Modern Optics
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Lecture – 12
Wave propagation in anisotropic media

So, we have seen plane Waves in Anisotropic Medium and as very interesting to note that the electric field vectors experience two different refractive indices in an anisotropic medium; particularly in uniaxial medium we have seen that the ray velocity and the wave velocity they are different.

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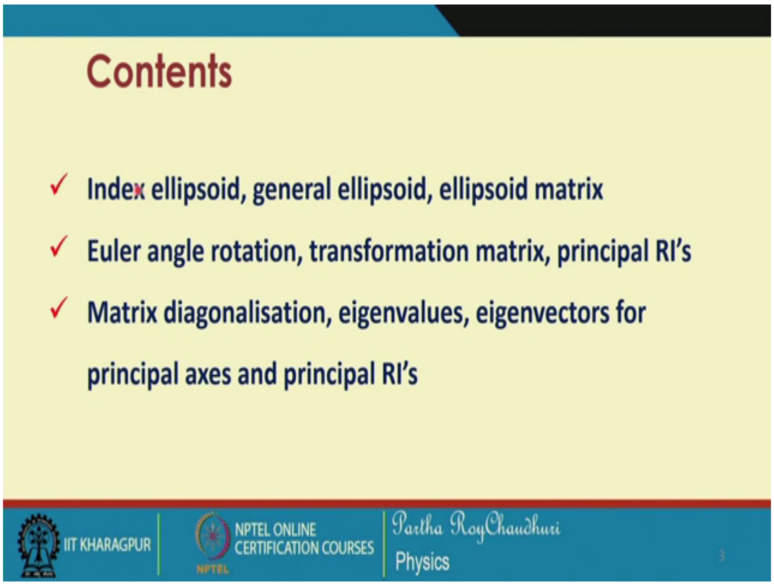

Under this heading we will be discussing today that this index ellipsoid, the general representation of ellipsoid which is the basis of understanding electromagnetic waves propagation in anisotropic medium. Then we will look at the ellipsoid matrix, we have seen the basic equation of ellipsoid while discussing this propagation in anisotropic medium.

Then we will see that how this ellipsoid can be rotated through Euler angle rotation to find out the principal axes system, this will give rise to the transformation matrix. Then there will be another way of doing this transformation by matrix diagonalisation then, because it is a matrix so we will end up with eigenvalues, eigenvectors.

And incidentally we will see that it is very interesting that the two methods of Euler angle rotation and matrix method will give you the same result in terms of the eigenvalues and the transformation matrix or the eigenvectors.

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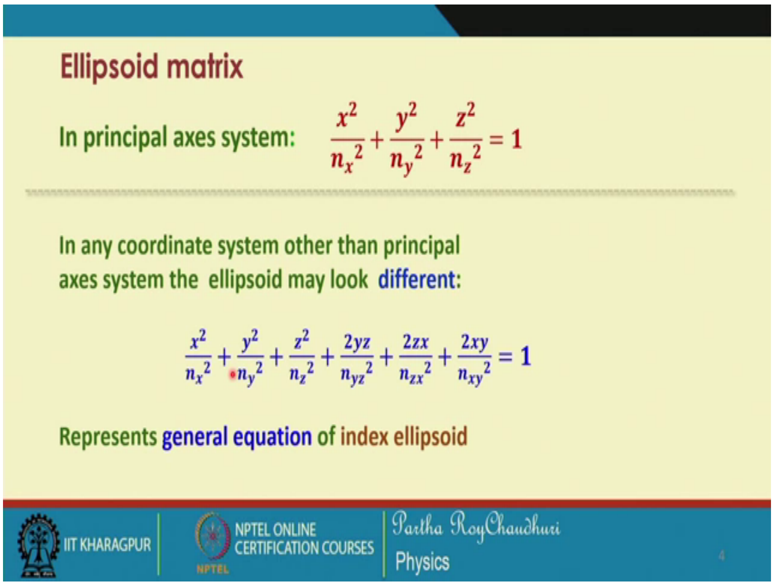

So, let us look at this index ellipsoid matrix; that is in general if you have an anisotropic medium which is characterized by the 3 refractive indices n x n y and n z; we can write this index ellipsoid in this form. Note that this is if the principle axes system, because you have 3 you know crystallographic axes along which the refractive indices are characterized as n x n y and n z, but in any other coordinate system which is different from the crystallographic axes, crystallographic coordinate system.

Then this ellipsoid may look different, that is you will have cross terms y z z x x y and that is actually the general equation of an ellipsoid. And, will see that this ellipsoid is rotated from this ellipsoid when you consider with respect to the coordinate axes fixed with this system and fixed with any other system.

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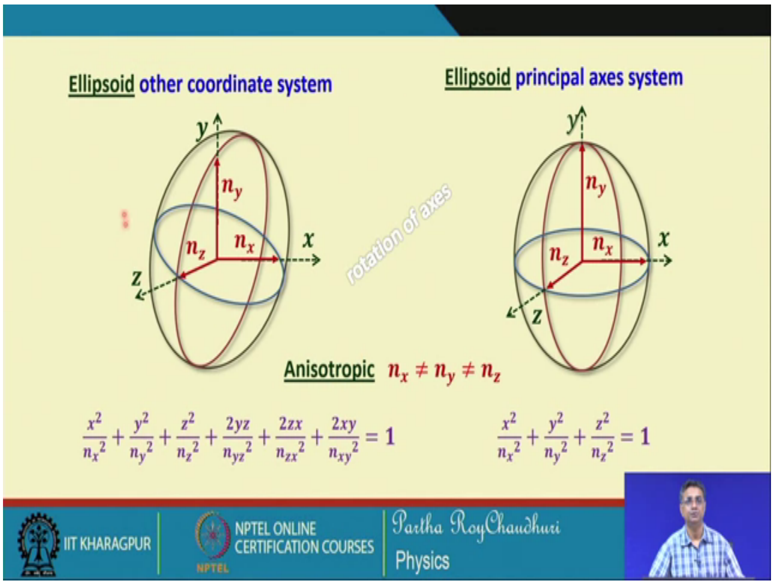

So, when we considered the general ellipsoid equation then it should it will look like this; you have an ellipsoid which is distorted, which is not distorted, which is having an inclination with the coordinate axes which is represented by the crystallographic axes that is the principle axes system.

So, in this system you can represent this x square by n x square plus y square by this so this is the ellipsoid equation and this is also the general equation for the ellipsoid and we considered that in general n x is not equal to n y is not equal to n z. Now, it simply suggest that this ellipsoid and this ellipsoid are the same, but it is rotated the axis is rotated or the ellipsoid is rotated with respect to the stationary axes system.

