Modern Optics
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Lecture – 02
Maxwell's equations and electromagnetic waves (Contd.)

So, in this class we will discuss the wave equations, for electromagnetic waves and their solutions.

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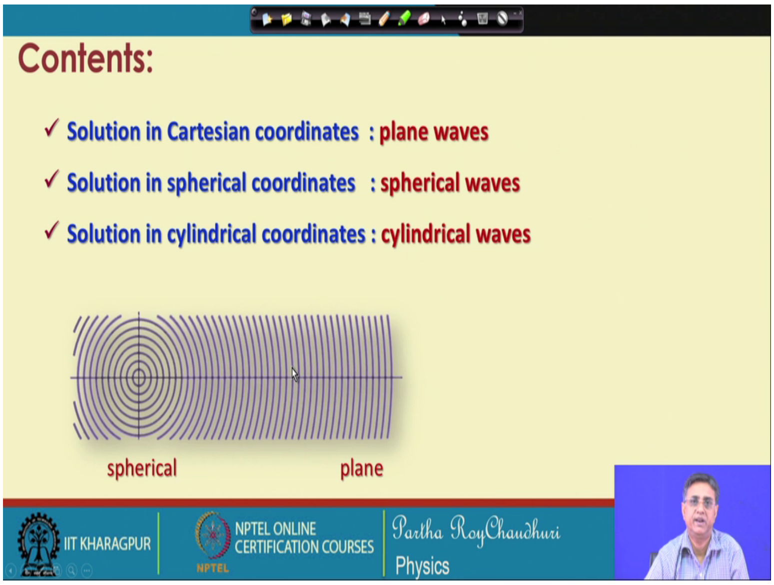

The content of this discussion is like this, the solution in the Cartesian coordinate system which will give rise to the plane waves or speaking otherwise that a plane wave satisfies the Cartesian system of coordinates. The solution in spherical coordinates which will give rise to spherical waves and the solution in cylindrical coordinates systems which will give out cylindrical waves.

So, these are the main three categories of electromagnetic waves, which are the direct outcome of the solutions of the Maxwell's equation the waves wave equation. You may recall that if you have a point source, then immediately around this point source; there are spherical waves, but as you move away from the spherical waves the curvature becomes less and less and you end up with a plane wave. But this if the source is a line source, then the waves corresponding waves which will come out from the source will be cylindrical waves. So, these three different kinds of waves we will try to analytical solved starting from the wave equation and will also look at the properties.

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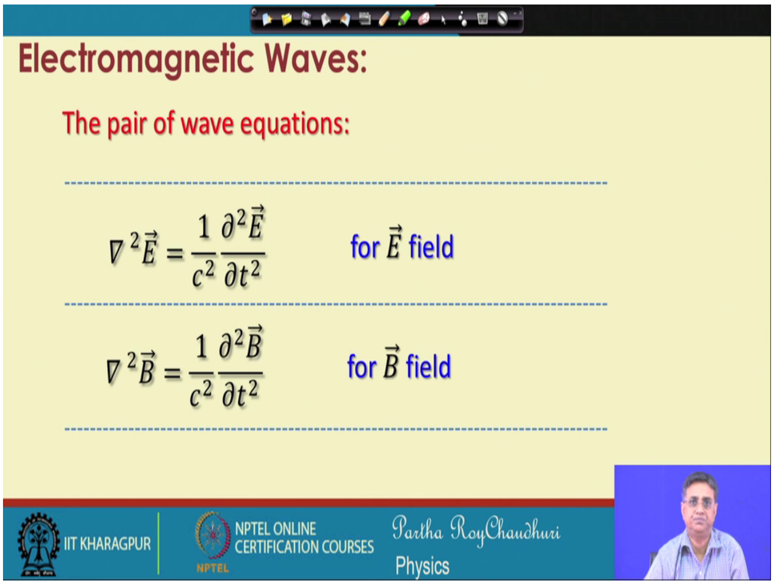

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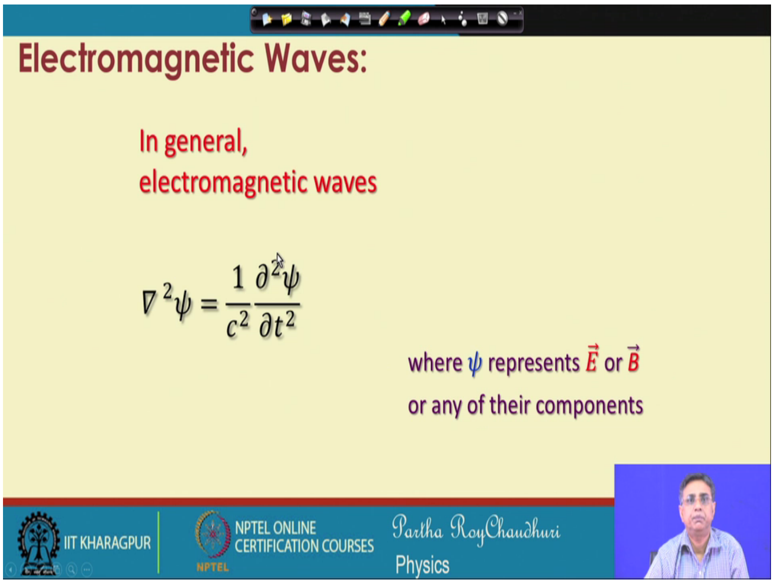

So, the pair of wave equations in free space is represented by an electric field equation and a magnetic field b vector equation, where c is the velocity of the electromagnetic waves in free space.

Now, from this set of two equations as we have seen earlier that it can be put into a general and compact form using psi, where this psi represents E or B fields or any of the component. So, it could vector wave equation or it could be a scalar wave equation.

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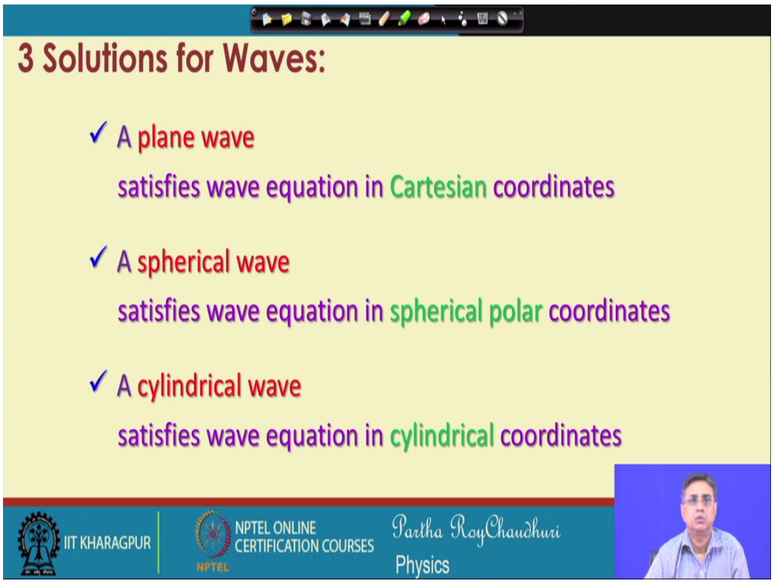

This equation when we solve that then we will see that if we solve it in the Cartesian coordinate systems, we will find that it will give a solution which is a plane wave and when we solve this equation in the spherical polar coordinates, then the resulting wave will be a spherical wave. In the same way if we solve it for cylindrical coordinate system the waves will be cylindrical wave these are very basic and very fundamental it is important to know the behavior of the waves.

