Modern Optics
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Lecture – 11
Wave propagation in anisotropic media (Contd.)

So, we are discussing Waves in anisotropic medium.

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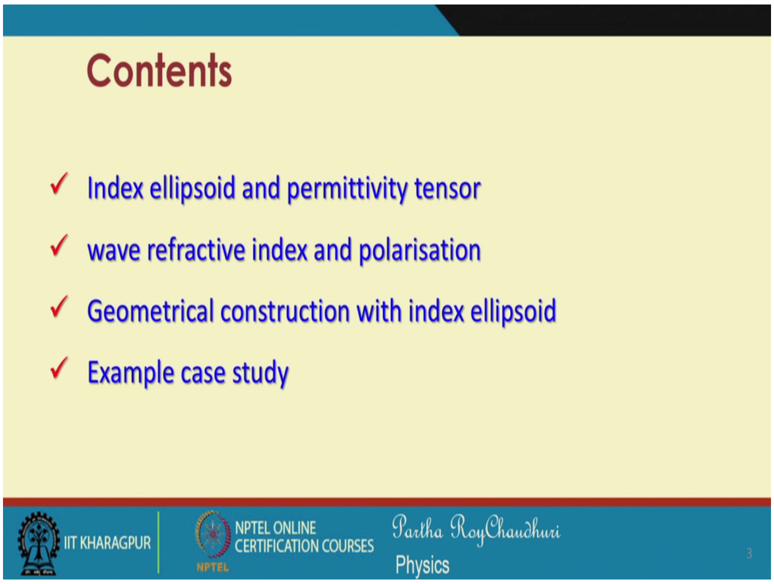

And we have seen that in general there will be 2 directions of polarizations. There will be 2 refractive indices seen by the way. And knowing the refractive indices from the solution of the determinant equation, we can identify the directions, we can identify the electric field and the direction of the displacement vector and the direction of propagation we can find all the relationship.

So, but this approach was starting with the Maxwell's equation, we wrote the eigenvalue equation, and from there we will look for the solution for the non-trivial values non-trivial solutions of the determinant equation. But there is another interesting geometrical approach, which we will discuss now on that is the method of using index ellipsoid. And from there in this context we will discuss the permittivity tensor. Wave refractive index and polarization these 2 properties will again determine from the index ellipsoid geometry.

So, for that we will analyze the geometrical construction, and then we will study an example case which will consolidate our idea. And understanding about how we can use the index ellipsoid for the purpose of identifying the refractive indices, the direction of polarization, the direction of electric field vectors, and given the direction of propagation of the electromagnetic waves in a non-isotropic medium.

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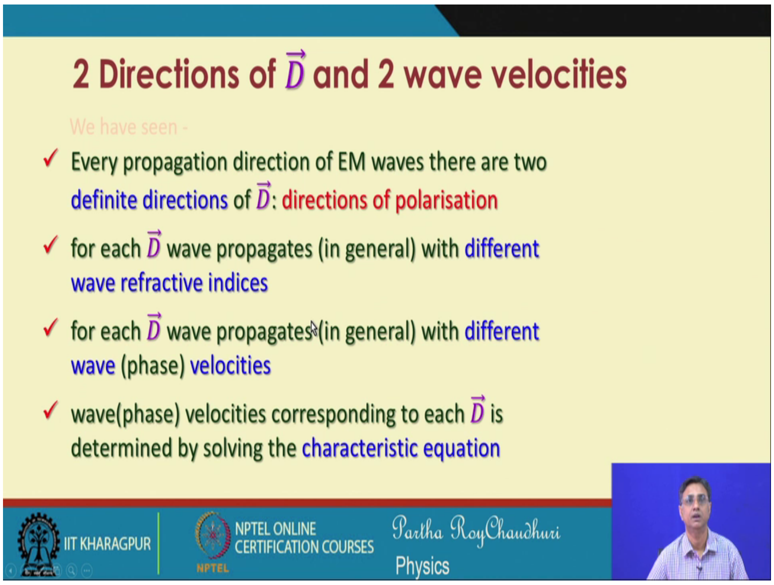

So, the 2 directions of D and 2 direct 2 wave velocities. This we have seen that every propagation direction of electromagnetic waves. There are 2 definite directions of D and these are the directions of polarization.

Now, for each D wave, for each D the wave propagates in general with different wave refractive indices. Since the wave propagates with different wave refractive indices because the refractive index and the wave velocity, they are connected with the free space velocity. So, for each D the wave propagates in general with different wave velocities or the phase velocities; that is c by n or is equal to the wave velocity or the phase velocity because the refractive indices are different.

So, the wave velocities are also different. Now these wave velocities corresponding to each D is determined by solving the characteristic equation. That is what we have seen in the earlier section by solving that matrix equation; that is in terms of that the putting the determinant of the matrix equal to 0.

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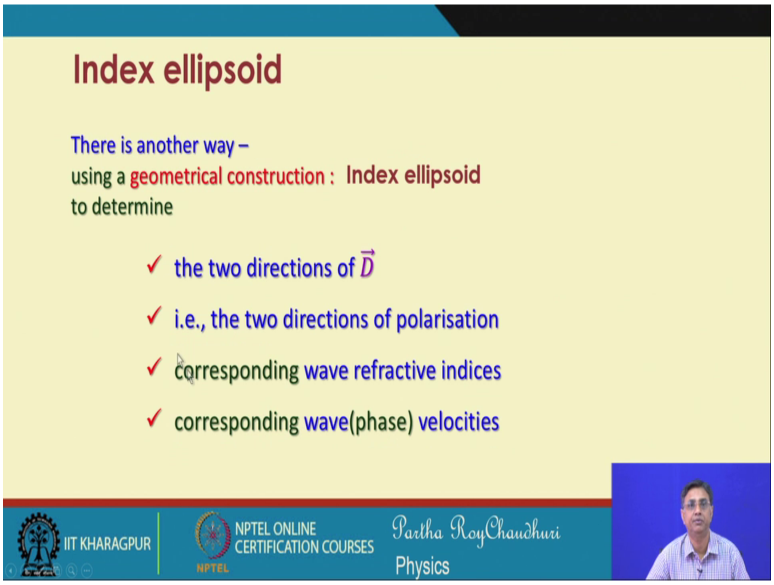

Now, we will make a different approach. That is a geometrical construction, and will be called as index ellipsoid construction. This will again help us to determine the same thing that is the 2 directions of the displacement vector; that is the 2 direction of the polarization when the direction of propagation of the electromagnetic wave is specified. Then the corresponding wave refractive indices which will come not from the solution of the Eigenvalue equation, but from this index ellipsoid, the minor axes and major axes half axes of this ellipsoid, and the corresponding wave velocities. So, all these quantities can again be found out by using this geometrical construction of index ellipsoid this is also called the optical indica matrix or indicatrix.

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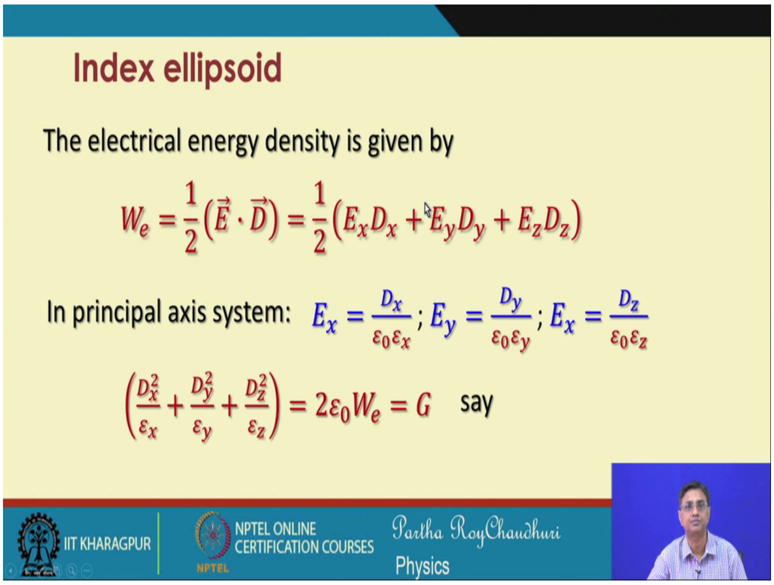

To do that let us recall that the electrical energy density of the electromagnetic waves is given by this quantity it is only the electrical energy density.