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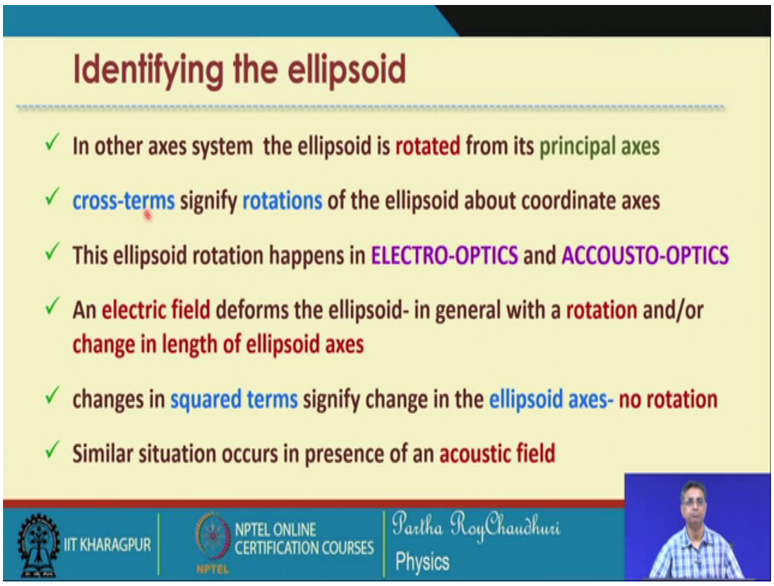

Now, how we identify the ellipsoid? In other axes system the ellipsoid is rotated from its principal axes system, that is what we have seen that you have an a you have a coordinate system which is called the principal axes system, but in any other coordinate system this ellipsoid will be rotated. That is if you rotate the coordinate system then you can again coincide, you can again make the coordinate system and the ellipsoid coincide. So, you can identify this from the cross terms that is y z z x and x y; cross term signify rotations of the ellipsoid about the coordinate axis.

The ellipsoid rotation this kind of situation we will very often encounter in electro optics and accousto optics. And, this is the basis of analyzing this electro optic, effect accousto optic effect and also magneto optic effect that is under some field the ellipsoid will be distorted will be deformed and then you will have to will have to rotate or will have to give some transformation. So, that we get back the principle axis system and you can see that how the principle refractive indices are changed in the new coordinate system.

So, that is the basis and we will be mostly discussing our electro optics and accousto optics under the basis that is why this discussion is very useful and very important. An electric field deforms the ellipsoid in general with a rotation or and or change in the length of the ellipsoid axes, hope we will also consider this situation. Changes in the, and this will be obvious in a situation of isotropic medium in electro optic system where there will be only change in the length of the ellipsoid minor or major axes length of the ellipsoid.

So, changes in the squared term signify change in the ellipsoid axes, but in that case there is no rotation. Similar situation all this things may also happen in the case of the presence of acoustic field. So, there may be a distortion of the ellipsoid in presence of the acoustic field, in presence of the electro optic electric filed or in presence of the magnetic field that also we will see later ok.

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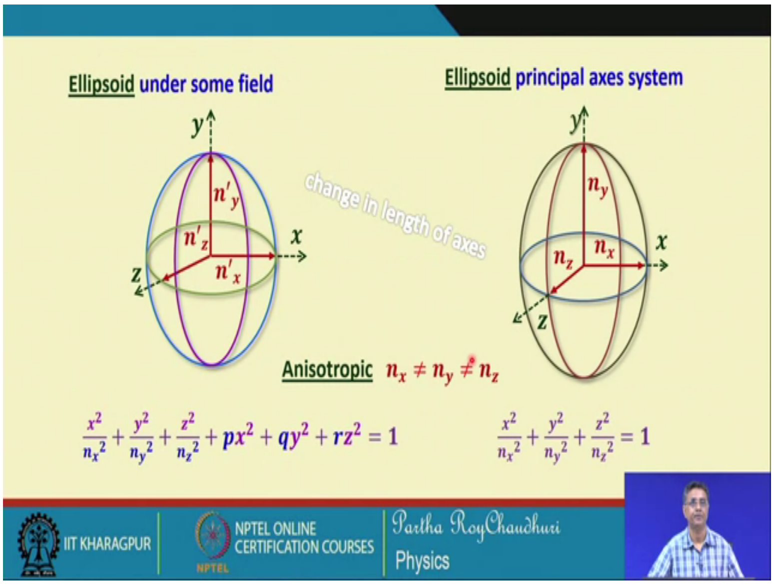

So, ellipsoid under some field; you can see that this is the ellipsoid under some filed there has been only a change in the length when you compare this ellipsoid with this one, but the axes are the same the principal axes are all the same, but there has been a change in the length of the of the minor and major axes of the ellipsoid. That is you see p x, square q y, square and r square because, the new ellipsoid equation involves only the square terms, only the square term x square y square and z square as a result there is no rotation. But, if there are cross term that is x y y z z x; that means, there is a rotation so will understand this situation.

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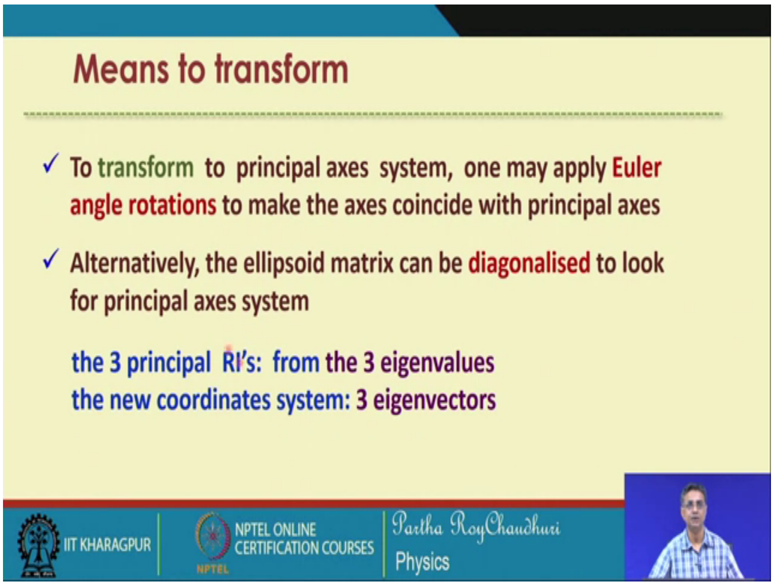

Now, the means to transform this ellipsoid to the principle axis system we will consider that there are cross terms that is we consider that the ellipsoid is changed under the action of some electric field or acoustic field or magnetic field we will understand this phenomenon. So, in that case one may apply this, Euler angle rotations to make the axes coincide with the principal axis, but alternatively one can do that matrix diagonalisation of the ellipsoid matrix to look for the principal axes system.