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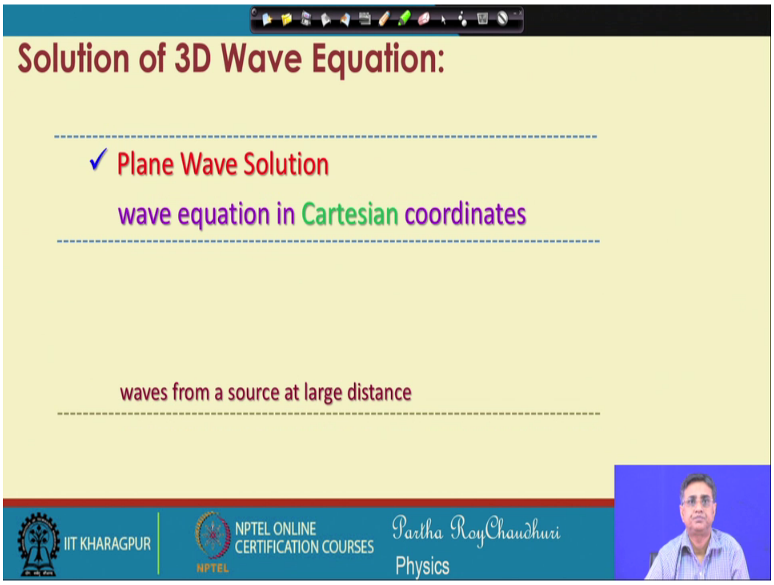

Therefore for a plane wave solution we start with the Cartesian system of coordinates. As I have mention that electromagnetic wave at a large distance from the source becomes almost a plane wave.

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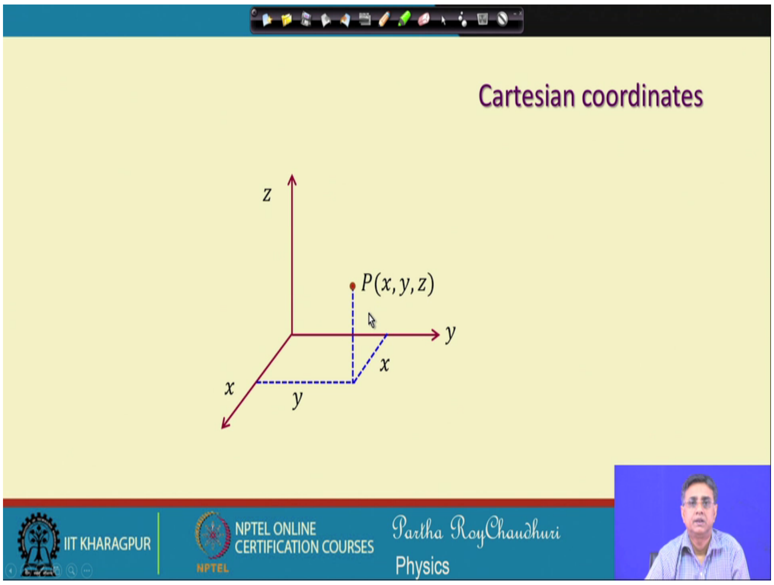

So, in Cartesian coordinate system as you know we are very familiar, that your point is represented by three components x y and z.

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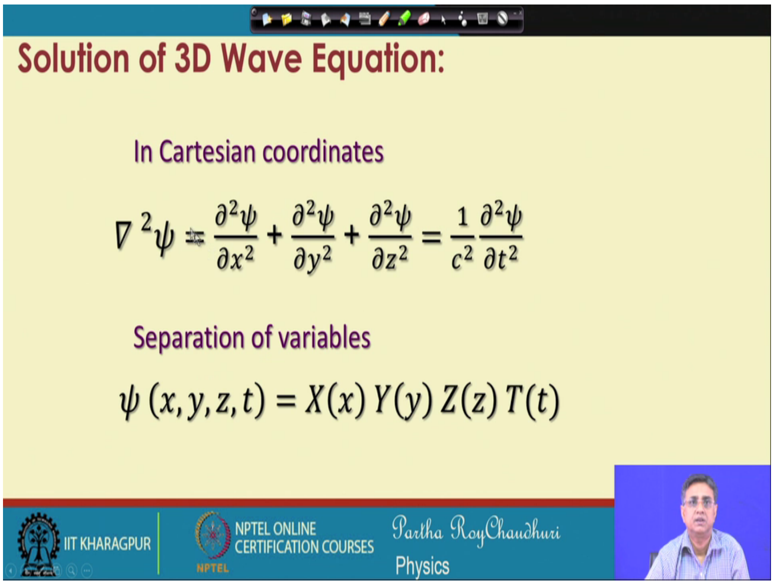

So, in this system the Laplacian is represented by del square del x square, del square del y square plus del square del z square. So, in this Cartesian coordinate system we can write the wave equation in this form that the three components equal to 1 by c square and the time derivative double derivative of times. So, for such equation it is well known that the separation of variables should be applicable to find out the solutions. So, expressing the psi which is a function of x, y and z and t for this case. We can write this psi of x, y, z and t as psi of psi x of x y of y z of z and t of t because these are the four independent variables associated with this equation.

Now, if I replace substitute psi by this x y z and t, then the first quantity will give you del square x by del x square multiplied by this quantity. In the same way the second quantity will give you del square capital y by del y square multiplied by the other three quantities and so on. For this quantity this will be del square capital T by del t square into 1 by c square multiplied by the three x y z parameters.

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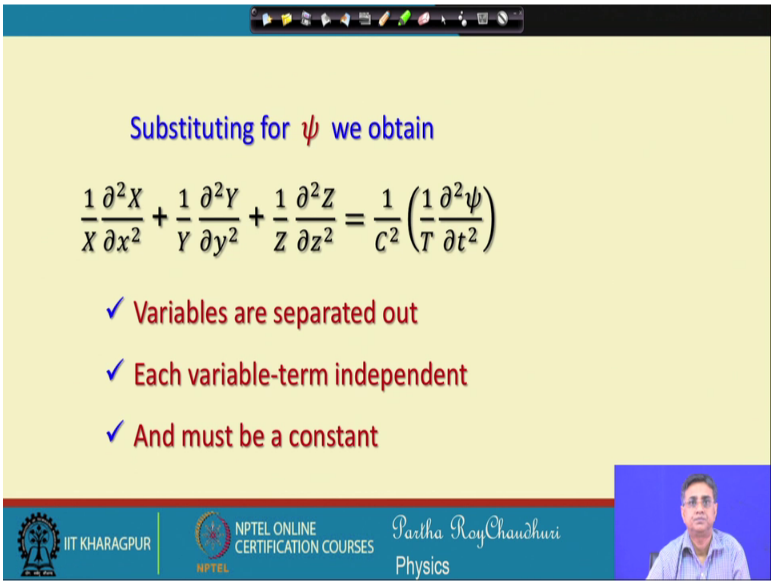

And now if you divide that equation, if you divide the equation where you have already substituted this value by psi that is by x y and z, then the equation will take the form of this that is 1 by x del square x by x square and so on. You can see that the variables are now separated out, each variable term is now independent they are not connected they are coupled with any of the other. So, if it is so, each of the terms are independent then they must be equal to a constant. Putting that constant for each of them, this I put equal to minus k x square this one equal to minus k y square.

