average of the electrical energy density associated with electric field will be equal to one upon four E epsilon E naught square where E naught is the amplitude of the electric field.

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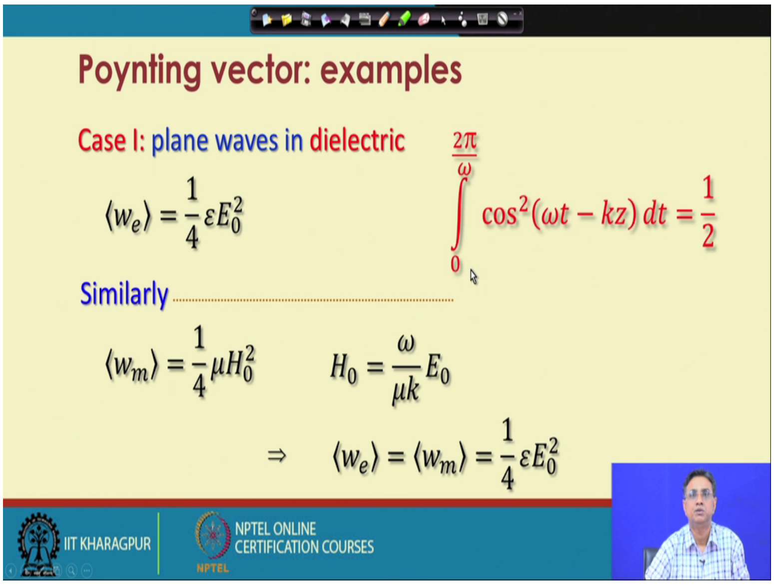

In the same way if I consider the magnetic field; energy density associated with the magnetic field, then we can calculate to be equal to 1 upon 4 mu H 0 square where H 0 is the amplitude of the magnetic field. This we could arrive at using this relation H 0 equal to omega by mu into k into E 0. So, if you substitute this, we can see that the electrical energy density, the energy density associated with the electric field and that associated with the magnetic field the average value of that over a complete cycle will be the same that is 1 upon of 4 mu epsilon naught square.

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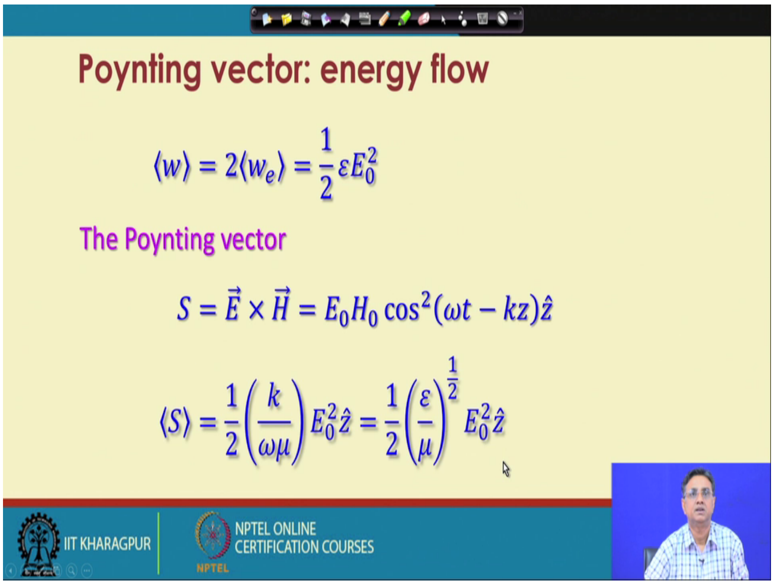

So, I mean known this fact, then we can write the total energy density average value of that which represents energy. So, will be equal to twice the contribution is both from the electric field and the magnetic field which will be equal to half E E 0 epsilon naught E 0 square which is again consistent with the with the our previous information that the Poynting vector S, there should be a mod S is equal to mod of E cross H which is equal to E 0 H 0 cosine square omega t minus z, since if you take mod then this z cap is not required. So, in that case the average value of the Poynting vector S will be equal to half of k by omega mu and E 0 square z that will give you this value. So, we could so that the flow of energy which is given by the Poynting vector is equal to this. So, these two are the same equation.