The 3 principle refractive indices will be the result and we the, we will get the 3 eigenvalues from the diagonalisation of the matrices. And we will also end up with 3 eigenvectors which will give you the transformation for the new coordinate system and we will see how it happens.

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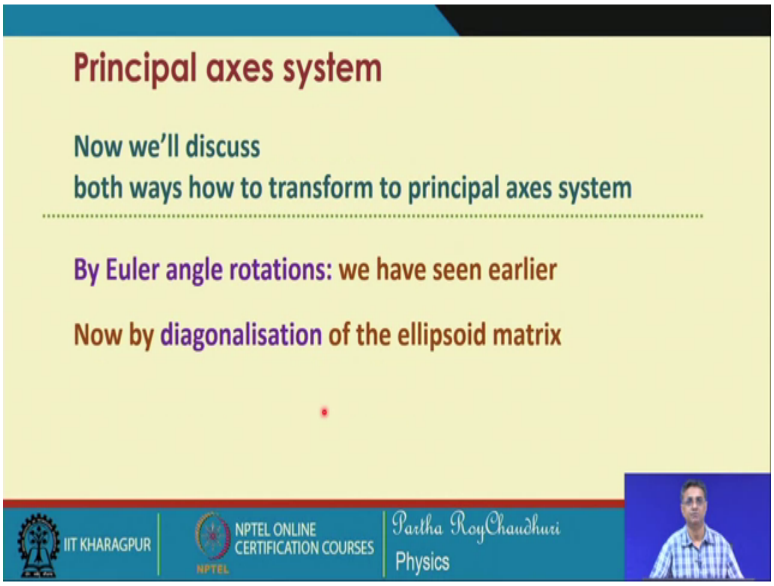

So, in both the ways we will taken example and will use both the both the methods to see; how the change coordinate, change ellipsoid can be made coincident with the principal axes system. So, first will consider you have already seen that Euler angle rotation, but will make a quick visit of this Euler angle rotation in order to make a comparison with the matrix diagonalisation process.

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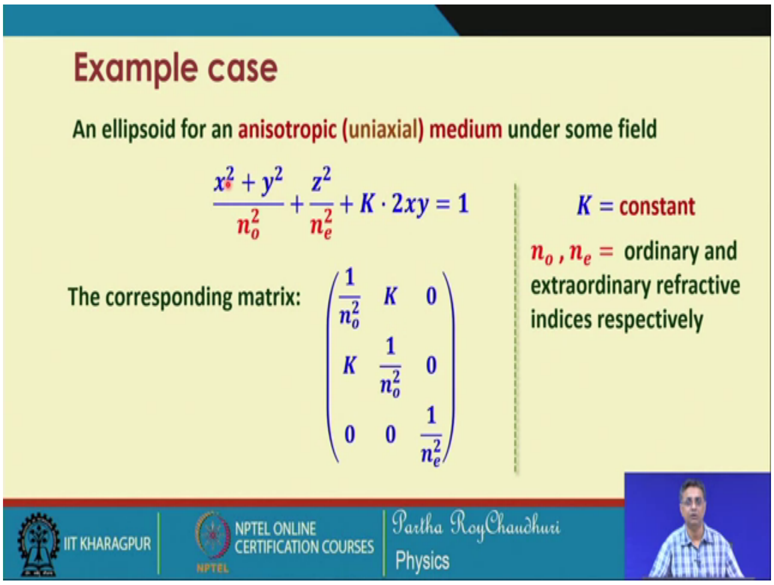

So, an ellipsoid for anisotropic medium uniaxial medium under some field will look like this. So, this may be equal to some electro optic coefficient multiplied by the electric field we will see that situation; which can be put as a constant K n o and n e they are the ordinary and extraordinary refractive indices and then we are very familiar with this ellipsoid which is for an uniaxial medium, but this is under the principle axis principle axis system that is the crystallographic axes system.

Now, corresponding to this equation we can write the ellipsoid matrix as 1 by n o square 1 by n o square by 1 by n. So, these are the diagonal elements of this the K matrix K elements will appear in this of diagonal positions.

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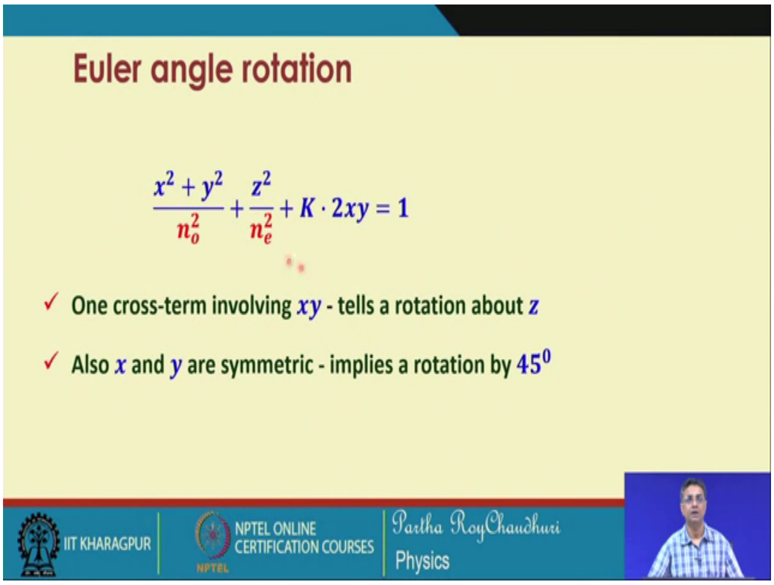

Now, this Euler angle rotation while applying this will have to do some inspection that because the cross term involves only x and y, therefore, it tells you that there should be rotation along z, because it is independent of z the cross terms are not involving any z. Then because it involves x and y which is you know symmetric if you change x by y or y by x the equation does not change; that means, it implies that the rotation will be by 45 degree and we have seen this in the earlier discussion also.