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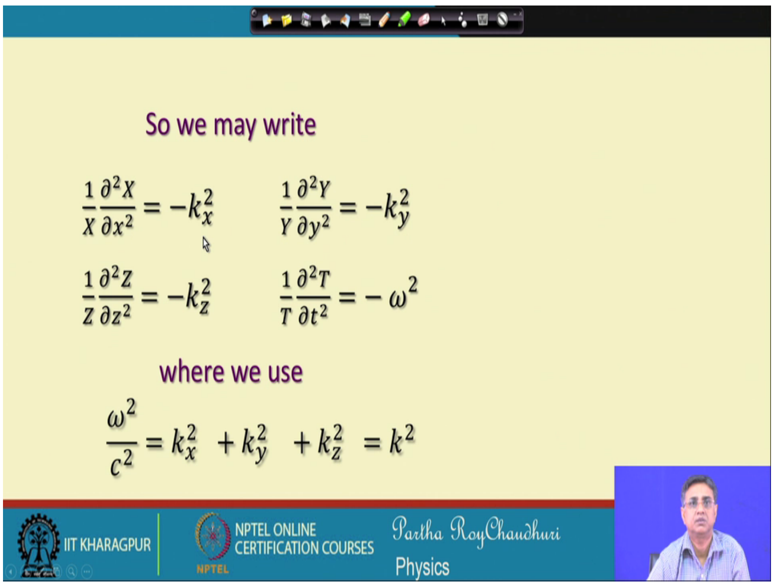

You can see here I can put this minus k x square k y square k z square and put this I use minus omega square. Where, this then if you if you put it back minus k x square minus k y square minus k z square equal to minus omega square into by c square. So, that is what appears here. So, omega square by c square equal to this, but this is equal to k.

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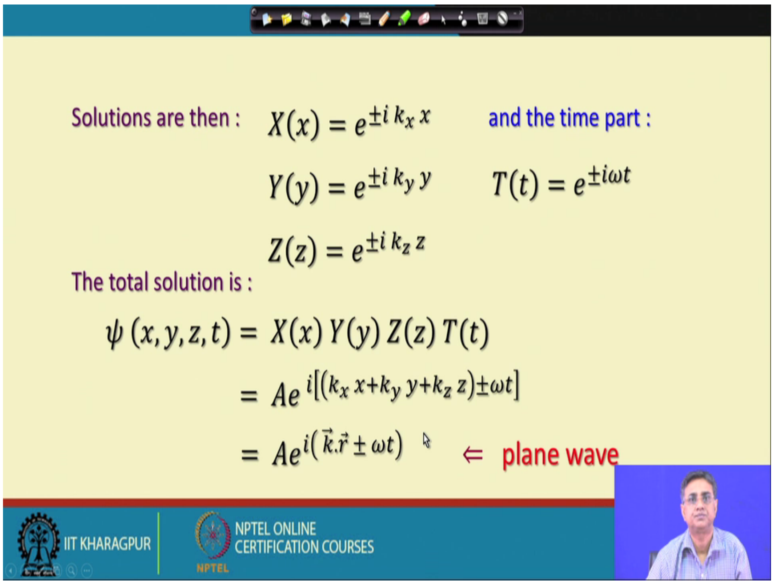

So, the solutions of each of this equation solution of each of this equation is of this form x equal to this, y equal to e to the power plus minus i k y y and similar z equal to e to the power of plus minus i k z z and the time part will become t of t equal to e to the power of plus minus i omega t.

So, the total solution is now you have started from this equation so, I put them back I get that A is a constant, and then you multiply these three quantities which will appear in this form and this is the plane wave solution.

So, starting with the Cartesian coordinate system for the Laplacian, we end up with the solution of psi. Because psi represents the electromagnetic waves the e vector or any of the components of psi the v vector or any of the components of B in general each component will satisfy this plane wave equation.

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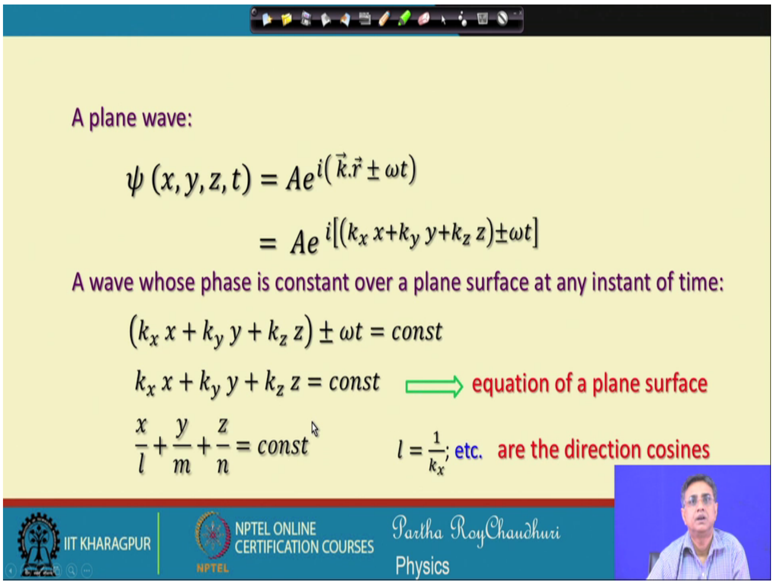

Now, a plane wave is usually written in this form and if you write this, this is the compact form using this r. So, if you decompose this r, k dot r as k x x plus k y y plus k z z then this quantity represents the phase of the of the electromagnetic wave. Now, if you look at the definition of the plane wave, a wave whose phase is constant over a plane surface at any instant of time is a plane wave.

So, to look for that, I have this phase which is this quantity and now that has to be constant. So, we look for the locus, where this phase remains constant. Now because at a given time we will look for this phase, constant phase plane so this omega t is a constant as a result this quantity is also constant this is the well-known equation of a plane. If you write in the form of the direction cosines, then x by l, y by m and z by n equal to constant where, this l m and n are represented by 1 by k x 1 by k y and so on. So, these are the direction cosines of. So, the reciprocal of the components of the propagation vector represents the direction cosine for the plane electromagnetic waves.

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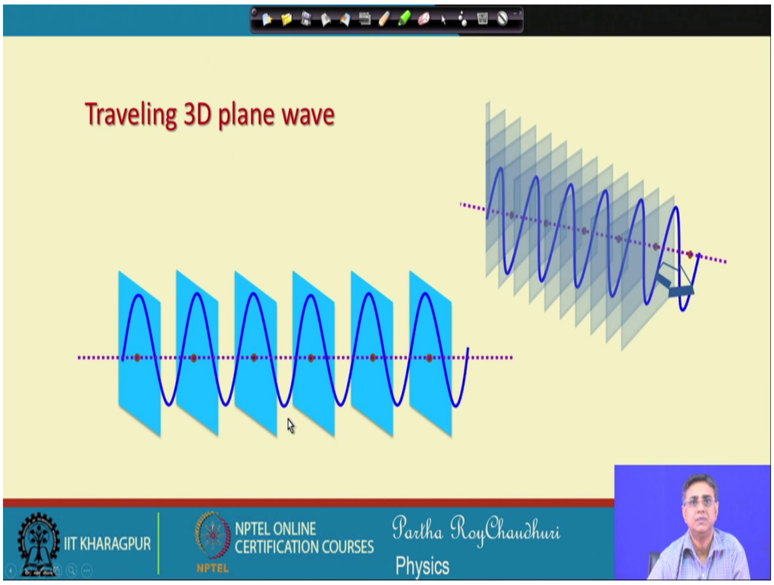

So, electromagnetic waves as you see that the phase remains constant over a particular plane, these are the periods over reach the phase remains constant is a propagation of (Refer Time: 11:24) pictorially.