So, that is why we use omega e. So, this is equal to half of E dot D, which if you expand in this form, then it will become half of E x D x plus E y D y plus E z D z. In the principal axes system this quantity is valid. Otherwise you would end up with the permittivity components the off diagonal components. So, in the principal axes system E x will be equal to D x by epsilon 0 E x. E y will be like a similar term and E z this should be E z will be equal to D z by e 0 epsilon 0 E z.

So, we can write this if we put into this equation the values of E x, E y and E z in this equation. Then we will get D x square by epsilon x D y square by epsilon y plus D z square by epsilon z, which will be equal to twice omega e into that epsilon naught which I call a quantity that is G.

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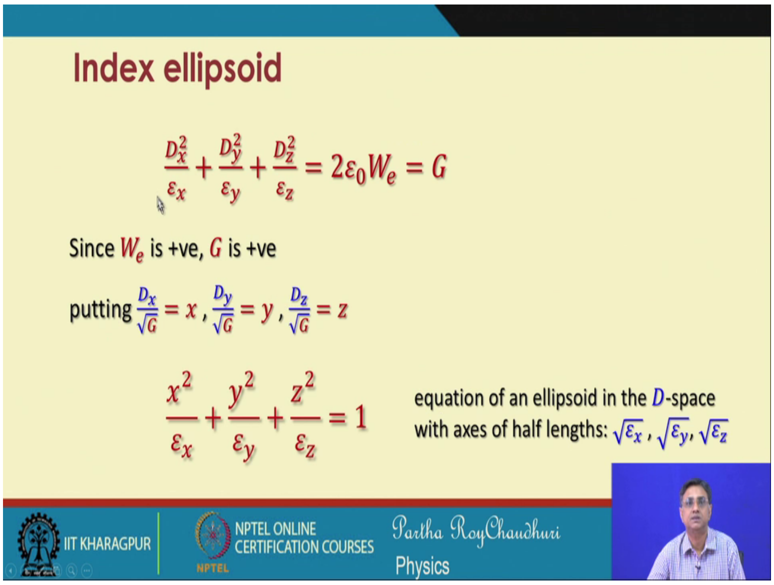

So, this equation that is D x square by epsilon x D y square by epsilon y and so, one is equal to G, if I again divide by G, then we can write that D x by G is equal to x.

If we assume that we assign this D x by G as equal to x, and similarly for D y by under root of G equal to y and so on, then this equation can be rewritten in this form; that x square by epsilon x plus y square by epsilon y plus z square by epsilon z is equal to 0 is equal to 1.

So, which is obviously, you can recognize that this is the equation of an ellipsoid in 3 dimension. So, equation of an ellipsoid in the D space is represented by this equation because x actually incorporates this D x y takes care of D y and z this D z so, with the axes of half lengths epsilon x epsilon y epsilon z. So, we define an ellipsoid in the D space whose half lengths of the axes will be epsilon x epsilon y epsilon z.

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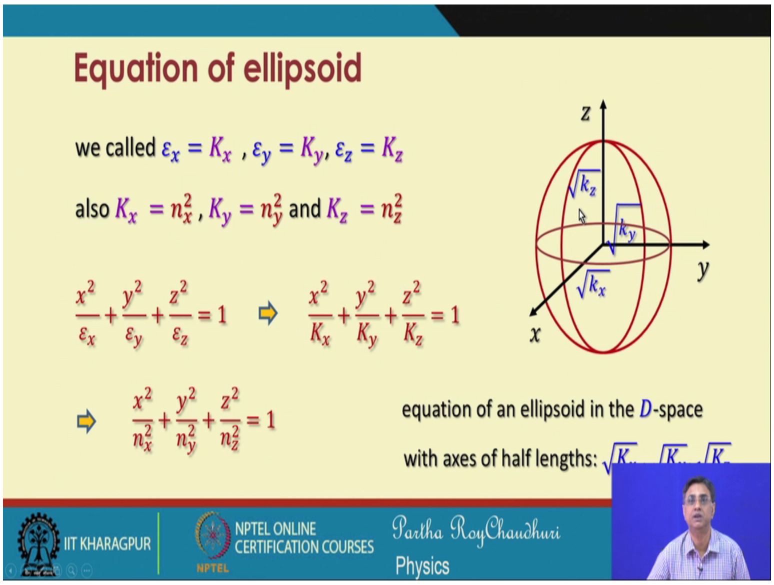

So, how does it look like? You can refer to this figure we have an ellipsoid where the half length of the axes will be epsilon x under root of that under root of epsilon y and this. So, this should be capital K, capital K, capital K. So, epsilon x equal to K x if K y and K z; if I represent in this way using this K x, K y and K z, then we can write K x equal to n x square K y equal to n y square and K z equal to n z square. Because this epsilon x is equal to n x square, epsilon y equal to n y square, epsilon z equal to n z square.

So, we rewrite this equation as x square by epsilon x or n x square y square by epsilon y z square by epsilon z equal to 1. Or you can write in this form or you can write in this form. In any case, if I represent this ellipsoid equation by using n x n y and n z, then it is not the square root of K x, but it is only the n x that will represent the half lengths. And that will be the refractive index seen by the wave when the wave is travelling along the z direction that individual cases will undertake and discuss.

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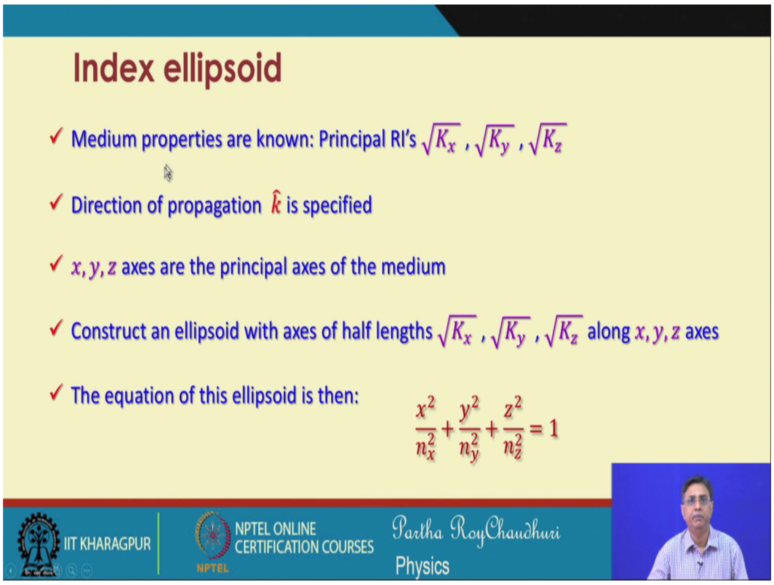

So, the index ellipsoid, the medium property is when they are known; that is, when we know the principal permittivity values principle refractive indices that is under root of K x and root of K y and K z. These quantities are known, then the direction of propagation k is also specified in such a medium; x, y, z these axes are the principal axes system of the medium which we have to reorient to coincide with the Eigen axis of the medium. After doing this because these are the these are the parameters, these are the quantities which are known to us which are available, then we will construct an ellipsoid with the axes of half lengths of under root of K x, under root of K y and under root of K z along x y and z axes. Then we will use this in this equation that is the equation of the ellipsoid. So, all that we have to do that, we have the medium and the medium within the medium we have specified the Eigen axis which is the axes of our reference that is x y z, these are the principal axes of the medium.