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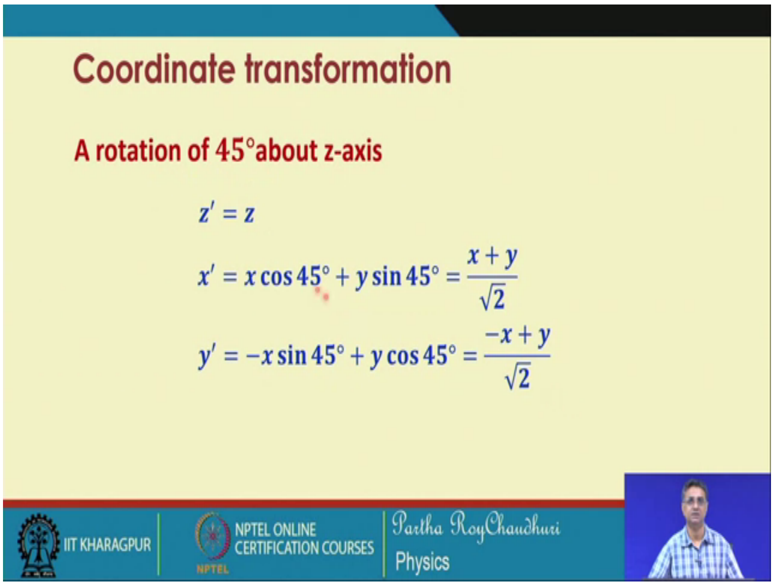

So, a rotation about by 45 degree about the z axis will you will this transformation from this Euler angle rotation if you write that matrix so, z plane will be equal to z, x will be represented in terms of this 45 degree of x and 45 degree of y and similarly y prime will be represent so, this is the transformed coordinate x dash y dash and z dash but, we need the reverse transformation.

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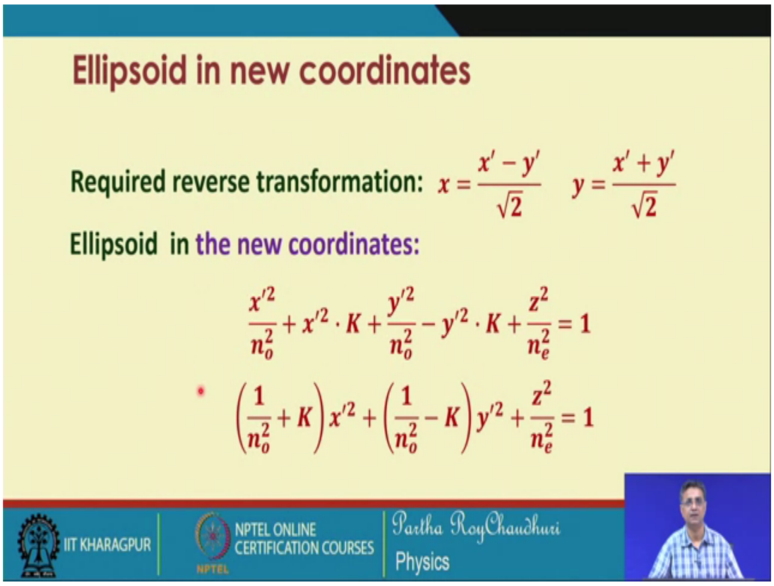

So, knowing this if you put the values of this from these two equation x dash to x plus y y dash to minus x plus y and z to z dash we can actually write the reverse transformation in this form. So, the ellipsoid in the new coordinate system will look like this because in place of x if I substitute this value of this so, square of this will appear square of this will appear in place of these equation x square plus y square by n o square z will remain z prime as it is so we will have a change in only the x square and y square. So, that is what we have trying to write.

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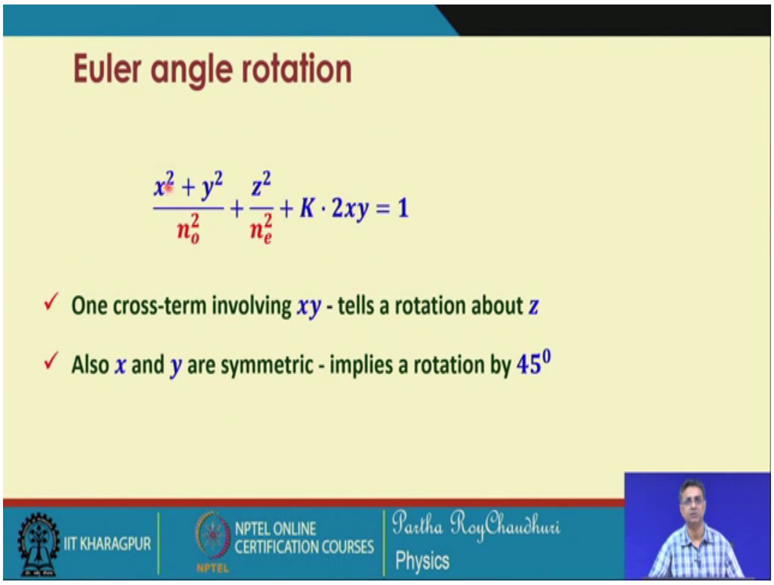

So, effectively we can write this because you have 1 by n o square plus 1 so plus K so we put two of them in one bracket. So, this is the new coefficient which is attached to x dash square and this should be equivalent to 1 by n x dash square and this should be equivalent to, this should be equal to 1 by n y dash square and z this n e remains as it is. So, we started with a medium which was which was uniaxial because x and y refractive indices where same, but now we get the refractive indices which is which is which are all of them are different for x y and z.

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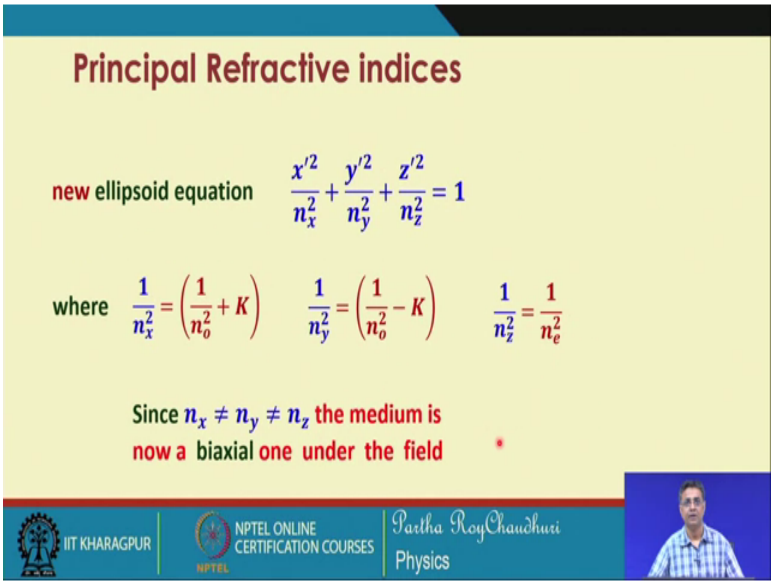

So, that represents a biaxial system so the new ellipsoid in this case we can write as we can write as n x dash square by n x x x prime square by this y prime square by n y square z prime square by n z square equal to so this is again the known form of the ellipsoid, so; that means, under Euler angle rotation we have been able to bring the ellipsoid back into the principle axes system and we have used that n x square is equal to 1 upon n square plus K and similarly n y square equal to.