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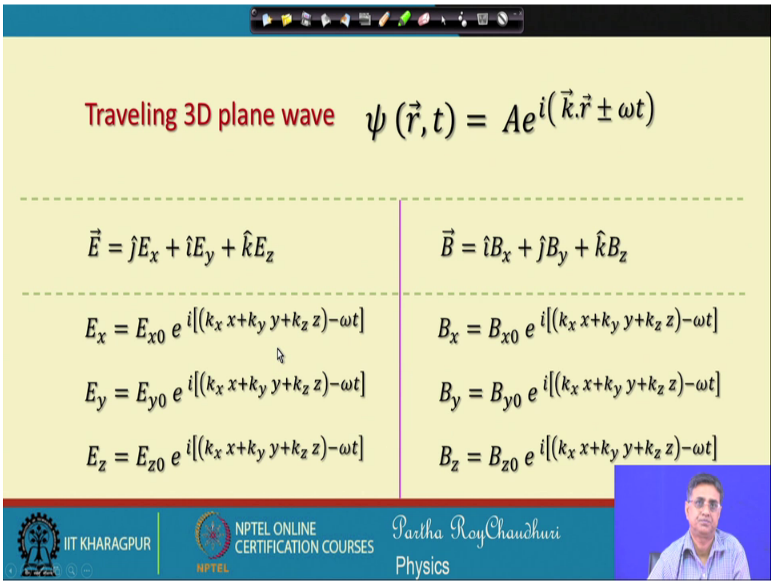

A traveling three dimensional plane wave is represented by this I have seen this now, if I look at the wave very explicitly, E vector will be represented by assumed i E x j E y k E z and similarly B vector will be represented by this. But each of the components E x E y and E z will satisfy the plane wave solution. So, I can write this equation in this form, for

E y I can write in this form and similarly for E z I can write this form and for B x B y B z I also get this expressions.

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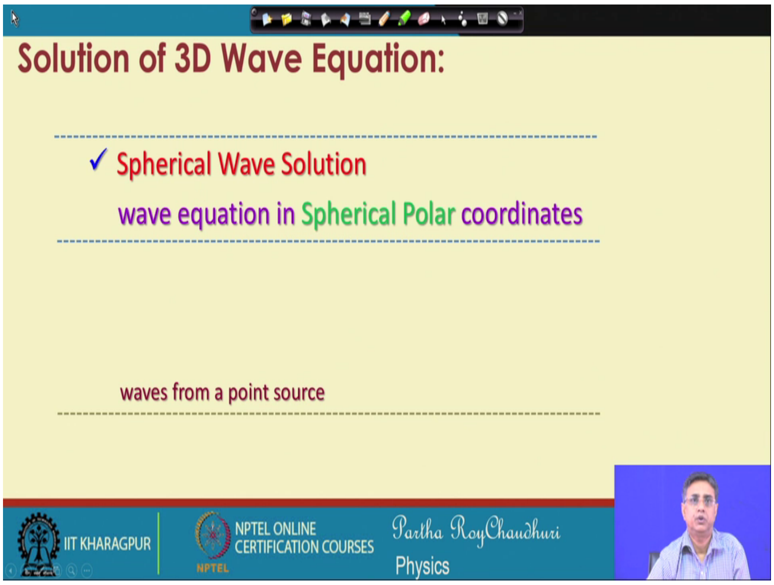

Now, we will look for the solution of the three dimensional wave equation, for the a spherical wave solution in spherical polar coordinates; waves from a point source is the spherical wave source.

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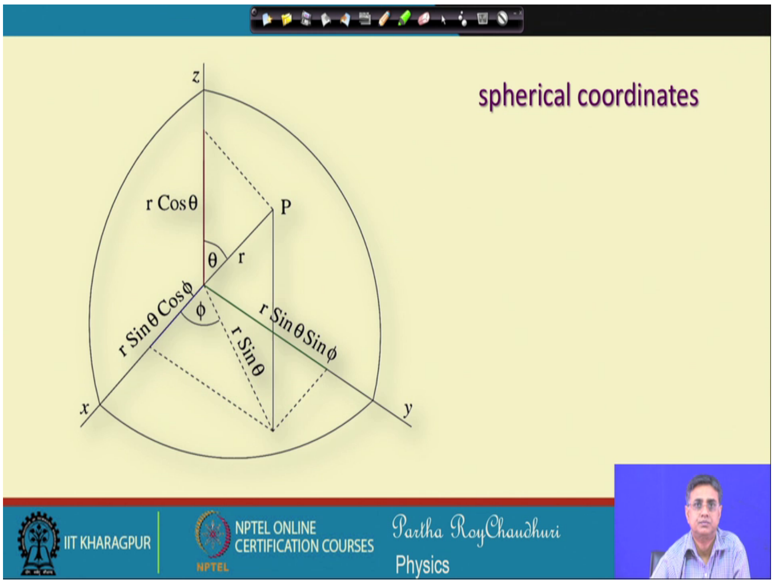

Look at this spherical polar coordinate system, you have r theta is the inclination and phi the azimuth r is the radius. So, any point p is represented and the spherical system in this way.

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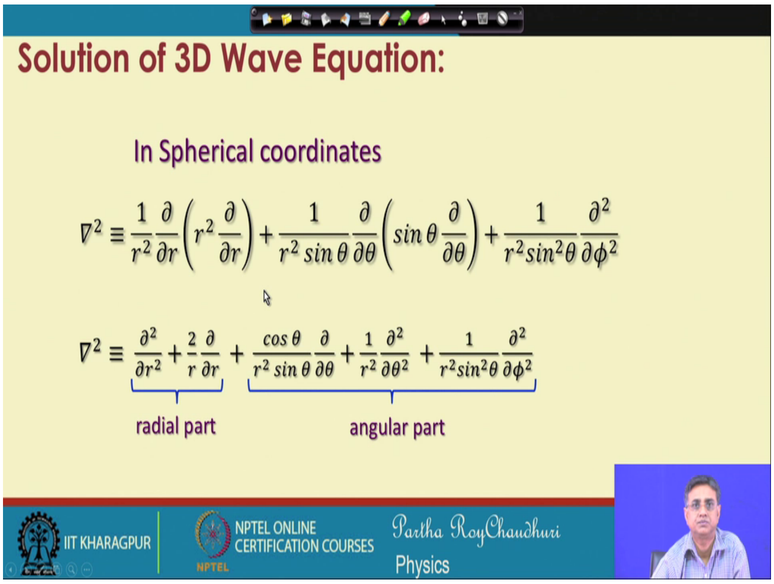

In spherical coordinates the Laplacian can be written in this from where r is the radius vector, theta is the inclination and phi is the azimuth angle. So, look at this part is purely the radial part and this part represents the angular part.

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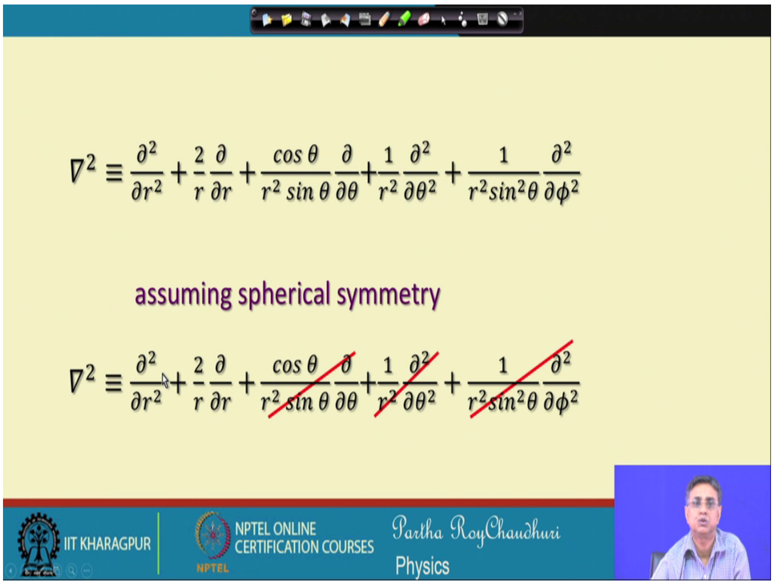

So, for a wave which is spherically symmetric that is there is no dependents on the angular part, we can remove these three terms from the Laplacian. So, the Laplacian simplifies to this form, that is del square del r square plus 2 upon r del r. We will use this reduced Laplacian to solve the wave equation in the spherical geometry.