Then about that principal axes, we construct an ellipsoid whose half lengths of the ellipsoid will be equal to epsilon x epsilon y and epsilon z square root of that that is n x, n y and n z. So, this is all that we would like to do.

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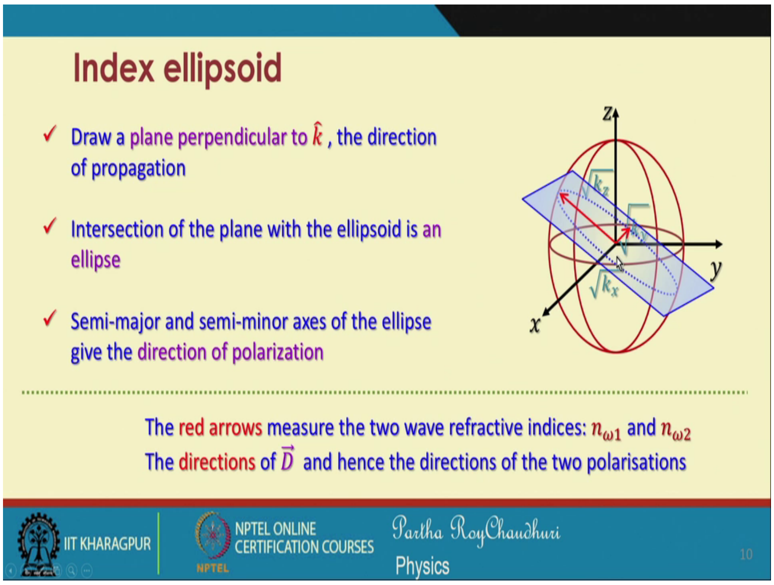

So, let us look at this that you have this ellipsoid constructed which has semi axes half lengths equal to under root of K x, for y direction it is under root of K y and for z direction it is under root of K z. So, these are the half lengths or the semi axes of the ellipsoid. Now you consider the propagation of a wave along any general direction. If I know, if I specify the direction of propagation of the wave, then we construct a plane which is perpendicular to the direction of the propagation of the wave. And this plane will now intersect the ellipsoid in a circle; you can see that there is a circle dotted circle which represents the intersection of the plane with the ellipsoid. Now the semi major and minor axes of the ellipse side will be the direction of polarization. This is as simple as that.

So, by geometrical construction we can identify the polarization directions and the values of the magnitude of this vector, the magnitude of this vector we will represent with appropriate scaling the solutions that is n omega 1 and n omega 2 we will see that. So, these are the directions of polarizations and this will give us the values of the refractive indices. So, the rate arrows measure the 2 wave refractive indices n omega 1 and n omega 2, the direction of D and hence the direction of the 2 polarizations are also indicated by the direction of the red arrows.

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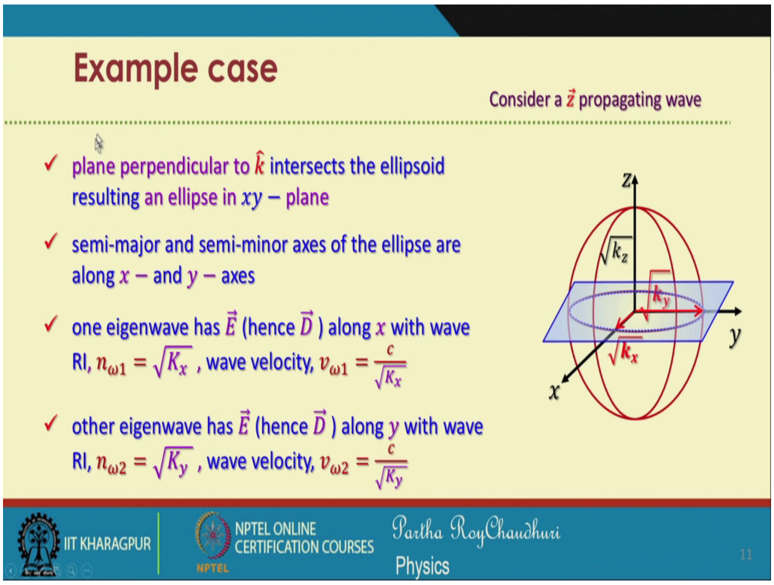

So, let us study an example situation that the plane perpendicular to k intersects the ellipsoid resulting an ellipse in the x y plane; that means that the wave is travelling along the z direction. Since the wave is travelling in the z direction, then we construct a plane which is perpendicular to z and this will intersect the ellipsoid in a circle, the circle will lie in the x y plane. Now not the circle the ellipsoid the ellipse will lie in the x y plane.

So, the major axes semi major axes and minor x semi minor axes of the ellipsoid will represent the 2 refractive indices. And also the direction of the polarization, that is what semi major and semi minor axes of the ellipse are along x and y axes. One Eigen wave has E hence D along the x because this is the principal axes system. So, the reduction of the electric field and the direction of the displacement vector there in the same direction. And they are related by this by multiplied by this quantity. And for this it will be D and E will be related by this constant under root of K y.

So, one Eigen vector will be along this k axes, that along the x axis that is n omega 1 will be under root of this. And the wave velocity corresponding to that Eigen vector will be given by n omega 1 equal to c by under root of K x. The other Eigen wave has the electric field E and hence the D along y axes. And the refractive index seen by this wave which is travelling along the z direction will be equal to under root of K y. The wave velocity for this wave will be c by under root of K y.

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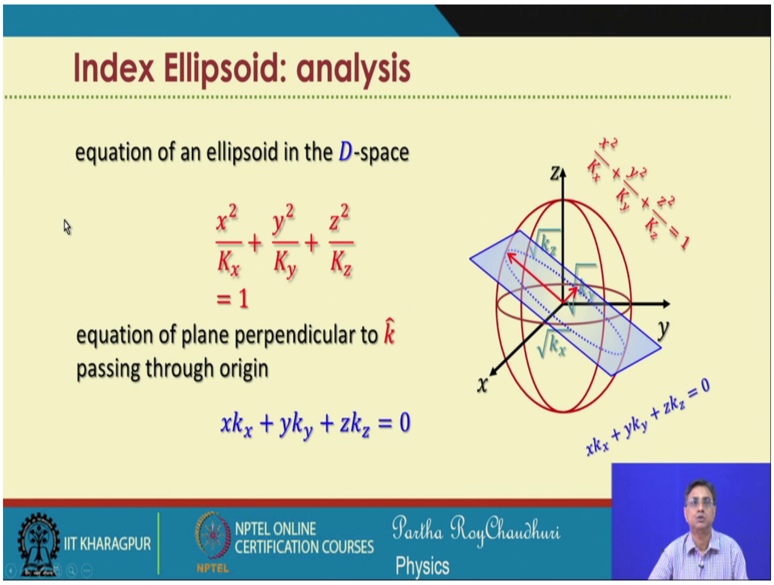

So, the equation of an ellipse in the D space is equal to this, we have seen this equation of the plane perpendicular to k passing through the origin is this. Now all that we are trying to do is that we would like to show that the ellipsoid intercepts the plane in an ellipse. And how we can mathematically extract from this construction that the semi major and semi minor axes of the ellipse will correspond to the quantities that we are looking for, that is the displacement vectors or the electric fields, and the solutions the roots that is the 2 wave refractive indices.

So, to do that let us write down the equation of the ellipsoid in terms of K x K y and K z which is equal to 1. Equation of the plane perpendicular to k can be written in this form we have seen while discussing the plane wave solution, that the phase factor must be describing the phase front must be describing a plane that is K x into x y into K y z into K z will be equal to 0.