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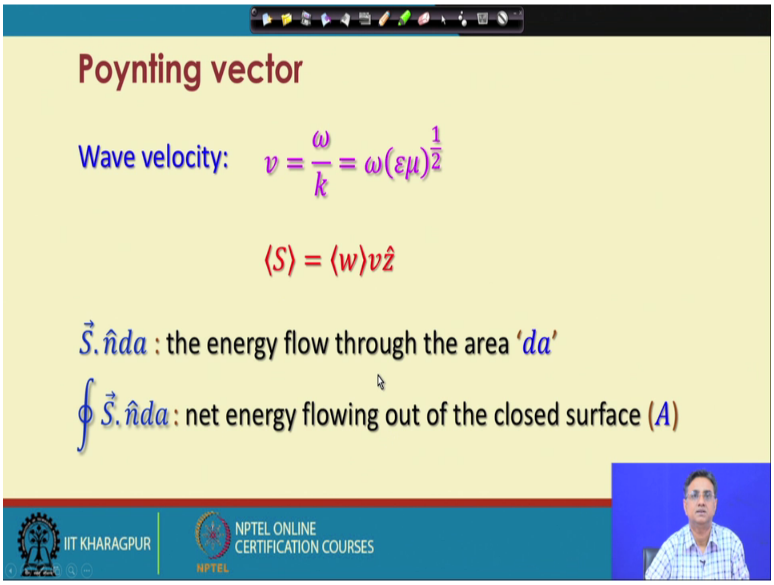

Now, since the wave velocity of an electromagnetic wave can be given by v equal to omega by k and k can be replaced by mu epsilon naught to the power of half. So, we can write that the average value of the Poynting vector is equal to the average energy density into the velocity that means, the average energy that is associated with the both the electric and magnetic field and over a distance which is given by the velocity will be equal. So, a S dot n d a that is the energy flow through the area perpendicular to the area d a is represented by this which will account for the net energy flowing out of the closed surface a.

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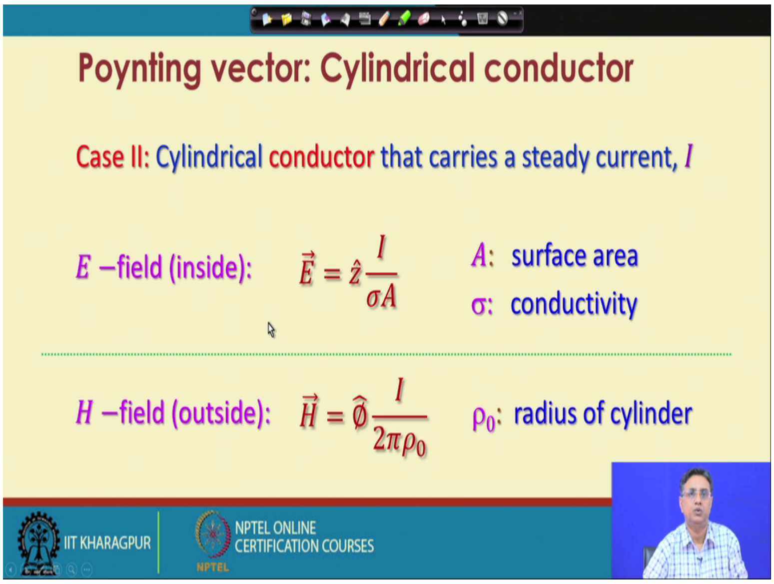

Now, we will consider another interesting case where we will see the Poynting vector through a cylindrical conductor that carries a steady current, I. Now the electric field inside this conductor is given by this E equal to z cap I by sigma into A, this is because this Ohms law if you write J into sigma is equal to E. So, sigma into E is equal to z into J.

So, in place of J we have written I by A where A is a surface area and sigma is a conductivity of the conductor and the magnetic field outside this conductor can be represented by H equal to phi I by twice pi rho. So, z and phi they are mutually orthogonal z is along the axis of the conductor phi is in the as azimuth. So, if you take E cross H that should represent a direction, which will be perpendicular to z and phi and in the outward direction. So, this will be evidently along the rho direction that is the radius vector direction of the cylindrical coordinate system representing the cylindrical conductor.