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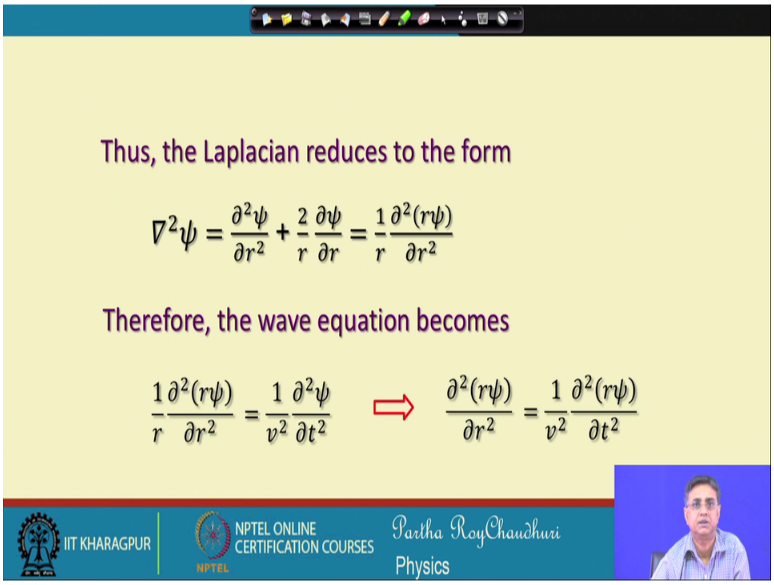

So, the Laplacian reduced to reduces to this form, which can be written in this form also by doing that the manipulation. Therefore, the wave equation becomes 1 upon r del square r psi del r square equal to 1 by v square del square psi del t square, where v is the velocity of the electromagnetic wave. So, this equation becomes using this we can write this equation in this form. So, this is the reduced wave equation for in the spherical coordinates for the electromagnetic waves.

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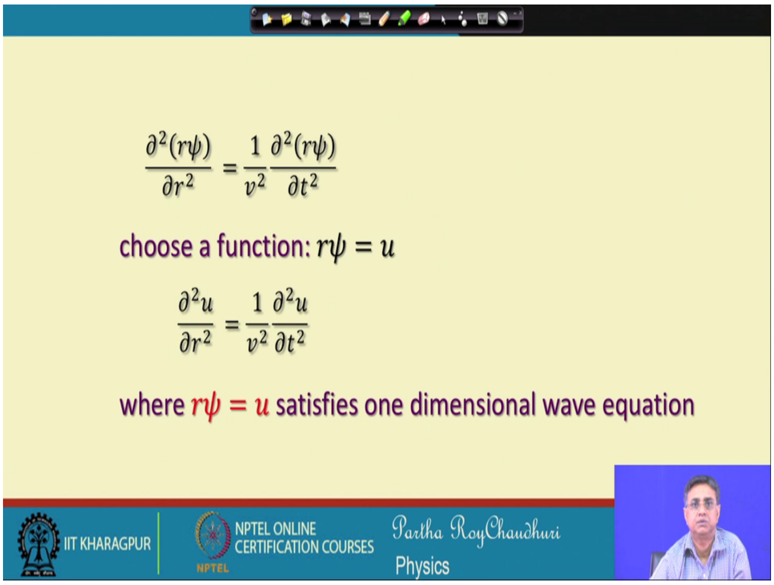

Now, I use this equation for that we choose a function r into psi equal to u, which is also a function of r, then if I substitute into this equation, it becomes del square u by del r square and 1 by u square del square u by del t square. Look at this equation this is only the function of r, this is function of time. So, where r in psi equal to u satisfies one dimensional equation. So, this represents a one dimensional wave equation.

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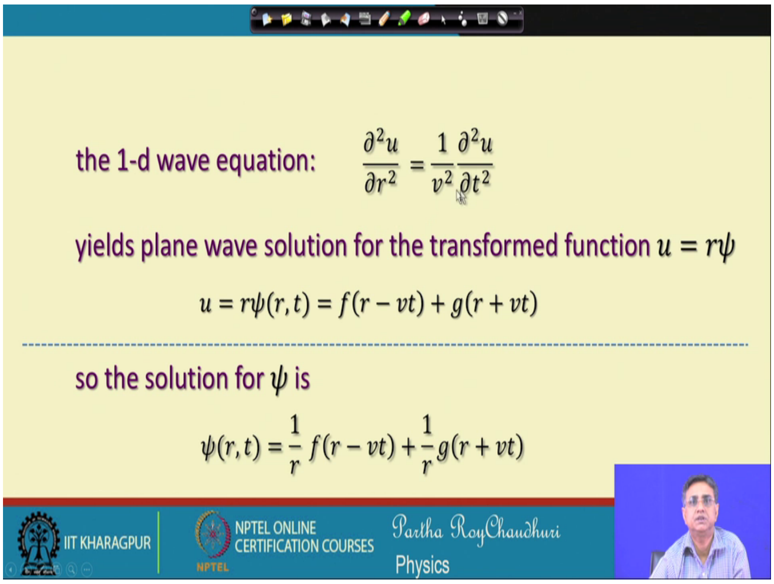

The one dimensional wave equation this yields plane wave solution for the transformed function this. So, by doing this through this root, we get a plane wave solution, but if we.

So, the plane wave solution will have a form like this u the r of this, where r minus v t plus g of r plus v t. So, the solution for psi is like this, but this is now the spherical wave solution.

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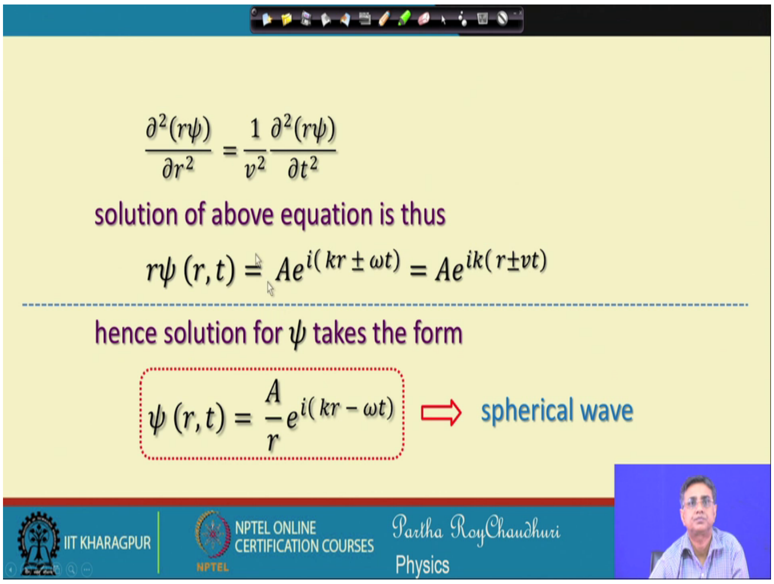

So, the solution of the above equation is thus r into psi will become like this, and hence the solution for psi takes the form of psi of r t equal to A by r e to the power of i k r minus omega t. You can see that if you are far away from the source then the intensity which is the mode of this function falls by r square, and hence the amplitude falls by 1 upon r. So, this is consistent and this represents the spherical wave.