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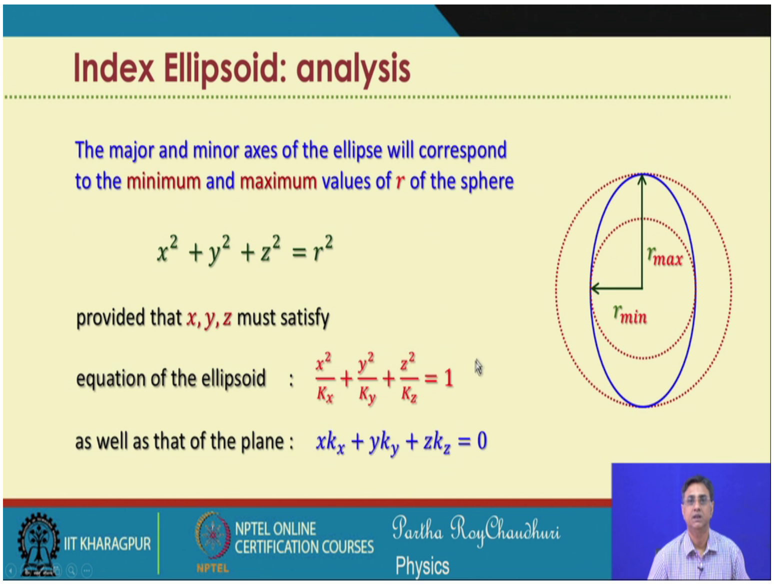

And because it describes an ellipse so, about that ellipse, if we think of the maximum value of the of the axes, that is the semi major axes and semi minor axes because they are different. So, this will correspond to 2 spheres or 2 circles in 2 dimensions rather 2 dimensions. So, this will correspond to a circle of radius r max, and the minimum value that is semi minor axes will correspond to a circle of r min. Therefore, we can write a general equation representing all ellipses within this by this equation x square, sorry, all spheres within this because this is all in 3 dimension. So, x square plus y square plus z square equal to r square; provided that x, y, z must satisfy this equation.

So, this condition has to be satisfied and as well as this condition has also to be satisfied. So, we have an imaginary sphere this will correspond to 2 spheres which will give you the value of r max and r min, and the spheres will simultaneously satisfy these 2 equations.

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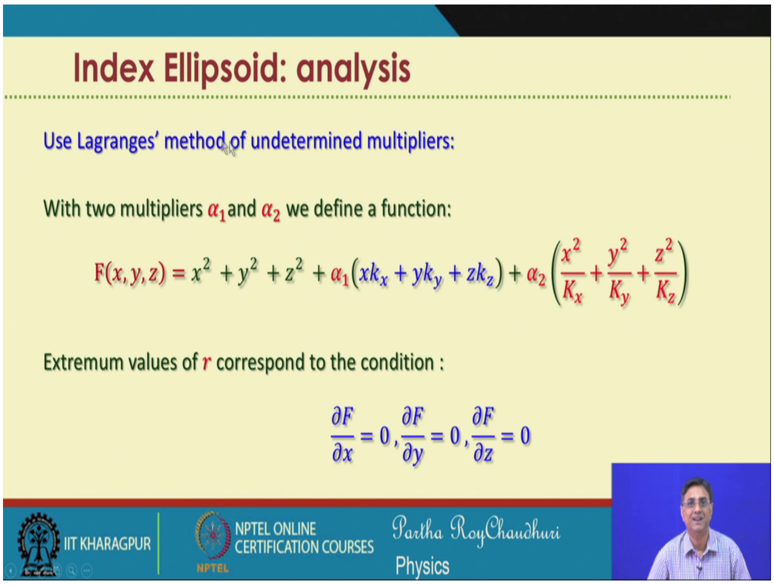

So, we can write a function which involves all the 3 conditions that there is a sphere. There is the equation of an ellipsoid and equation of a plane.

So, we put together and write a function, and we will use the Lagrange's method of undetermined multipliers to find out the solutions for n omega values that is the K x K y and in terms of K x K y and K z. So, we define this function f of x, y, z equal to x square plus y square of z plus z square, the left hand side of the sphere, then some constant into this part of the plane, and some constant alpha 2 into this part of the ellipsoid. So, put together we write this function. And to look for the extremum values of this function we know that del f del x must be equal to 0, del f del y must be equal to 0, and del f del z will be.

So, these 3 conditions will be simultaneously satisfied when that will correspond to the condition for the extremum value that is minimum and maximum value of the r. That is x square plus y square plus z square we can write this equal to r.

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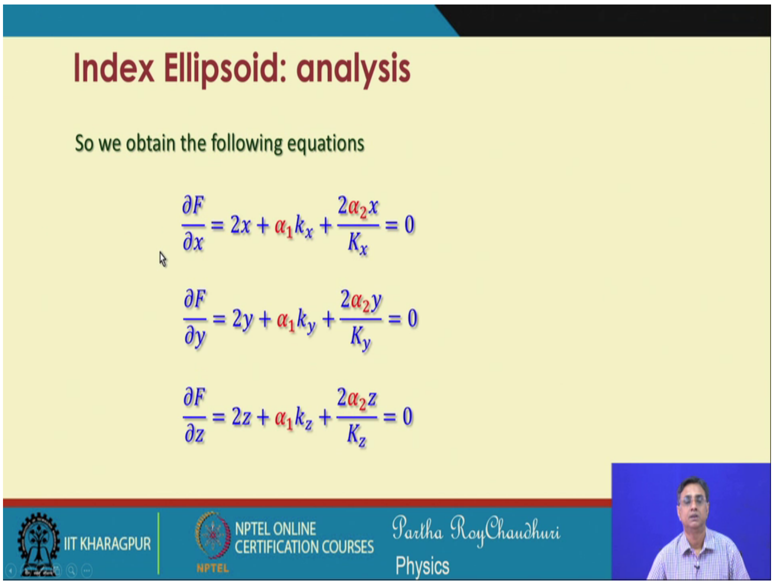

So, if I take the differential of the function f that is del f del x, this will be equal to this will be equal to 2 x, and in this case it will be equal to alpha 1 x, and this will give you 2 x into alpha 2. That is what I have written del f del x equal to 2 x, alpha 1 x and 2 alpha 2 x by K x; which will be put to equal to 0. Similarly, for del f del y we can write a similar equation and in the same way for del f del z we can write the third equation.

Now we have 3 equations and the task lies is to find the values of this constant unknown constant that is alpha 1 and alpha 2.

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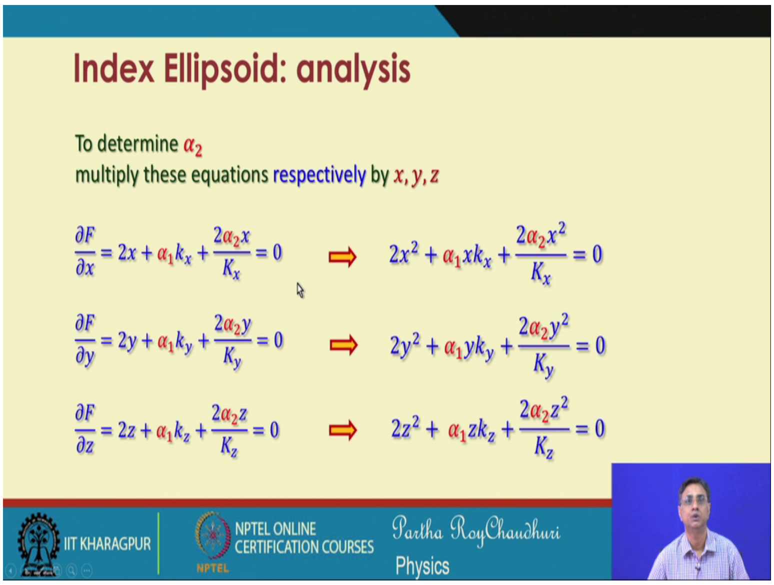

And in order to do that we reorganize this equation 2 x plus alpha 1 K x plus 2 alpha 2 x by K x equal to 0, this equation this equation and this equation we add them. But before adding we multiply this equation with this equations with x, y and z respectively. So, if you multiply this equation with x, we get 2 x square alpha 1 x K x plus 2 x square alpha 2 K x. And similarly this will be 2 y square and this will be alpha 1 y K y this will be 2 alpha 2 y square by K y is equal to 0 and so on for the third equation. So, this will result in this form.