So, let us remember that n x dash the new principle, new effect n x representation for the principle axis system is equal to this and n y for the new principal axes system will be equal to this and n z equal to this. So, these are the 3 values of this n x n y and n z we will see the same result when will apply this diagonalisation method.

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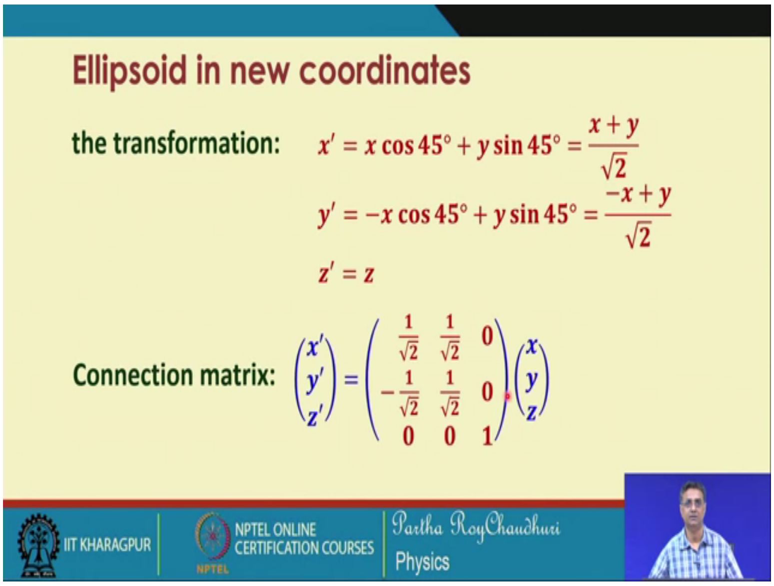

Also remember that this transformation which we imposed that is x dash equal to this, y dash equal to this, which is corresponding to this matrix this will correspond to this matrix, because this will give you 1 by root 2 this will be 1 by root 2 this will be minus 1 by root 2 and z for this will become will be equal to 0. So, this is the transformation matrix from x dash y dash z dash to from x y z to x dash y dash z dash, but we need the reverse transformation: that is we have to express this x y z in terms of x prime, y prime and z prime.

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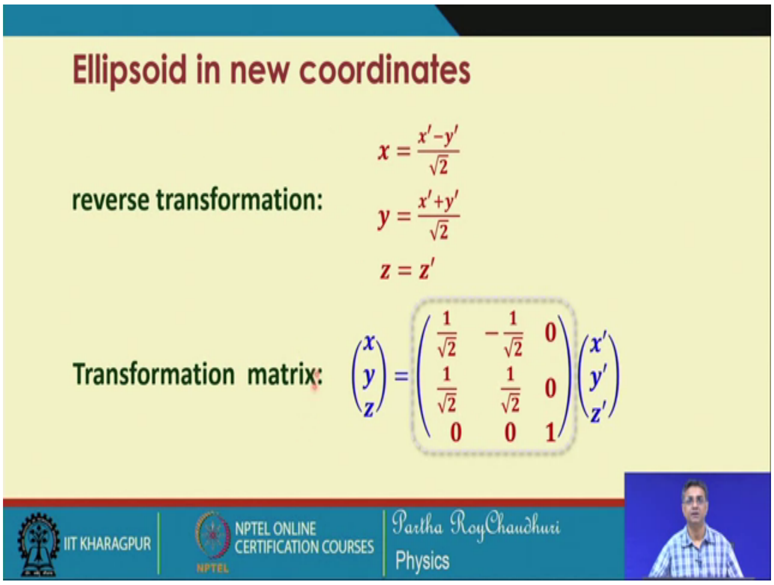

So, then the matrix will look like this x y z will become 1 by root 2 minus 1 because in that case will use this relation so these are the transformation matrix from x-x dash frame whereas, this is the transformation matrix which is from x dash frame to x frame. But, we have used this so, this equation will remember and we will compare with the result that we have obtained from the method of diagonalisation ok.

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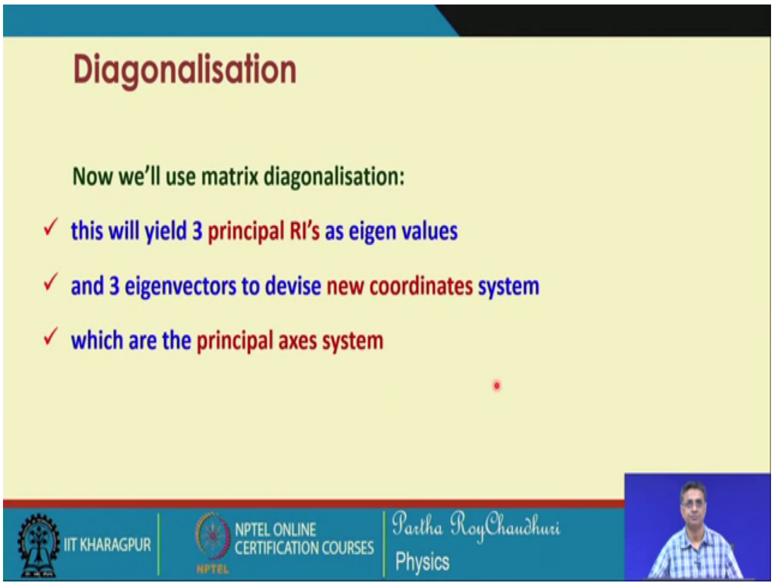

Now, we will use this matrix diagonalisation this will yield 3 principle refractive indices as eigenvalues these 3 eigenvectors will be used to device the new coordinate system: which will correspond to the transformation and this will give you the new principal axis system.

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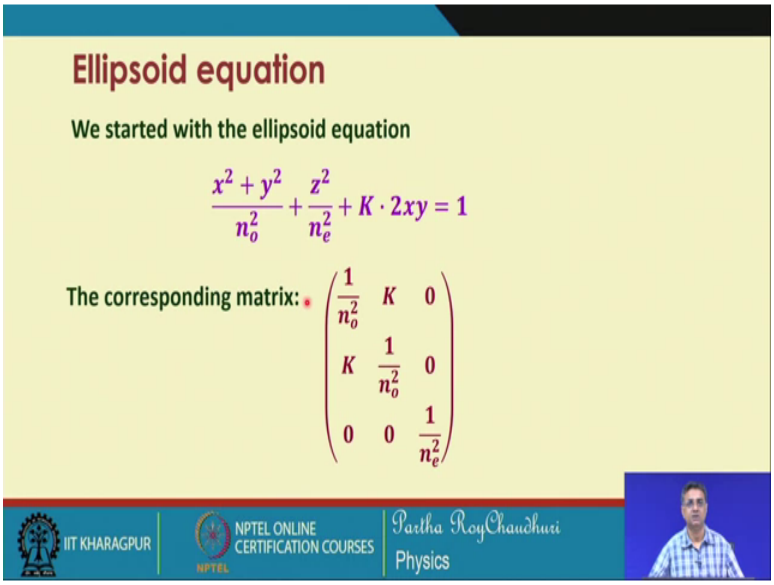

So, this is the mathematical operation that will perform with this ellipsoid matrix index ellipsoid matrix.