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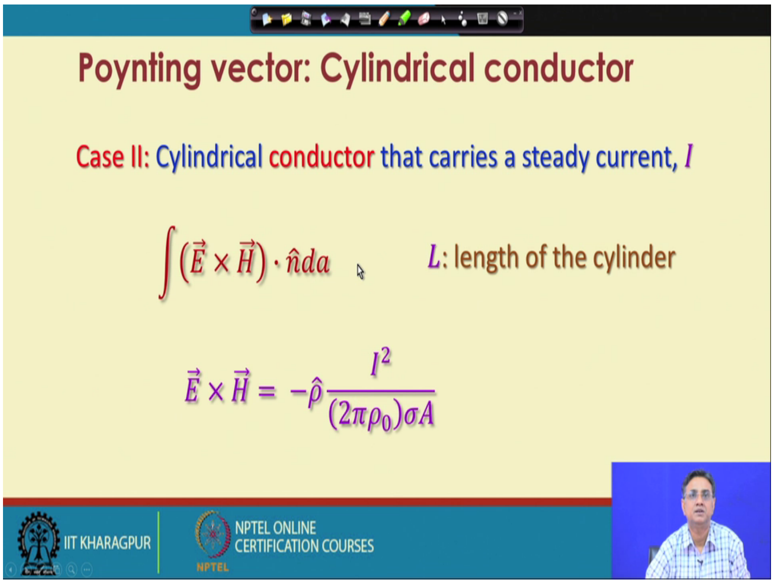

So, let us see that for the cylindrical conductor that carries a current I, we can write the Poynting vector in this way E cross H dot n d a where we consider a certain length L of the cylinder.

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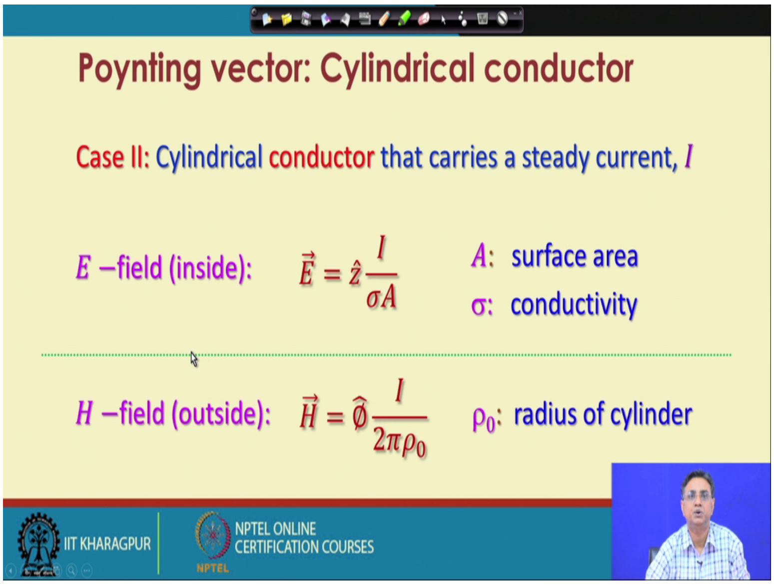

So, E cross H if we put together, this E cross H and take the cross product of these two quantities, then we can represent that E cross H equal to minus rho cap unit vector of rho that is the radius vector direction by into I square by twice pi rho sigma into A.

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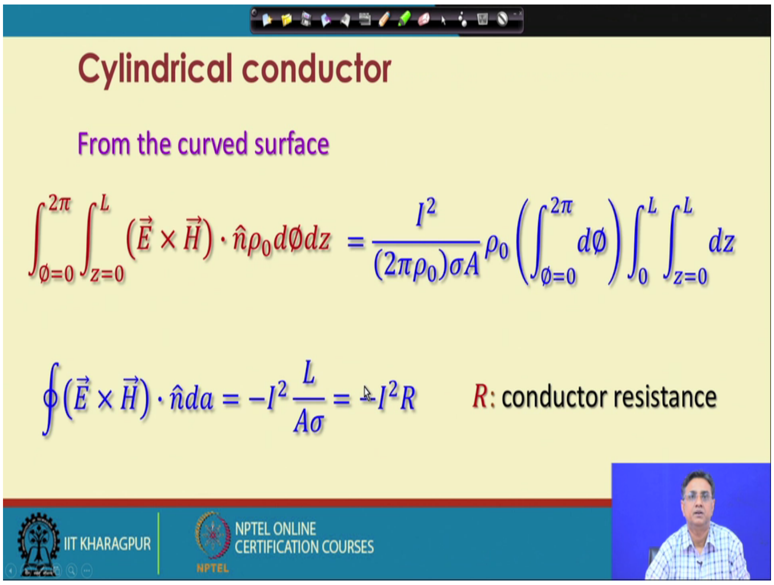

So, this is the value of the Poynting vector and from the curved surface; if you consider the Poynting vector, then we can see that we have to take the integration for phi which is about the complete azimuth that is from 0 to 2 pi and for a certain length L of the conductor which will be from z equal to 0 to L then, we also write the area elementary area in cylindrical coordinate systems that is equal to rho d phi d z and that should be equal to this quantity is the constant; we have to take the integration of phi equal to 0 to 2 pi and L and for z it will be z equal to 0 to L. So, this is a mistake.