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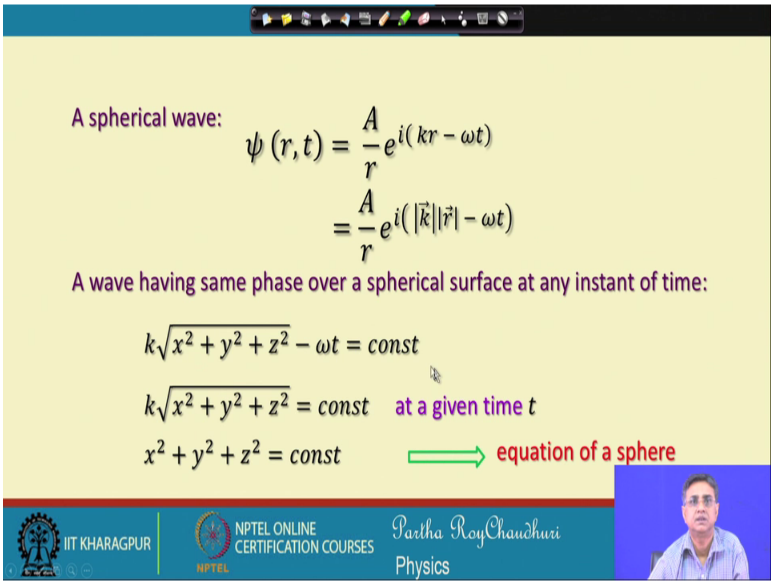

There is so for a spherical wave we write psi of r t equal to this.

Now, let us check that the phase of the electromagnetic waves is a constant over a spherical surface. So, k is now the mod of k and r is the magnitude of r. A wave having the same phase over a constant spherical surface at any instant of time will represent a spherical wave. So, because it is constant at a given time therefore, we set omega into t is also a constant. Therefore, k into under root of x square plus y square plus z square which is the mod value of this r vector is equal to constant at a given time. Therefore, if you square both sides we get x square plus y square plus z square equal to constant and this is the equation of a sphere. So, all this is very consistent.

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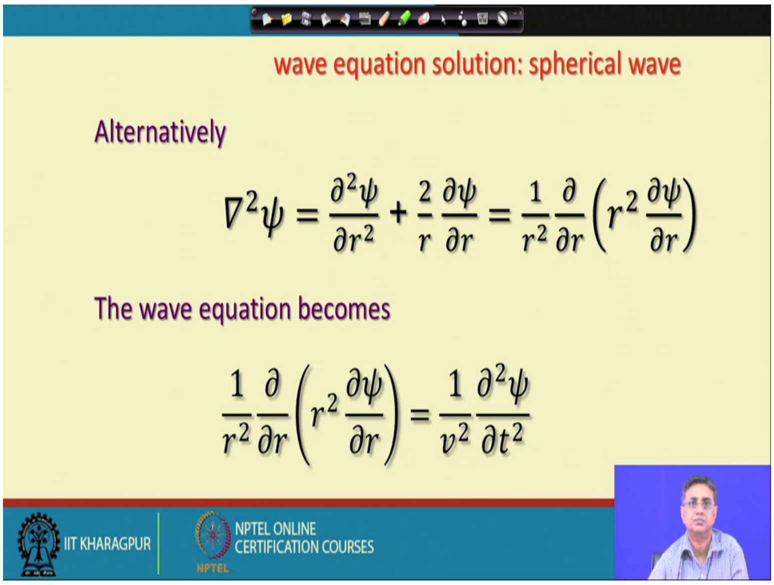

Alternatively there is an alternative way of arriving at the spherical wave solution, the Laplacian reduced form of the Laplacian you have seen like this, which can be reorganized into this form and then the wave equation becomes like this 1 by r square del del r of r square del psi del r, which is equal to 1 by v square del square psi del t square. Now, will solve this wave equation to get the.

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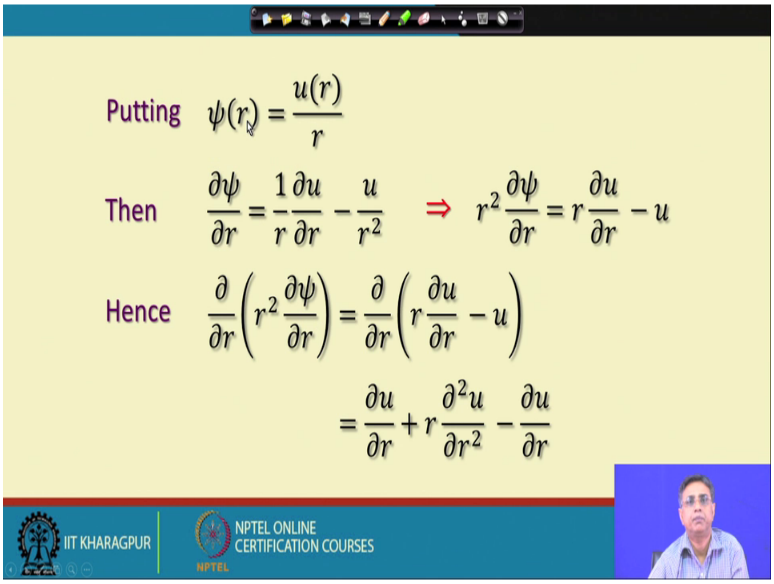

So, in this case we put there is one step that let us put psi of r equal to u of r by r. So, the equation del psi del r now becomes 1 by r del u del r minus u by r square. Just del del r if you do of this you will get this equation. Now, this equation will give you when multiplied by r you get r square del psi del r equal to r del u del r minus u, hence del del r of r square del psi del r is equal to this, which will give you two terms which are identical, but opposite in sign. So, they will cancel these two terms will cancel.

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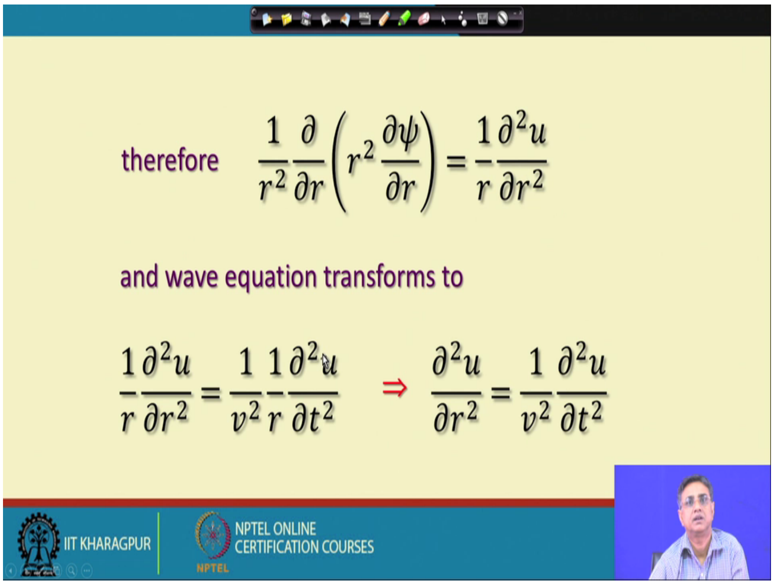

So, this left hand side now becomes r del 2 u by del r square; del 2 u by del r square this quantity and the wave equation now transforms to this form. So, because 1 by r cancels from either side.