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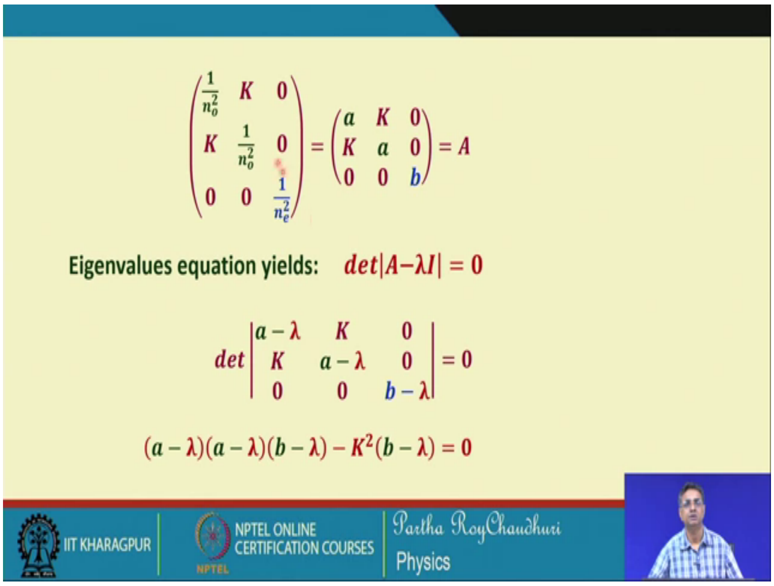

So, this matrix we just write for 1 by n o square equal to a 1 by n o square equal to a and 1 by n e square equal to b. Therefore, this matrix equation for diagonalisation we have to write this eigenvalue equation which is equal to determinant of this matrix a minus lambda I equal to which is the eigenvalue matrix is equal to 0. Which is again very well known in the basic mathematics of matrix diagonalisation and by doing so, we write this determinant of this equal to 0 because a minus lambda, a minus lambda, b minus lambda this determinant will be equal to 0. If you open up the bracket will get this a minus lambda a minus lambda b minus lambda and K square b minus lambda equal to 0 just from here K this into this ok.

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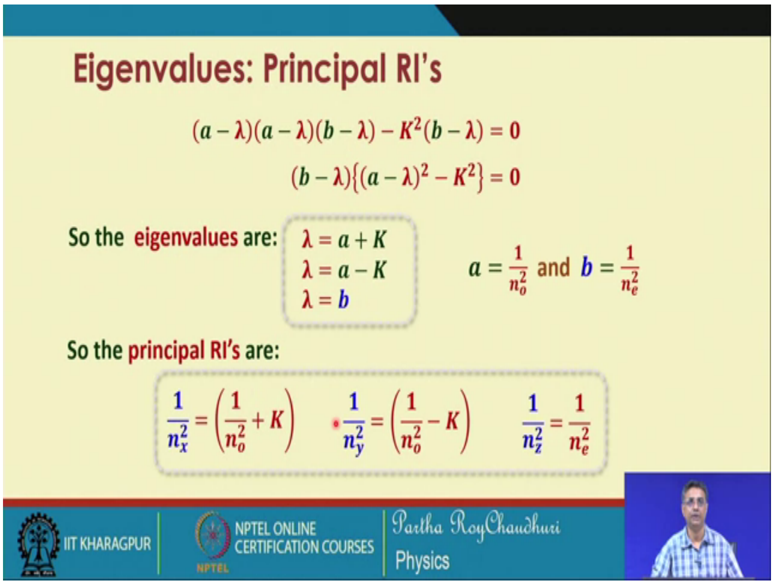

So, from this equation if you take b minus lambda bracketed out then you will have a minus lambda square minus K square equal to 0 this gives you two possibilities that this bracket may correspond to equal to 0 or this bracket b minus lambda may correspond to equal to 0. So, the eigenvalues are corresponding to this bracket will be equal to a plus K lambda and the other value will be a minus K which is obvious from here a minus lambda equal to plus minus K. So, a these are the two roots of the that coming that results from the 1st bracket and lambda equal to b it comes from this bracket b equal to lambda.

Therefore, we get the eigenvalues because lambda equal to a lambda because we have substituted a equal to 1 by n o square and b equal to 1 by n e square if we get it back then you can write this the eigenvalues lambda equal to 1 by n o square plus K this will be 1 by n o square minus K, which are exactly the same that we obtained from the Euler angle rotation. So, this is your new n x square 1 upon n x square and this is 1 upon 1 n y square which is the new principle refractive indices this 1 by n z square n e square that remains the same.

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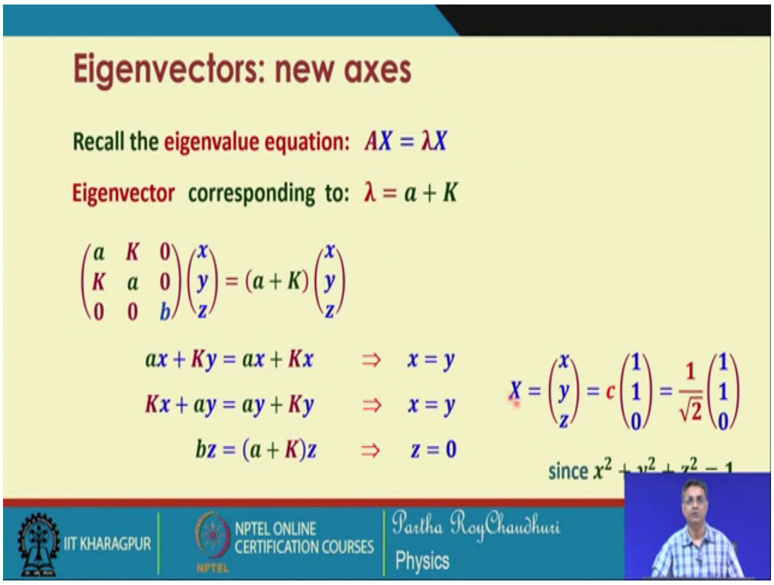

Now, will have to look for the eigenvectors recall that eigenvalue equation A X equal to lambda X and the eigenvector corresponding to this lambda a plus K, which is the first eigenvector we have called a plus K, if I use this eigenvector then we can write this equation this equation as a x equal to the eigenvector eigenvalue a plus K x.