Now, we can add these 3 equations, the left hand side and the right hand side.

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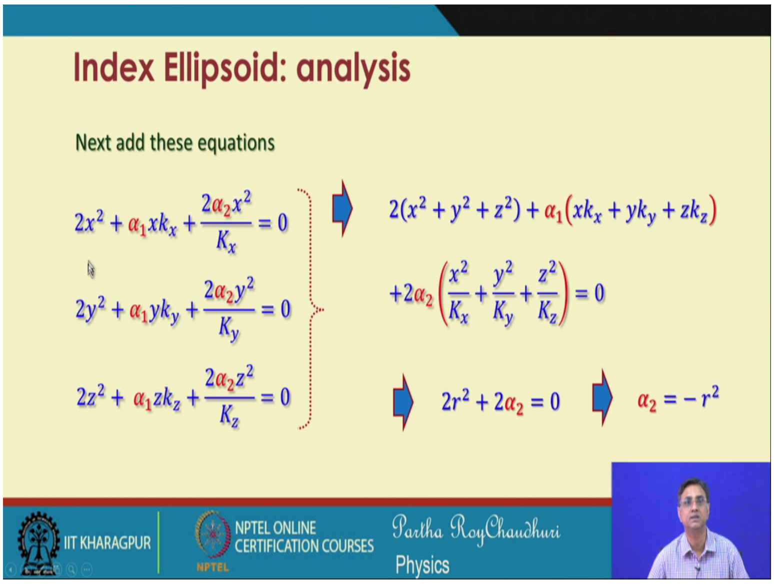

So, as a result when you add these 2 equations, 2 into x square plus y square plus z square which is equal to r square. So, 2 r square plus alpha 1 x K x alpha 1 x K x y K y z K z will be put inside one bracket, but this is equal to 0. This quantity is equal to 0, because this is the condition the equation of the plane which is perpendicular to the direction of propagation, but in and intersecting the ellipsoid. And this is the equation of the ellipsoid. So, the value of this quantity is equal to 1, and the value of this quantity is equal to 0. Knowing that and the value of this quantity is equal to r square. So, we can write that 2 r square is equal to 2 alpha 1 into 1.

So, that is what I have written. 2 r square plus 2 alpha 2 is equal to 0; which gives you that the alpha 2 value equal to minus r square. So, we have been able to find out one value of the 2 unknown constants that is alpha 2 equal to minus r square. And the task lies next is to identify to calculate the value of alpha 1. So, determine to determine the value of alpha 1, we have to multiply these equations respectively by K x K y and K z.

So, this minimum extremum condition equations will be multiplied by K x, K y and K z if you multiply by K x, this will give you K x x, this will give you K y y and this will give you K z z. So, put together they will again represent the plane the equation of the plane and which will be put equal to 0. And this one will be equal to alpha 1 x K x, alpha 1 y K y, alpha 1 z K z. So, we find this set of 3 equations when we multiplied by K x, K y and K z.

Now, we will add these 3 equations, which will give me that this 2 into x K x plus y K y plus z K z. And for the second terms when added up you get alpha 1 K x square plus K y square plus K z square. And for the third term you get this 2 alpha 2 x K x by capital K x y K y by capital K y and so on which is equal to 0. So, this part this is equal to 0 because this represents the equation of the plane and this quantity K x square plus K y square plus K z square is equal to 1. Because K x square plus K y square plus K z square equal to k square equal to 1.

So, I get alpha 1 equal to 2 into alpha 2 multiplied by this quantity is equal to 0. So, we can write the value of alpha 2 in terms of alpha 1 in terms of alpha 2. And this, but alpha 2 is equal to minus r square so, that is again known.

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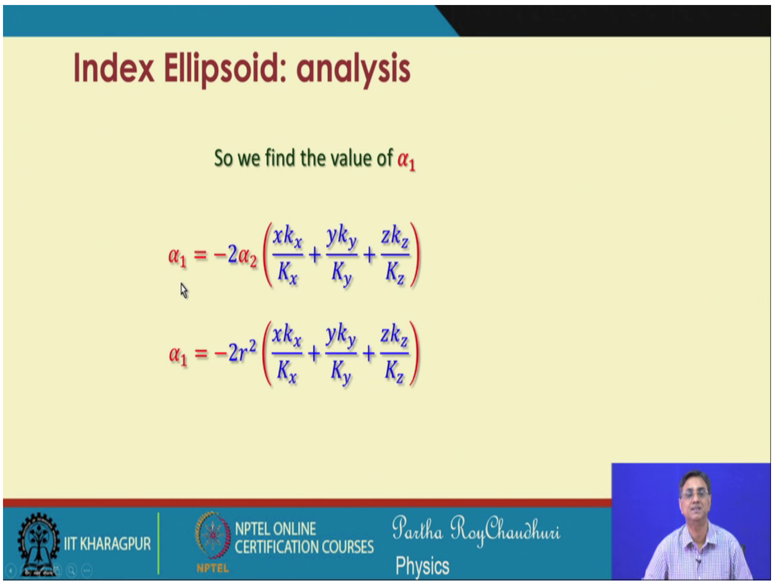

So, we can write this equation alpha 1 equal to minus 2 alpha 2 into this or we can write alpha 2 in terms of r square. So, we have been able to determine the 2 constants alpha 1 and alpha 2 exclusively in terms of the known parameters that is x K x and capital K x etcetera.

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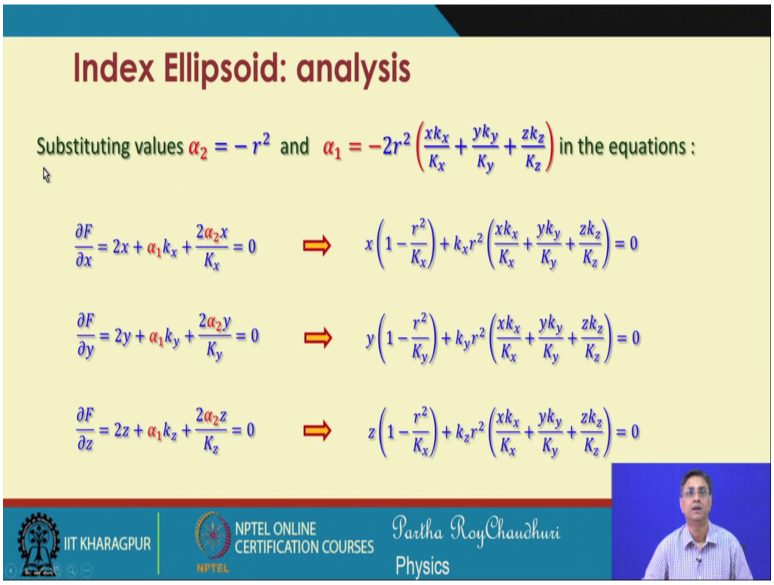

Now, we substitute these values alpha 2 alpha 1 back into these equations. So, if we substitute in this equation I will get this equation, and when I substitute this extremum condition equation, I will get this equation which is the y component, and when I substitute into this equation I will get. So, again I get a set of 3 equations corresponding to the 3 extremum conditions, these 3 equations will constitute the Eigen value equation.