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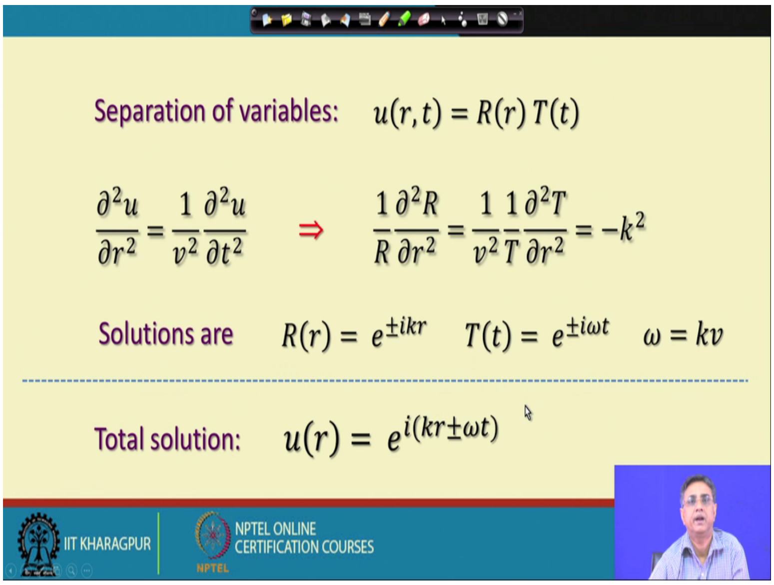

So, this is again a well-known equation and we can use the separation of variables because r is a representation for the radial function and t is for the time part. So, if I substitute this u equal to R of r and T of t into this equation, then we will get this there will be t multiplied by this and r multiplied by this. Now if you divide both sides by R and T I get this equation. Now, each term is independent and they must be constant so, using the same approach, we can write each of them is equal to minus k square.

So, the solutions are now this equal to minus k square this equation is a is again well known equation whose solution is R of r equal to e to the power of plus minus i k r, and t this equal to minus k square will give a solution, which is e to the power of plus minus i omega t where, I have used this omega equal to k into v the frequency of the wave. So, the total solution takes the form of u of r equal to e to the power of i k r plus minus omega t.

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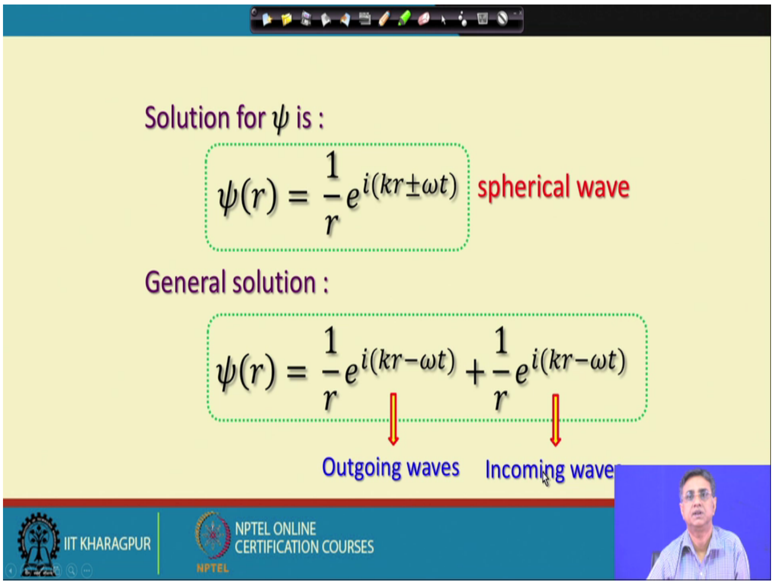

But we started with psi. So, to get back with a with an equation of psi we should write psi of r equal to 1 by r e to the power of. So, again will end up with the spherical wave and the general solution of this wave is psi of r 1 by r e to the power of i k r minus omega t plus e to the power of this should be plus. So, this is spherically outgoing waves and if you put a plus here it should be spherically incoming wave.

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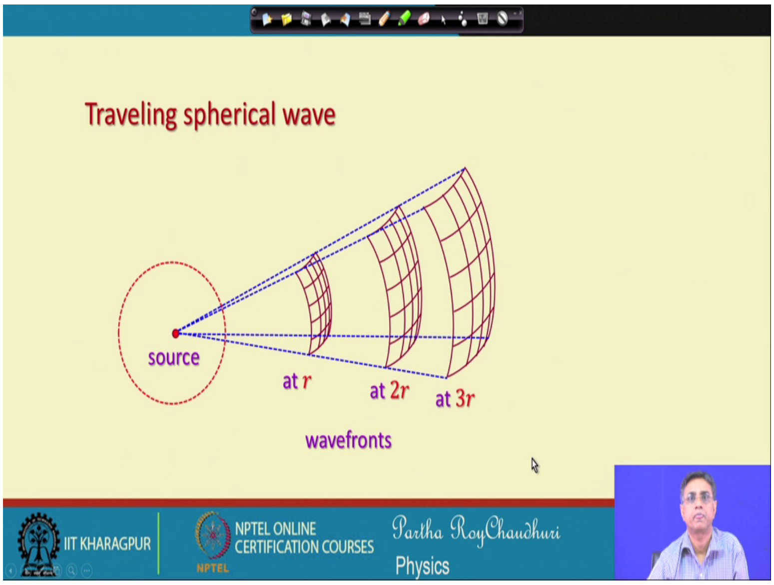

A traveling spherical waves you have a source, if you move radially outwards then they will form the wave front, which are the surface of the spheres. So, at a distance r the phases will be constant, at a distance 2 r the phases are constant and so on.

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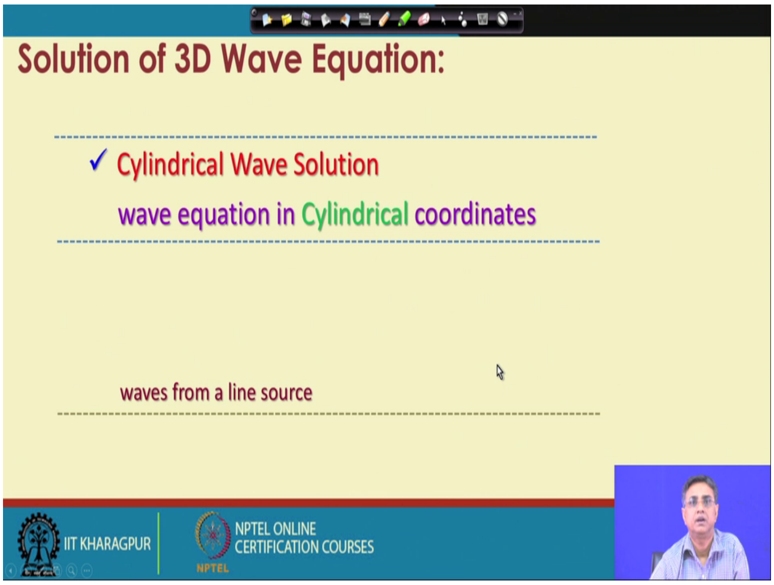

Now, we look at the cylindrical wave; cylindrical wave is realized from a line source if you have a linear chain of radiating dipoles or linear chain of radiating system, then all around the line in a cylindrical surface the phases will be constant.

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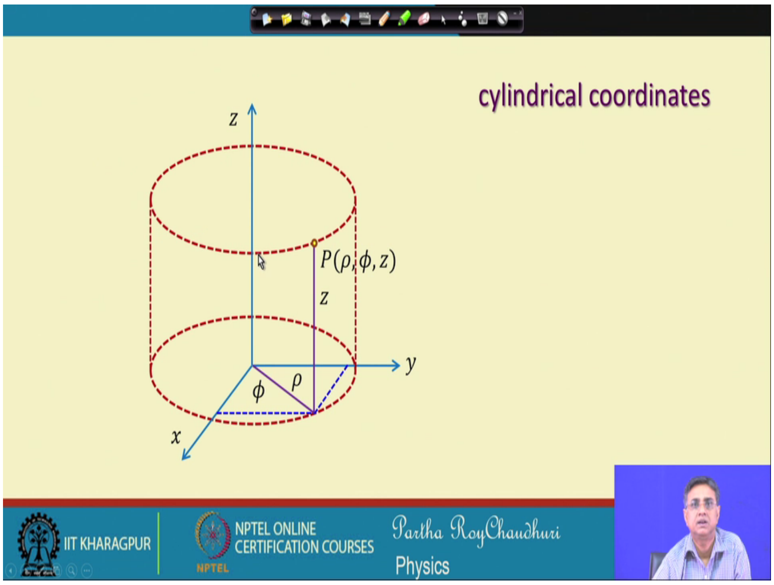

So, look at this cylindrical coordinates if I have my source sitting here, then on the surface of the cylinder the phases will be constant. The representation is this is the azimuth phi, this distance this is rho and the height is z is a usual notation for cylindrical coordinate system.