So, by doing that E cross H dot n d a a equal to minus I square L by A sigma; that L by A sigma is equal to the resistance, for resistance we write rho L by A and in terms of conductivity, we write the resistance L by A sigma. So, this quantity represents that minus I square R. So, this is a known quantity which appears in the textbook that this is the joules loss heating loss.

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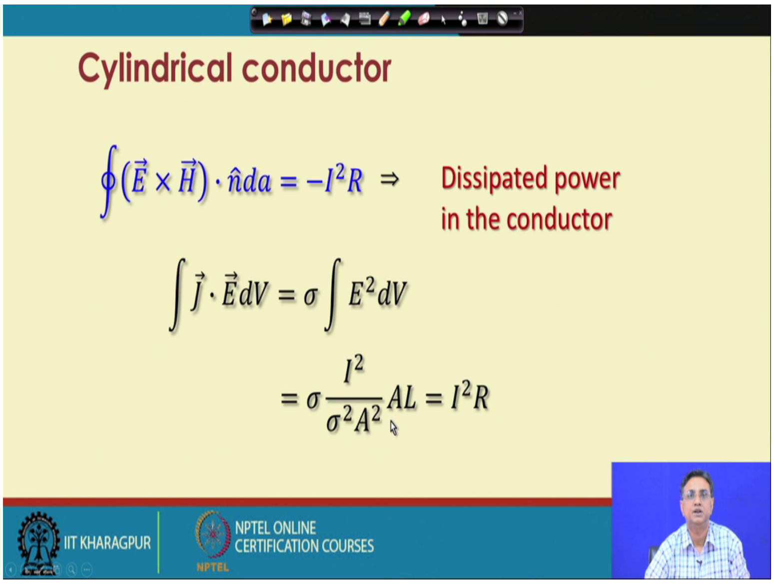

So, in the case of a cylindrical conductor which carries a current I, we can see that the Poynting vector if you calculate the Poynting vector, but the curved surface; then it really gives you I square R with the negative sign, it tells you that this amount of energy is dissipated out and away from the conductor. And J dot E d v will be equal to sigma E square d V. So, this quantity is the loss this quantity is the loss. So, that is equal to sigma integration of E square d V; if I substitute for E square, then we again end up with the same value that is I square of R; that means, the energy which is flowing out is equal to the dissipation of through at the conductor which is carrying a current I.

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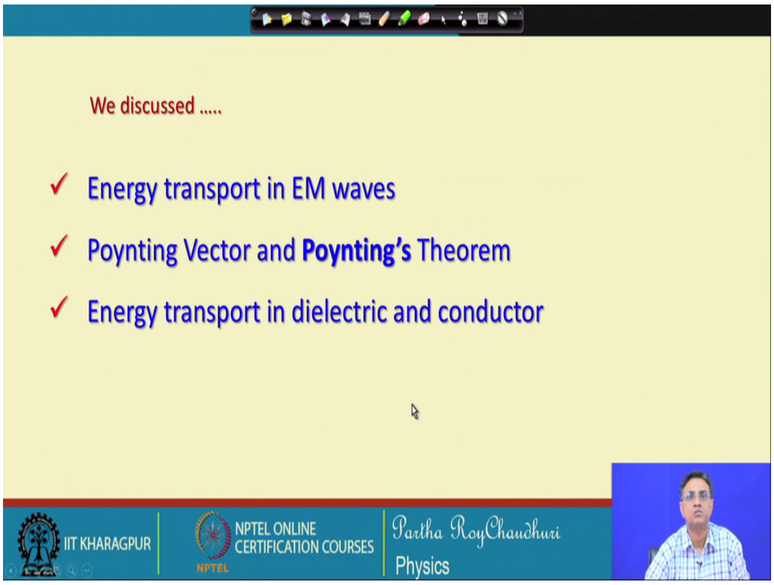

So, by this discussion we have considered the energy transport in electromagnetic waves which is primarily represented by the Poynting vector and through the Poynting's theorem, that the total amount of electromagnetic energy that is flowing inside a certain volume of space through which the electromagnetic wave is propagating will be equal to the amount of energy that is leaving that space plus the amount of energy which is lost within that volume. So, energy transport in the case of a dielectric medium as well as in the case of a conducting medium we have also calculated.