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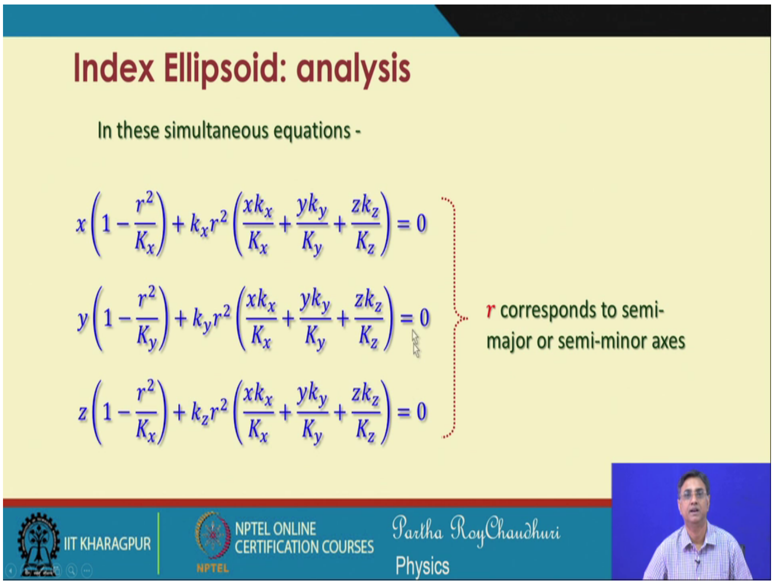

So, this r corresponds to the semi major and minor axes. In this set of 3 simultaneous equations, you can see r square r square appearing everywhere, here also r square.

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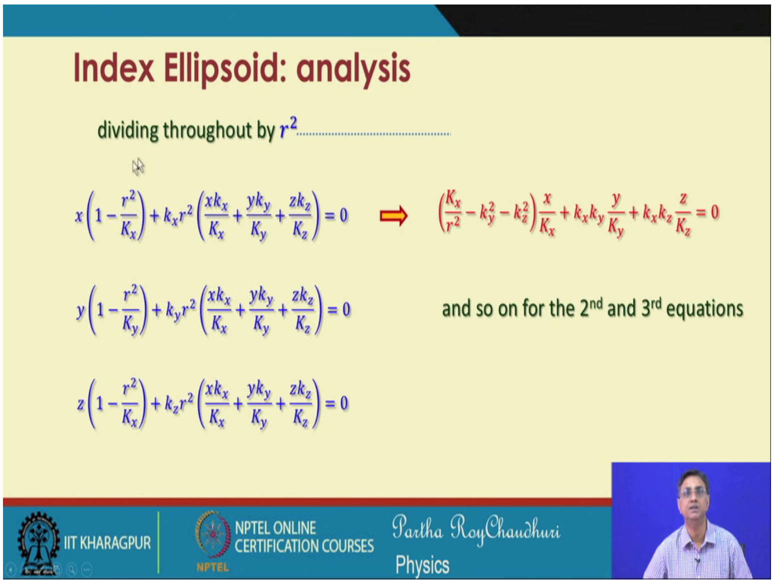

Now, if I divide this equation throughout by r square, then we can write this equation in this form. This first equation when you divide by r square you will get K x by r square K y square K z square and so on, for the second and third. So, you will again get a set of 3 equations where the r square now appear in the denominator.

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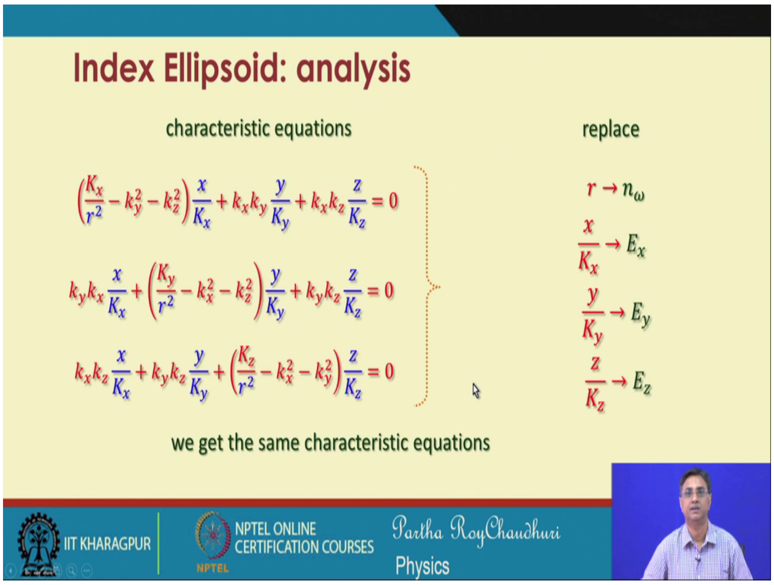

So, remember that this equation is again the same equation which we calculated from the Eigen value equation matrix equation from starting from the Maxwell's equation. And to identify that let us replace r by n omega and x by capital K x by E x and so on, then this equation will take the shape of the same equation the form of the same equation that is K x by n omega square go back, and see that K x by r square will become K x by n omega square and so on.

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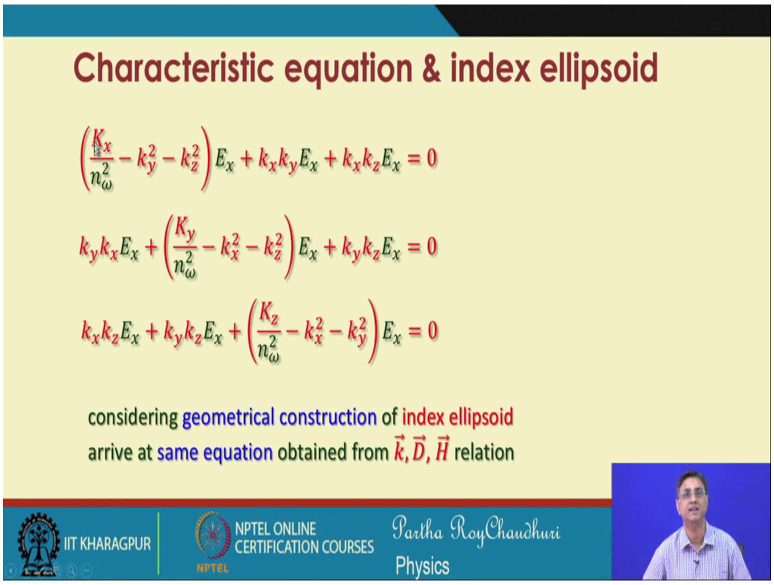

So, we get this equation which is the same equation that we obtained from the characteristic matrix equation, but this time we have obtained this equation using the geometrical construction of the index ellipsoid. And we arrived at the same equation in terms of k, D, H relation.