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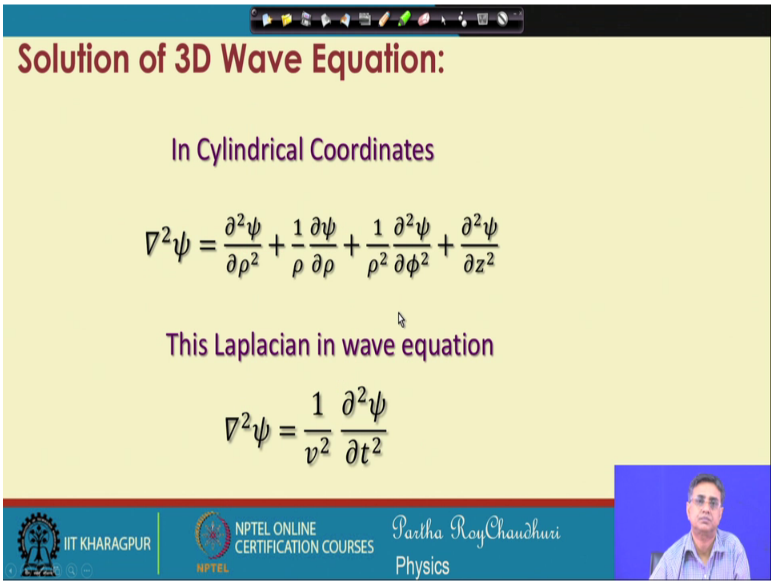

So, the Laplacian in cylindrical coordinate system comes through this equation for psi. So, del square psi will be involving del square del rho square 1 by r rho del rho and the azimuth part. The Laplacian in the wave equation, this Laplacian gives you the wave equation in this form.

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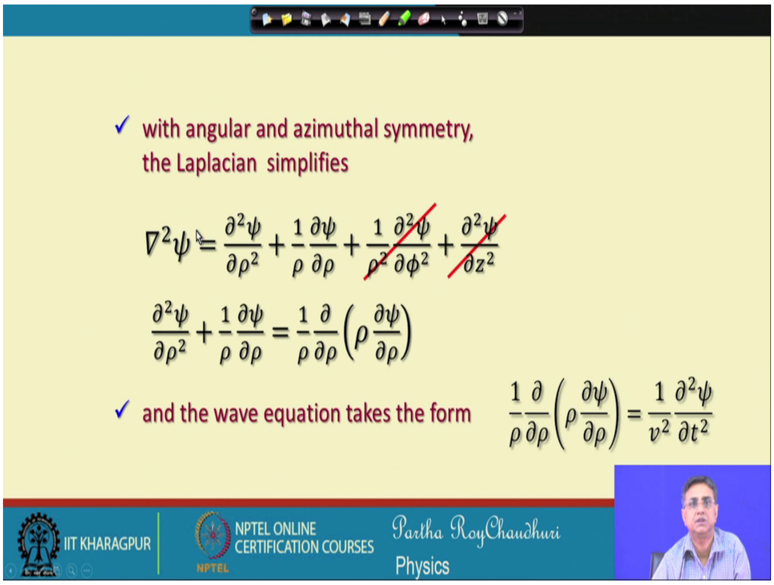

With angular and azimuthal symmetry, that is if I assume that it is independent of the azimuthal symmetry, we can write this equation only as a function of rho and the wave equation takes the form of this. Because you can remove these two terms because of the independent azimuthal symmetry the solutions of this wave equation.

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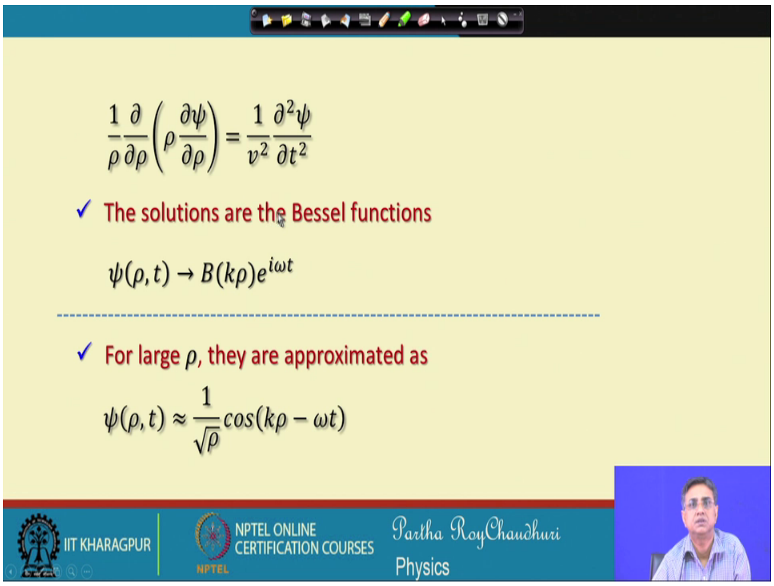

The solutions of this equation are well known Bessel functions and can be represented by this where this Bessel function has the argument of this propagation constant, and the radial coordinate rho times the e to the power of i omega t. For large distances rho this cylindrical waves are approximated as psi of rho t is equal to this.

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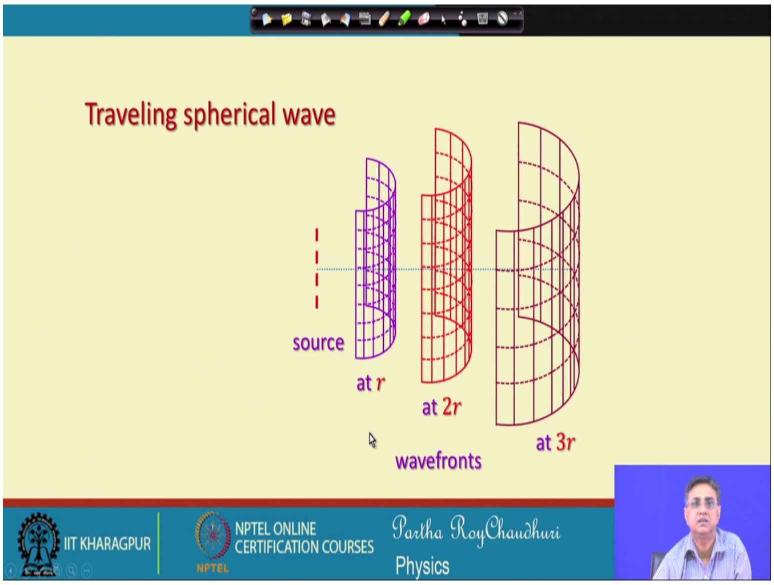

So, 1 upon under root of rho this is again very consistent. Travelling spherical waves look like this we have a line source, there is a cylindrical surface with constant phase at some larger distance, the phases are constant over a cylindrical surface and so on and so forth. So, by doing this we have discussed different types of waves particularly the spherical wave, the plannar wave and cylindrical waves, which we have solved from starting from the Maxwell's equation and the wave equation.

Thank you.