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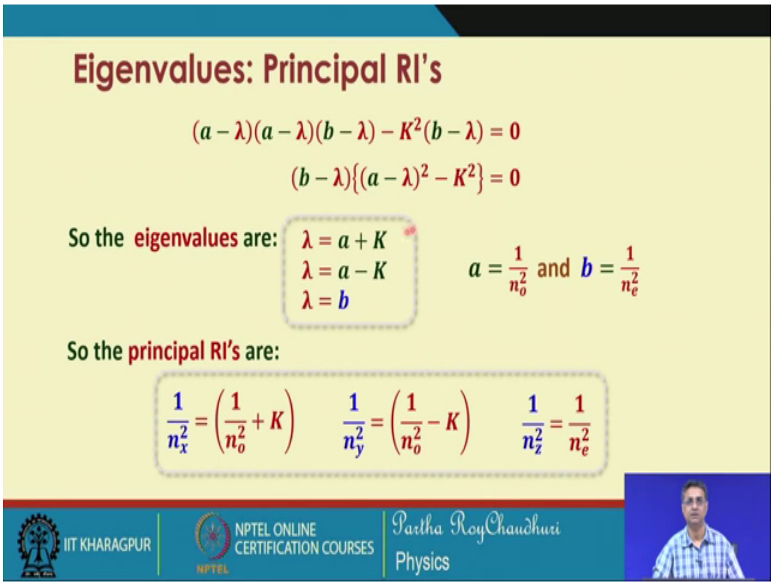

So, if I just take the 3 simultaneous equation obtained from this matrix equation a x plus K y equal to a x plus K x this simply says that if you removed a x from either side then you will end up with K y equal to K x; that means, x must be equal to y, so, that is what is implied by writing x equal to y. From the 2nd equation a y and a y will cancel from either side, which will give you K x equal to K y and that yields the same equation that is the same relation that is x is equal y, but from the 3rd equation b z equal to a plus K into z.

So, it tells you that because a plus K cannot be equal to b so therefore, b z must be equal to 0, therefore, you get the eigenvector x equal to x y and z because x equal to y so putting x equal to x equal to y equal to c we can write that 1 1 0 c if you take outside, now, since x square plus, y square plus, z square equal to 1 then you can write this normalized eigenvector 1 by root 2 1 1 0.

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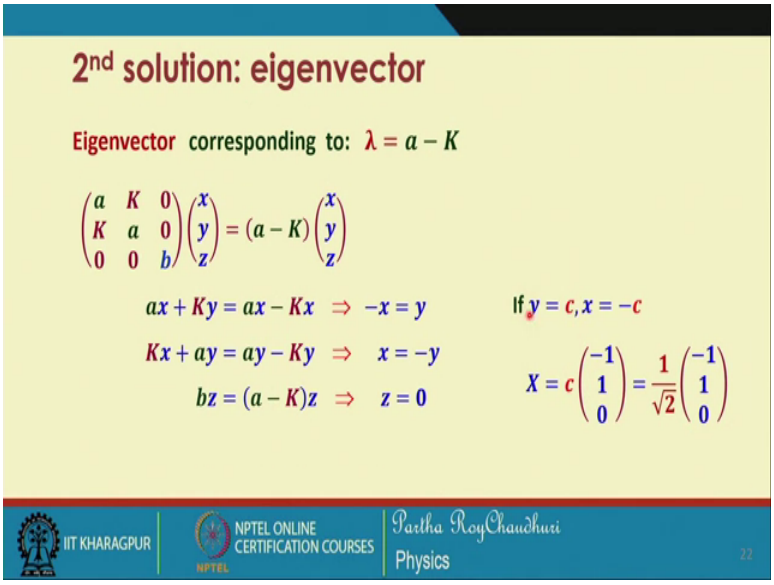

And in the same way we will considered the 2nd eigenvalue that is lambda equal to a minus K we will substitute in the eigenvalue equation a a b x y z the eigenvalue and again the eigenvector. Therefore, the we again get the 3 equations simultaneous equations and these are a x plus K y equal to you can remove a x from either side that tells you that K y equal to minus K x which is very straight forward that minus x equal to y and from the 2nd equation if you removed a y you get K x equal to minus K y that also gives you the same relation that is K x equal to minus y.

Now, if y equal to c the next will become equal to minus c and if you put it back or the 3rd one also gives you that z is equal to 0. In the same way we found in the 1st eigenvalue case that is b and a minus K they are unlikely to be equal in that case z must be equal to 0 so we get this eigenvector corresponding to the eigenvalue a minus K.

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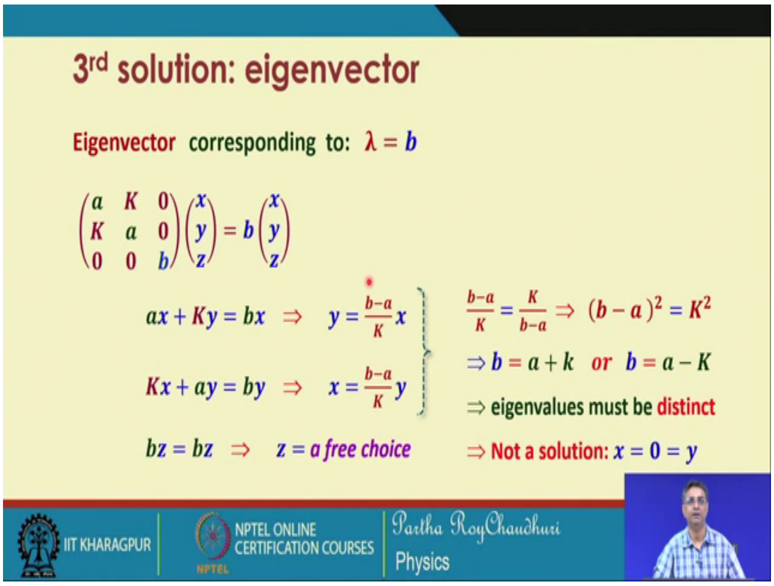

Let us now see the 3rd eigenvalue that is corresponding to lambda equal to b, we have this eigenvalue equation in this case this is the eigenvalue and this is the eigenvector the 3 simultaneous equations are this. So, from the 1st equation a x plus K y equal to b x what does it mean? It means that y equal to b minus a by K into x, from the 2nd equation we get K x plus a y equal to b y this means x is equal to b minus a by K into y, which are very difficult I mean which will give you this equation b minus K equal to K by b minus a so that gives you b minus a square equal to K square.