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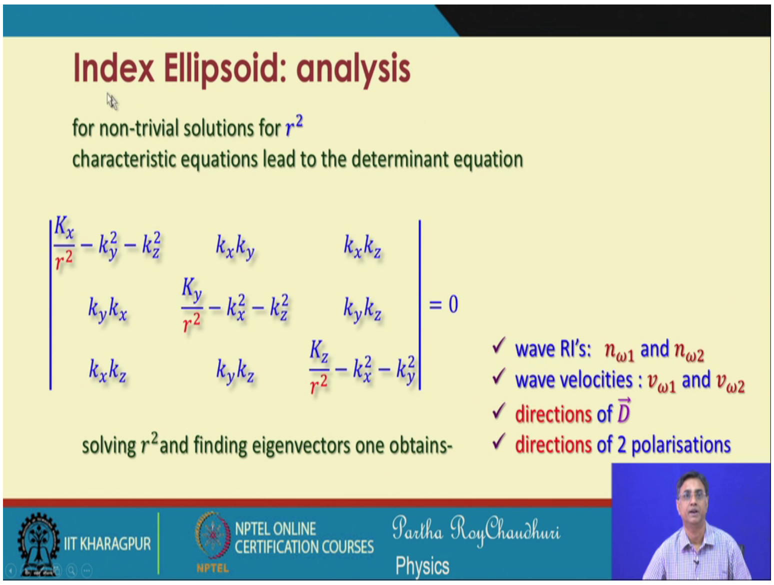

Ok now again for non-trivial solution for r square, we will put this equal to 0. And solving for r square and finding the Eigen vectors, we can obtain the n omega 1 and n omega 2 because this r square is nothing but the representative of n omega square. So, this the from the in the geometrical case it is r square, but in the case of eigenvalue equation obtained from the matrix it was n omega square.

So, if I can measure the value of r, then I will get the in omega 1 and n omega 2. Knowing these 2 values, I can find out the velocities the directions of D and the directions of polarization.

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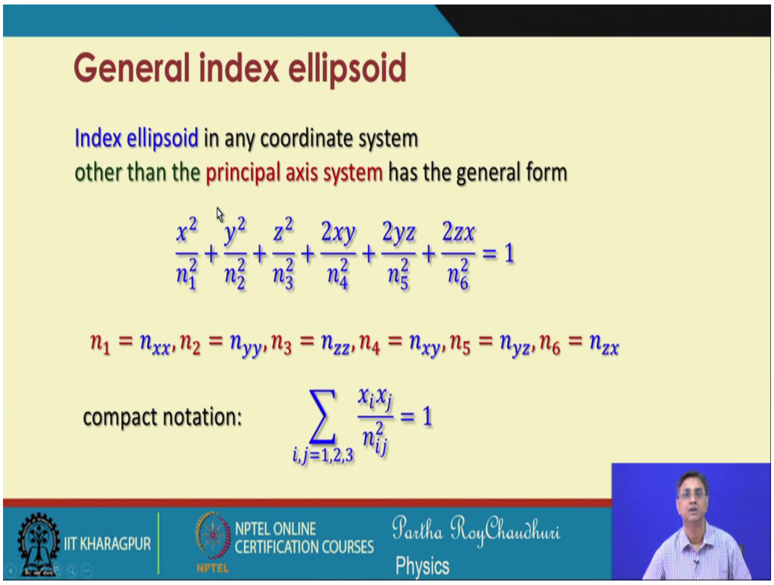

So, the case that we have discussed is the one which corresponds to the principal axes system, the index ellipsoid. But in any general coordinate system, we can write this index ellipsoid which will also involve the cross terms with x y, y z and z x and the of diagonal elements of the permittivity tensors.

So, that is n x equal to n x x n y y and so on. And in a compact notation, we can use this equation, we can rewrite in this form. And this is the form which we will be using extensively throughout our discussion in the case of electro optics and acousta optics their modulators and devices.

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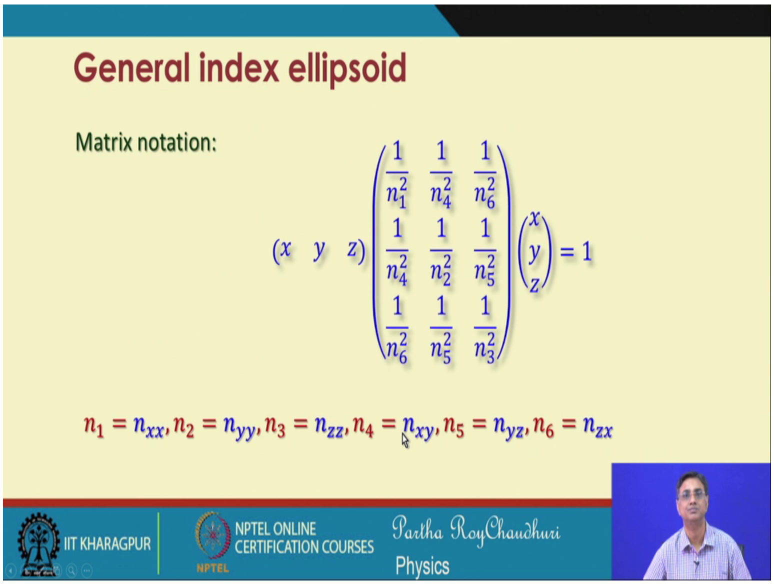

So, in the matrix notation this compact equation can be written in this form where this n x, n y, n z etcetera all are defined.

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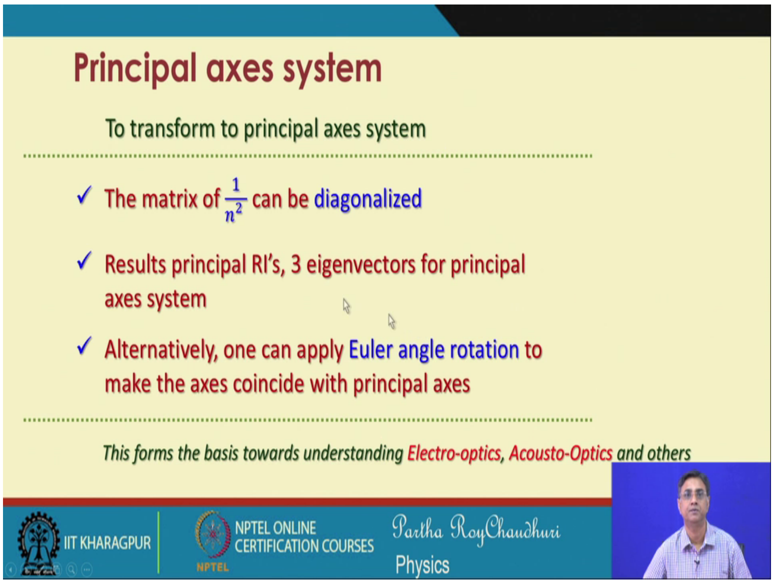

So, the matrix of 1 by n square can be diagonalized. So, the idea is that if we are given a medium where the medium we do not know about the principal axes system of the medium. And we want to identify that using the knowing the permittivity tensor which is in general not diagonal, then we can diagonalize and by diagonalizing we can find out the eigenvalues that the eigenvalues. And from there we can find out the Eigen vectors which will correspond to the direction of the electric field or the D field in the principal axis system.

So, there are 2 ways of doing that by diagonalizing this matrix there is an alternative way that is by using the Euler angle rotation of the of the axes system to coincide with the principal axes system which will also give us the and this form will form the basis of understanding this electro optics and acousta optics. So, this discussion is very much relevant to understand the electro optic effect and the acousta optic effects and so on.

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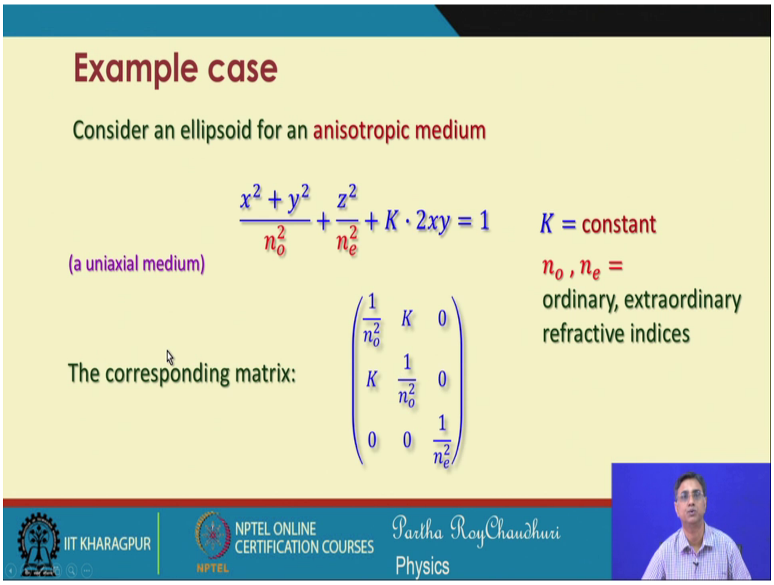

Consider an ellipsoid which is of this form, this is an example case, but we will study it again where you have this medium permittivities and then if you diagonalize this equation we can find out that if we apply a Euler angle rotation by inspection because it involves only the cross term in x and y.

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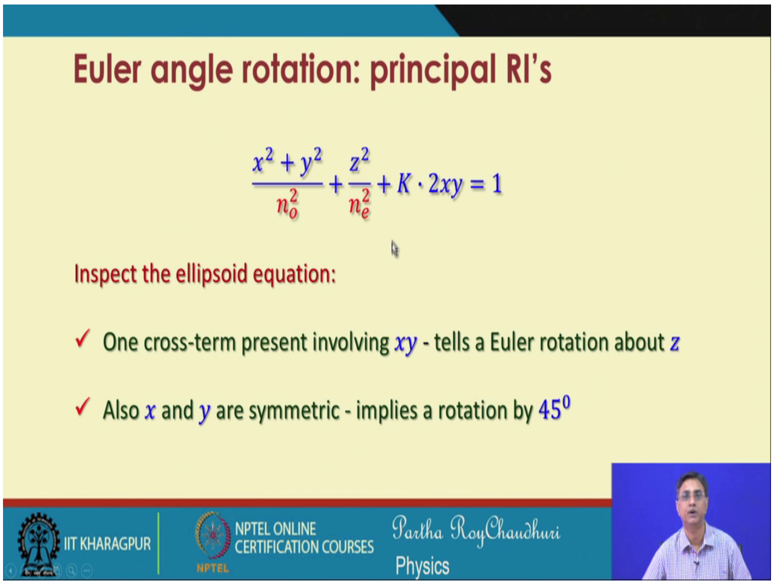

So, it requires a rotation in x and y along the z direction and because it is symmetric.

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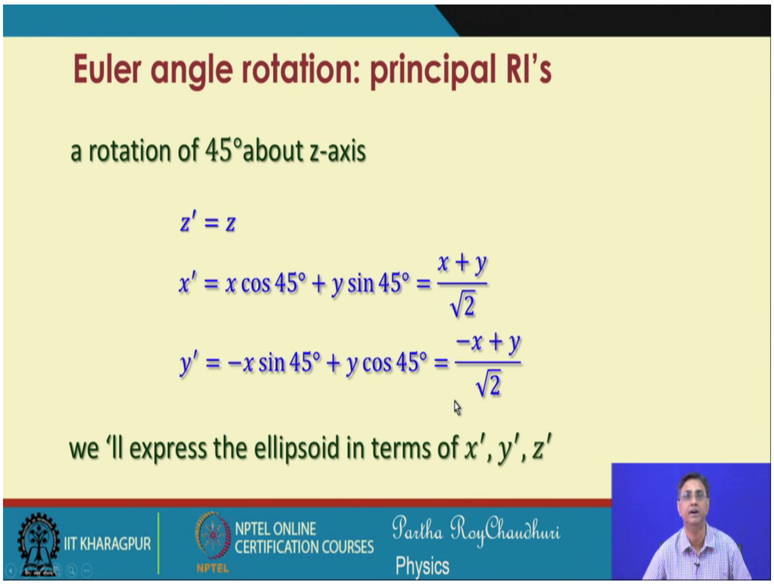

So, a rotation by 45 degree is required, if I do that then the coordinate transformation will be related by this equation. But we want to express this x, y, z in terms of x dash, y dash and z dash.

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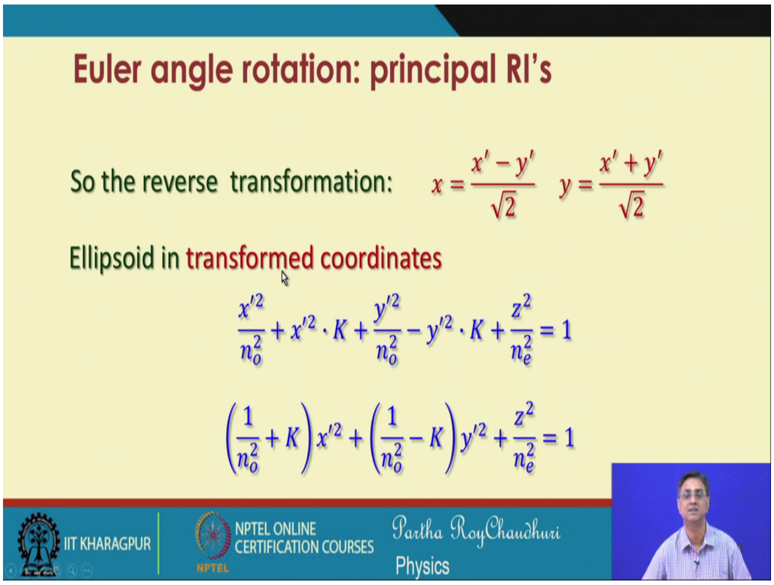

And by doing that we get this x equal to this and y equal to this, and z is equal to z prime. And if we substitute back into the ellipsoid equation, we will end up with this which will give me the new ellipsoid equation, which is n x, n y, n z; these are the terms which involved.

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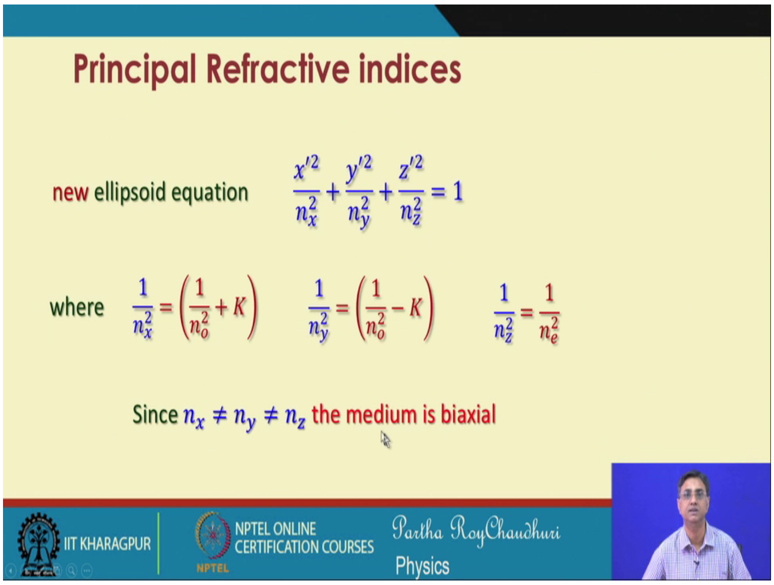

So, this n x will be represented by this n y will be represented by this and n z by this. Now we can see that we find that the medium is a biaxial medium, we will discuss this case alongside the diagonalization case, when we will take up in the example cases study next time.

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So, in this section, we have discussed the index ellipsoid, the construction and the geometry of the index ellipsoid, and the corresponding permittivity tensor by this geometrical construction we have discussed how to extract the wave refractive indices, and the polarization directions, the value of the electric field and the displacement field. Then we talked about the example cases, which we will discuss in more details in the next section.

Thank you very much.