Therefore, you will have which is indeed very difficult to give one unique value of x and y b will be equal to a plus K or b is equal to a minus K so both the possibilities the connection with a and b are including both the values a plus K or a minus K. But, eigenvalues must be distinct and they should not merge they should be separate, so as a result this x this cannot give you any solution. So therefore, x equal to 0 and y is also equal to 0 and z will have a free choice, because on either side b z equal to b z so; that means, you can take z can assume any value so you have a free choice of z.

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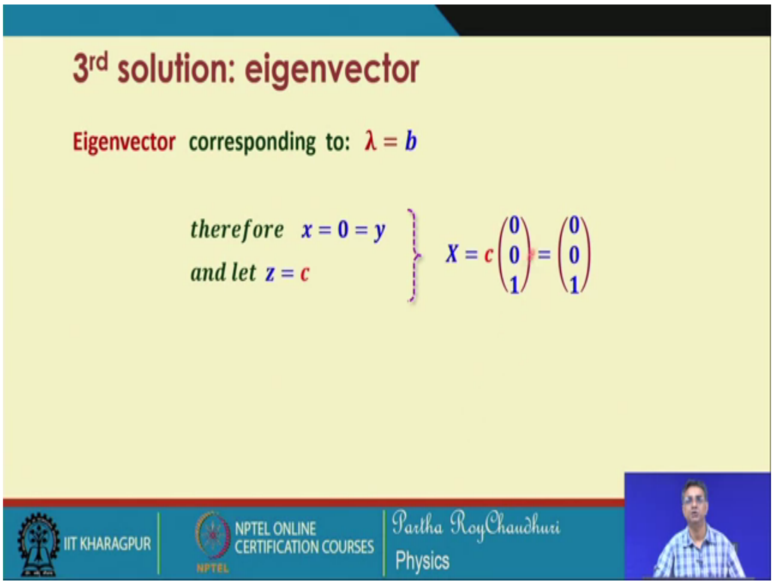

If I use this relation that x equal to 0 y equal to 0 and z equal to c then we can write that eigenvector x equal to c 0 0 1 and under normalization we can write this equation like 0 0 1.

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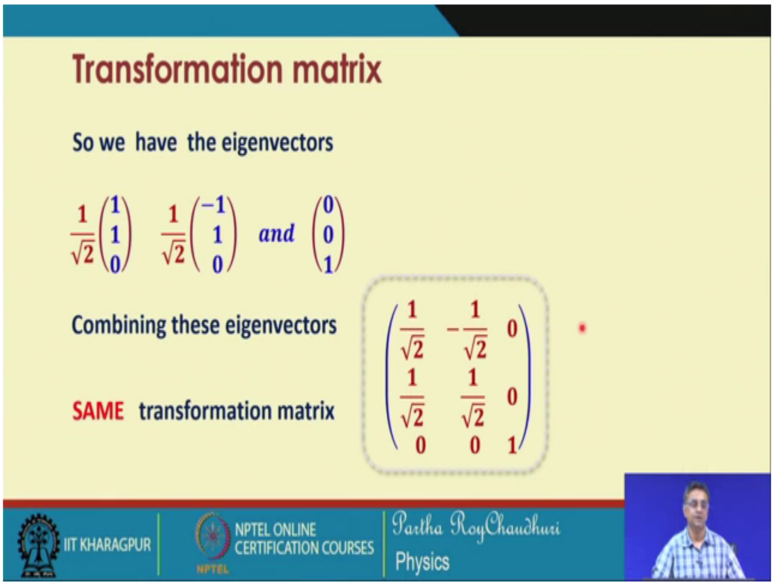

Now, we have 3 eigenvalues and 3 eigenvectors then if I combine this 3 eigenvectors then we can write this equation this matrix as 1 by root 2 minus 1 by root 2 you can see this from here 1 upon root 2 1 upon root 2 0 here 1 upon minus 1 upon root 2 plus 1 upon root 2 0 and this will be 0 0 1. So, I just put combine all 3 eigenvectors together which is nothing, but the transformation matrix that we obtained from the Euler angle rotation.

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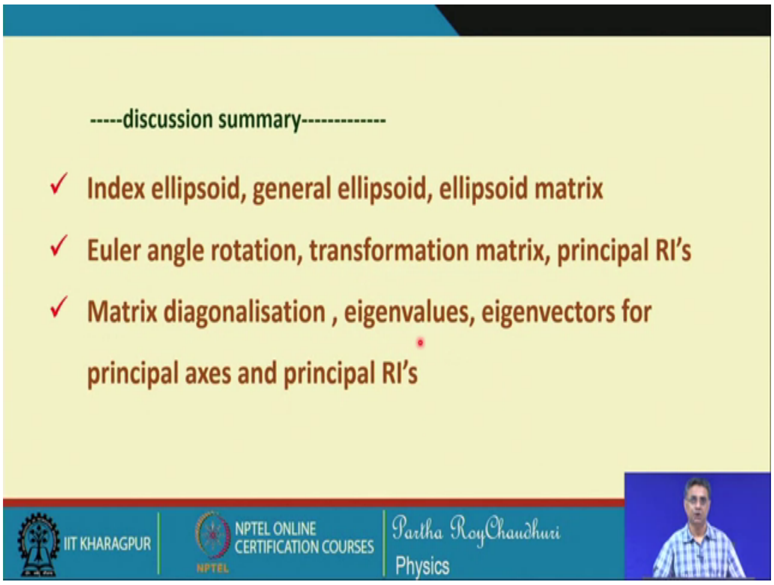

So, we have discussed in this context that a general index ellipsoid which is very useful in analyzing the waves electromagnetic waves in anisotropic medium and how this ellipsoid will look in general coordinate system which is different from the principle axes system. This ellipsoid will associate itself with a, with an ellipsoid matrix. So, there are beings of bringing the ellipsoid which is otherwise not looking very simple to bring it back to the principle axes system. There are methods one is this Euler angle rotation, that if I can give the proper rotation to the coordinate axes then, we can make the coordinate axes coincide with the principle axes system; that is the one which are associate with the crystallographic axes.

The other method is to use this matrix transformation and by diagonalising the matrices we can and again get the eigenvalues and eigenvectors. We have seen that by using Euler angle rotation, we get the principal refractive indices and we also find the transformation matrix by matrix diagonalisation. We also till date have found that the 3 principle refractive indices which are the same as we obtained from here and these 3 eigenvectors were combined together to get the transformation matrix.

So, this discussion is very useful and we will be very often on using in the analysis of electro optic effect, the modulators, switching of the modulators by considering just diagonalisation or this Euler angle inspection for evaluating the electro optic, magneto optic and accousto optic effects.

Thank you very much.